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# THE LAMINAR FLOW SOLUTION IN THE PLANE BY EIGENMODE EXPANSION

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**Abstract:** *The paper deals with non-stationary laminar flow solution of an incompressible fluid. The Navier-Stokes equation is used for the description of this motion and it is solved by means of an expansion into a series of eigenmodes of vibration. A mathematical model, which assumes planar flow with specific boundary conditions, can be generalized to the spatial problem with different types of boundary conditions.*

**Keywords:** ***Navier-Stokes equation, modal analysis, eigenmodes***

## 1. Introduction

The paper proposes a solution to a non-stationary laminar flow of incompressible fluid using an expansion into a series of eigenmodes of vibration. Pressure drop is chosen as the boundary conditions for this problem.

## 2. Problem definition

The Navier-Stokes equation was used as the mathematical model of the aforementioned planar flow in a pipe of length $L$ (axis $x_1$) and height $H$ (axis $x_2$):

$$\rho \frac{\partial c_1}{\partial t} + \frac{\partial p}{\partial x_1} - \mu \frac{\partial^2 c_1}{\partial x_2^2} = 0. \tag{1}$$

The flow with the pressure drop was considered and hence the pressure at the ends of pipe was chosen as the boundary conditions. In order to solve this problem, initial conditions for velocity and pressure are required. Initial velocity was chosen as zero and initial pressure was prescribed by using a general function $\varphi$, which depends only on the position in axis $x_1$.

Equation (1) was then transformed into the following form:

$$\frac{\partial c_1}{\partial t} - \nu \frac{\partial^2 c_1}{\partial x_2^2} = \frac{p_1(t) - p_2(t)}{\rho L}. \tag{2}$$

The solution of the homogeneous part of this equation was considered to be of the form:

$$c_1(x_2, t) = e^{st}\, w(x_2), \tag{3}$$

where $s$ is the eigenvalue and $w$ is the eigenvector of velocity.

Using equation (3), the partial differential equation (2) was transformed to the ordinary differential equation with zero boundary conditions for the eigenmode shape of velocity. The eigenvalue $s$ was found to be a negative real number and the terms of the eigenvector were found to be orthogonal. To determine the form of $k$-th eigenvector, terms of the eigenvectors were considered to be actually orthonormal.

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Using these properties and also boundary conditions, an equation for the eigenvalue and also the form of the eigenmodes of velocity were then found:

$$s_k = -\nu\left(\frac{k\pi}{H}\right)^2 \quad \text{and} \quad w_k = \sqrt{\frac{2}{H}}\, sin\left(\frac{k\pi}{H}\, x_2\right) \tag{4}$$

## 2.1. Eigenmode expansion

In the previous section, only the homogeneous part of equation (2) was discussed. Here the complete solution of that equation will be considered:

$$\frac{\partial c_1}{\partial t} - \nu\,\frac{\partial^2 c_1}{\partial x_2^2} = \frac{p_1(t) - p_2(t)}{\rho L}, \tag{5}$$

accompanied by the zero boundary and initial conditions.

Solution of this equation was approached by using an expansion into a series of eigenmodes of vibration in the form of linear combination of time dependent function $a_k(t)$ and eigenmode shapes of velocity $w_k(x_2)$:

$$c_1(x_2, t) = \sum_{k=1}^{\infty} a_k(t)\, w_k(x_2). \tag{6}$$

The following expression was obtained:

$$c_1(x_2, t) = \frac{2H^2 \Delta p}{\pi^3 \rho L \nu} \sum_{k=1}^{\infty} \frac{1 - (-1)^k}{k^3}\,(1 - e^{s_k t})\; sin\left(\frac{k\pi}{H}\, x_2\right). \tag{7}$$

From equation (7) it can be easily seen that the velocity profile is composed only from odd eigenmode shapes of velocity, because the time dependent function $a_k(t)$ for even eigenmodes is zero.

## 3. Conclusions

This paper described the solution of the Navier-Stokes equation for non-stationary flow of an incompressible fluid. This solution is based on an expansion into a series of eigenmodes of vibration. The proposed approach considers the solution of velocity function as a combination of eigenmode shapes of velocity and time dependent function. Specific solution to the presented problem depends only on the odd eigenmode shapes of velocity since even eigenmodes are zero.


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