18th International Conference
ENGINEERING MECHANICS 2012
Svratka, Czech Republic, May 14 – 17, 2012

pp. 278–279
Paper #304

STUDY OF COMPUTATIONAL EFFICIENCY OF NUMERICAL QUADRATURE SCHEMES IN THE ISOGEOMETRIC ANALYSIS

D. Rypl, B. Patzák [*]

Abstract: *Isogeometric analysis has been recently introduced as a viable alternative to the standard, polynomial-based finite element analysis. One of the fundamental performance issues of the isogeometric analysis is the quadrature of individual components of the discretized governing differential equation which may become computationally prohibitive because the evaluation of the high degree basis functions and/or their derivatives at individual integration points is quite demanding. The aim of this paper is to compare computational efficiency of several numerical quadrature concepts which are nowadays available in the isogeometric analysis. Their performance is assessed on the assembly of stiffness matrix of B-spline based problems with special geometrical arrangement allowing to determine minimum number of integration points leading to exact results.*

Keywords: ***Isogeometric analysis, numerical quadrature, Gaussian quadrature, Bezier extraction, half-point rule.***

1. Introduction

The concept of the isogeometric analysis (IGA) (see paper Hughes (2005)), initially motivated by the gap between the computer aided design (CAD) and the finite element analysis (FEA), builds upon the concept of isoparametric elements, in which the same shape functions are used to approximate the geometry and the solution on a single finite element. The IGA, as its name suggests, goes one step further because it employs the same functions for the description of the geometry and for the approximation of the solution space on that geometry. This implies that the isogeometric mesh (discretization for computational purposes) of the CAD geometry encapsulates the exact geometry no matter how coarse the mesh actually is. As a consequence, the need to have a separate representation for the original CAD model and another one for the actual computational geometry is completely eliminated. It has been shown that the IGA outperforms the classical FEA in various aspects, which is the consequence of several important advantages of the IGA compared to the FEA. On the other hand, the computational effort of the IGA, especially when using higher order basis functions, seems to exceed that for the FEA. One of the fundamental performance issues of the IGA is the integration of individual components of the discretized governing differential equation. The capability of the IGA to adopt basis functions of high degree together with the (generally) rational form of those basis functions implies that high order numerical quadrature schemes must be employed. This becomes computationally very prohibitive because the evaluation of the high degree basis functions and/or their derivatives is quite demanding. The situation tends to be critical in 3D where the total number of integration points can increase dramatically.

2. Adopted quadrature schemes

Currently there are three quadrature concepts available for the IGA. The first, most natural one, performs the integration over individual non-zero knot spans (intervals of the underlying parametric space) on which the basis functions are continuous up to infinite order. Due to the tensor product structure of the basis functions on individual knot spans of a two- and three-dimensional B-spline patch, the Gaussian quadrature schemes used for (so much popular) quadrilateral and hexahedral finite elements can be readily adopted in the IGA. The second concept benefits from the fact that the smooth B-spline basis can be constructed as a linear combination of a C^0 Bernstein polynomials which are the basis functions on the

[*] Doc. Dr. Ing. Daniel Rypl, Prof. Dr. Ing. Bořek Patzák: Department of Mechanics, Faculty of Civil Engineering, Czech Technical University in Prague, Thákurova 7; 166 29, Prague; CZ, e-mail: daniel.rypl@fsv.cvut.cz, borek.patzak@fsv.cvut.cz

so-called Bezier element. Because the Bernstein polynomials are defined over the same parametric domain (typically from 0 to 1) and because the degree of Bernstein basis is the same for all Bezier elements within a single B-spline patch, the values of individual Bernstein basis functions and their derivatives are the same at individual Gauss integration points on all Bezier elements and can be therefore precomputed. Since there is typically as many Bezier elements as non-zero knot spans, this concept resembles the first one in the sense that the non-zero knot spans are used as the basic integration units. The last approach attempts to profit from the continuity of the basis functions between the adjacent knot spans. By taking into account the precise smoothness of the basis functions across boundaries of infinite number of uniform knot spans, a simple integration rule (so-called half-point rule) independent of the degree of the polynomial basis and having (in 1D) just one integration point per two knot spans has been derived. For practical purposes, however, integration rules corresponding to open non-uniform finite knot vector are desirable. These rules can be obtained by numerical solution of a system of non-linear equations which is computationally demanding and which is worth only if the rules are applied repeatedly many times. Therefore only rules on 2, 3, 4, or 5 consecutive uniform knot spans for few cases of degree of practical interest have been derived (see paper Hughes (2010)). Although these rules only approach the best possible performance, the savings, especially in 3D, are significant.

3. Comparison of quadrature schemes

Although the above quadrature schemes are used generally for (only approximate) integration of rational functions, they can handle precisely only polynomials. Therefore the examples on which the efficiency of above quadrature schemes have been assessed (using assembly of the stiffness matrix) are chosen to be one-, two-, and three-dimensional B-spline patches with orthogonal system of isoparametric curves with control points defined by Graville's coordinates. This ensures that all components as well as the determinant of the Jacobian matrix are constant and that the minimum number of integrations points leading to exact results can be safely determined. In order to enable application of half-point rules, the (open) knot vectors describing the parameterization of the B-spline patch are always uniform. Since the efficiency of these rules is dependent on the actual number of knot spans, the same number of knot spans is used for each spatial dimension. The results of the comparison, namely the timing to perform the assembly of stiffness matrix, can be found in the full paper.

4. Conclusions

The investigation of computational efficiency of the above quadrature approaches has revealed that the main source of the computational costs of the numerical quadrature is dependent on the spatial dimension qualitatively as well as quantitatively. While in 1D the prevailing costs are related to the expensive evaluation of basis functions and their derivatives and are increasing with the degree and consequently with the complexity of the B-spline basis functions, in 3D, the dominating costs are associated with the assembly of the contributions to the stiffness matrix at individual integration points, number of which as well as the size of the contributions is also growing with the degree. This implies that in 1D, faster algorithms are those which profit from the precomputed values of the basis functions and their derivatives (schemes based on the Bezier elements). In 3D, on the other hand, since the critical factor is the total number of integrations points, the quadrature rules that benefit from taking into account the continuity between the knot spans (such as half-point rule scheme) are the better ones. In 2D, both effects are combined. The numerical evidence shows, however, that the half-point rule scheme is more appropriate than other considered schemes.

Acknowledgments

This work was supported by the Technology Agency of the Czech Republic - Project No. TA02011196.

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