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ESTIMATION OF THE CRITICAL TIME STEP FOR EXPLICIT INTEGRATION

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Abstract: *Explicit integration plays a key role in many problems of linear and non-linear dynamics. For example, the finite element method applied to spatial discretization of continua leaves a system of ordinary differential equations to be solved, which is often done by the central difference method. This and similar explicit schemes suffer from magnification of the round-off errors if the time step exceeds certain fixed length known as the critical time step. The corresponding critical Courant number (*Cr*, dimensionless time step) is inversely proportional to the maximum natural frequency of the system. The well known recommendation* $\mathrm{Cr} = 1$ *is deemed as the best. In fact, for some configurations this choice may dangerously overestimate the true value. It was shown in an earlier paper by the same authors that by increasing the number of elements in the finite element mesh one will paradoxically improve the mesh's stability towards its theoretical limit. The present paper refines some details, presenting small scale numerical tests. The first test involves a long truss/bar consisting of one row of elements whose critical Courant number changes as elements are added one after another. Since this increases the critical number one may pick up a time step such that it is supercritical to a certain mesh but becomes subcritical by merely adding one element. In a similar fashion, a square area is tested in the second example, using different arrangements of edge supports. It is concluded that the usual setting,* $\mathrm{Cr} = 1$*, is not entirely safe.*

Keywords: ***Explicit integration, Courant number, Critical time step, Wave propagation, Dispersion***

A number of transient problems of wave dynamics airising in computational sciences have been solved by explicit integration methods. As a representative of those, the most widely used—the central difference method (CDM)—is adopted as a subject of detailed analysis. As is well known, this method is only conditinally stable, which places a constraint on the length of time step, one stipulating that its size should be limited twice the reciprocal of the maximum angular frequency, $\omega_{\max}$, of the computational grid. In the finite difference or finite element method one had better employ dimensionless time step, which is known as Courant's number, as

$$\mathrm{Cr} := \frac{c\Delta t}{H}$$

with c being the wave speed, Δt time step and H an element size. The classic Courant-Friedrichs-Lewy (CFL) condition, $\mathrm{Cr} \leq 1$, applicable to hyperbolic systems is often made use of. In terms of accuracy, the famous recommendation $\mathrm{Cr} = 1$ (or slightly less to be on the safe side) for linear finite elements is deemed to be best.

An extended survey of papers concerning accuracy and stability was presented by Plešek et al. (2010). This review paper and the references cited therein discussed various discretization approaches, including finite elements with consistent and lumped mass matrices. It was shown that the CFL condition could be derived from dispersion analysis but for some configurations it might dangerously overestimate the true critical time step. It was also shown that by increasing the number of elements, N, in the finite element mesh one would improve the mesh's stability towards $\mathrm{Cr}_{\mathrm{crit}} = 1$ as $N \to \infty$, which was rather a paradoxical finding.

The present paper refines these details, presenting small scale numerical tests, which exemplify some peculiarities. The first test involves a long truss/bar consisting of one row of elements whose critical Courant number changes as elements are added one after another. Since this increases the critical number one may pick up a time step such that it is supercritical to a certain mesh but becomes subcritical by

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merely adding one element. In a similar fashion, a square area is tested in the second example, using different arrangements of edge supports. It turns out that the numerical solutions to wave propagation may be strongly influenced by small variation of distant boundary conditions, which should normally be physically insignificant. Finally, the third illustration shows the direct numerical results relevant to the above mentioned choices of sub and supercritical times steps. The examples involving free bodies clearly demonstrated the way the vibration of corner elements changed the stability limits.

It might seem at first glance that, except illustrating certain mathematical principles, the present study bears little importance to real-world computation. On the one hand, today's engineering problems are extremely large (rendernig $N \to \infty$ effectively) and, on the other hand, one may safely use the upper bound by calculating the maximum eigenvalue of a single element, ω_e. This is manifested in reference (Fried, 1972) by the inequality

$$\omega_{\max} \leq \omega_e$$

It should be borne in mind that the latter estimate, is only useful for a structured mesh when all the elements have the same spectrum. For an unstructured mesh, this information is hardly available and one must resort to alternatives. It is namely under such circumstances that an analyst uses the CFL asymptotic limit derived from dispersion spectra often unaware of any risk.

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