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TIME-DISCRETE INTEGRATION OF FINITE DEFORMATION

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Abstract: *Some necessary implications for the time-discrete integration of finite deformations will be discussed together with particular schemes, when the geometrical structure of the space of Cauchy-Green deformation tensors, implicitly contained in the principle of virtual power, is taken into account. All these time-discrete schemes reflect this geometrical structure in that the actual integration of corresponding evolution equation of deformation process takes place in the subset of positive-definite symmetric matrices (with non-euclidean geometry) by definition, instead of in the linear space of symmetric matrices (with euclidean geometry) as usual.*

Keywords: ***Finite deformation, time-discrete integration, Runge – Kutta – Munthe-Kaas method.***

The conference paper is intended to draw attention to one of consequences, namely, the time-discrete approximation of finite deformation, when seeing a deformation process as a curve in the space of deformation tensors – in the sense of Noll and Seguin (2010), though using a rather different mathematical infrastructure and, moreover, employing natural geometry of this space, inherited from the principle of virtual power, see Fiala (2011). This approach provides exact and geometrically consistent procedure for linearization and integration of deformation process in time variable.

STARTING POINT: From the viewpoint of finite deformations, a deformation process can be represented pointwise by a trajectory $\mathbf{C} : I \to Sym^+(3,\mathbb{R})$ – the configuration space consisting of the set of all positive-definite symmetric matrices (*right Cauchy-Green deformation tensors*).

Note that $\partial\mathbf{C}_t = 2\mathbf{F}^T\mathbf{d}\mathbf{F} \in sym(3,\mathbb{R})$ – the linear vector space of all symmetric matrices, where $\mathbf{d}$ is the *rate-of-deformation* tensor (stretching) – symmetric velocity gradient, and $\mathbf{F}$ is deformation gradient. One can then prove (Fiala (2011)) the following proposition.

PROPOSITION: Within small deformations, a deformation process superposed on initially strained body, characterized by the initial deformation field $\mathbf{C}$, is represented by a trajectory in the linear vector space of all symmetric matrices $sym(3,\mathbb{R}) \equiv T_{\mathbf{C}}Sym^+(3,\mathbb{R})$ – the tangent space to the manifold $Sym^+(3,\mathbb{R})$ at a point $\mathbf{C}$, i.e. the space of all vectors emanating from $\mathbf{C}$.

Based on the *power of internal forces*, we introduce Riemannian metric on $Sym^+ \equiv Sym^+(3,\mathbb{R})$ to become a manifold with Riemannian geometry, so that we shall be able to analyse deformation process by means of tools of differential geometry. Similarly we set $sym \equiv sym(3,\mathbb{R})$. Let us consider the stress power

$$\frac{\delta E_i}{\delta t} := \int_{\mathcal{S}} (\sigma : d)\, dv = \int_{\mathcal{S}} g^{ik} g^{jl} \sigma_{kl}\, d_{ij}\; dv = \int_{\mathcal{B}} B_t^{ik} B_t^{jl} K_{kl}\, \tfrac{1}{2}\, \partial C_{t\,ij}\, dV = \tag{1}$$

$$= \int_{\mathcal{B}} \mathrm{tr}(\mathbf{B}_t\mathbf{K}_t\mathbf{B}_t(\tfrac{1}{2}\,\partial\mathbf{C}_t))\, dV = \int_{\mathcal{B}} \mathrm{tr}(\mathbf{C}_t^{-1}\mathbf{K}_t\mathbf{C}_t^{-1}(\tfrac{1}{2}\,\partial\mathbf{C}_t))\, dV = \int_{\mathcal{B}} \mathrm{tr}(\mathbf{P}_t(\tfrac{1}{2}\,\partial\mathbf{C}_t))\, dV, \tag{2}$$

where symbol σ, as usual, stands for the Cauchy stress field, $\mathbf{K}_t$ for the *convective stress* and $\mathbf{P}_t = \mathbf{C}_t^{-1}\mathbf{K}_t\mathbf{C}_t^{-1}$ for the *2nd Piola-Kirchhoff stress*.

Now, consulting the analytical mechanics, we can interpret

$$\mathbf{\Omega}_{\mathbf{C}}(\,.\,,.\,) := \mathrm{tr}\left(\mathbf{C}^{-1}(.)\mathbf{C}^{-1}(.)\right) \tag{3}$$

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as the Riemannian metric on Sym^+ at the point $\mathbf{C}$ – a particular deformation state, and, as a consequence, the convective stress $\mathbf{K}_t$ as the *vector* and the 2nd Piola-Kirchhoff stress $\mathbf{P}_t$ as the *covector fields* along deformation process $\mathbf{C}_t$. Interestingly – in view of the *logarithmic strain* $\log(\mathbf{C})$, a *geodesic* (i.e. straight line) $\mathbf{C}_t$ connecting two deformation states $\mathbf{C}_1$ and $\mathbf{C}_2$ then reads

$$\mathbf{C}_t = \mathrm{Exp}_{\mathbf{C}_0}(t\mathbf{H}) := \mathbf{C}_0 \exp(t\mathbf{C}_0^{-1}\mathbf{H}), \tag{4}$$

where $\mathbf{H} = \mathbf{C}_0 \log(\mathbf{C}_0^{-1}\mathbf{C}_1)$, and exp, log stands for matrix exponential, resp logarithm.

Now, we can draw conclusions of the geometrical structure of Sym^+ for the *time-discrete integration* of finite deformations. If we calculate, starting from a given deformation state of a body $\mathbf{C}_t$, a deformation increment $\partial\mathbf{C}_t$, based on linearized equations and prescribed increments of external loading and displacement, the new resultant deformation $\mathbf{C}_{t+\Delta t}$ then is obtained by mapping this deformation increment to the space of all deformations Sym^+ starting at the initial state.

$$\mathbf{C}_{t+\Delta t} = \mathrm{Exp}_{\mathbf{C}_t}(\Delta t\, \partial\mathbf{C}_t)\,. \tag{5}$$

In our context of Sym^+, the generalized exponential map (2) adds up an increment of deformation $\mathbf{H} \equiv \partial\mathbf{C}_0 \in T_{\mathbf{C}_0}Sym^+$ to the deformation $\mathbf{C}_0 \in Sym^+$, so that the resulting deformation $\mathbf{C}_1(\mathbf{H}) = \mathrm{Exp}_{\mathbf{C}_0}(\mathbf{H})$ stays in the space of deformations Sym^+. This would not be the case if we just set $\mathbf{C}_1(\mathbf{H}) = \mathbf{C}_0 + \mathbf{H}$ due to neglecting the "shape" of Sym^+ within the linear vector space of symmetric tensors sym.

CONSEQUENCE: Resulting deformation $\mathbf{C}_1(\mathbf{H})$ from adding an increment of deformation $\mathbf{H}$ to the deformation $\mathbf{C}_0$ is given by

$$\begin{aligned} \mathbf{H} \longmapsto \mathbf{C}_1(\mathbf{H}) &\equiv \mathrm{Exp}_{\mathbf{C}_0}(\mathbf{H}) := \mathbf{C}_0 \exp(\mathbf{C}_0^{-1}\,\mathbf{H}) = && (6)\\ &= \mathbf{C}_0 + \mathbf{H} + \tfrac{1}{2!}\,\mathbf{H}\mathbf{C}_0^{-1}\,\mathbf{H} + \tfrac{1}{3!}\,\mathbf{H}\mathbf{C}_0^{-1}\,\mathbf{H}\mathbf{C}_0^{-1}\,\mathbf{H} + \ldots && (7) \end{aligned}$$

The approach mentioned above is nothing but the forward or explicit Euler's scheme, only conditionally stable, for *evolution equation* of deformation process

$$\partial\mathbf{C}_t = 2\mathbf{C}_t\mathbf{D} \tag{8}$$

evolving on Sym^+, where $\mathbf{D} = \mathbf{F}^{-1}\mathbf{dF}$, which is constant along geodesics.

After having summed up basic facts related to time-discrete integration of finite deformations, we shall first discuss the geometry of the underlying configuration space Sym^+ and the properties of evolution equation of finite deformation on this space. Then we introduce methods of its solution in terms of Runge – Kutta – Munthe-Kaas (RKMK), and finely, briefly mention another closely related method – again based on Lie group approach.

Conclusions

We analysed the evolution equation for finite deformations. We proved that instead of considering it on the linear vector space of symmetric matrices sym, it actually evolves on its subset – the manifold of symmetric positive definite matrices $Sym^+ \subset sym$, so that the usual time-discrete integration schemes are inapplicable. However, thanks to the specific geometry of Sym^+, due to the principle of virtual power, the modified RK method, namely the Runge – Kutta – Munthe-Kaas method applies. Moreover, the closely related Magnus expansion method, based on the same geometric approach, might prove especially useful for highly-oscillatory problems, see Iserles et al. (2000).

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