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Partitioned Equations of Motion for Wave Propagation Problems in Solids

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Abstract: We present a decomposition of equations of motion in solids into a $curl$-zero (longitudinal) component and a $divergence$-zero (shear) component. Partitioned equations of motion are needed for accurate finite element numerical modelling of wave propagation problems. In that case, each equation of motion is integrated separately with corresponding critical time step size in explicit time integration. By this approach, dispersion behaviour of the finite element method and mainly spurious oscillations in numerical results can be suppressed.

Introduction

A novel explicit time integration method in finite element computations of wave propagation problems in solids has been presented in [1, 2, 3]. This technology is based on separate integration of longitudinal and shear waves with each critical time step size respecting different wave speeds. The mentioned time integration scheme is the three-time step scheme in the predictor-corrector form and it produces excellent results without spurious oscillations near theoretical wavefronts. In this paper, we give a theoretical framework for a wave orientated decomposition of the equations of motion in solids.

Decomposition of displacement fields and equations of motion

From the vector analysis [4], we know that a vector field $\mathbf{p}(\mathbf{x})$ with described boundary conditions, which is defined on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$, $\mathbf{x} \in \Omega$, and is twice continuously differentiable, can be decomposed into two parts: an $irrotational$ (curl-free) component which can be expressed by the gradient of a scalar function and a $rotational$ (divergence-free) component which can be expressed by the curl of a vector function. This decomposition of a vector field is called the $Helmholtz$ decomposition [4, 5].

In the Helmholtz decomposition, the vector field $\mathbf{p}$ is given by the sum of the gradient of a scalar potential ϕ and the curl of a vector potential $\boldsymbol{\psi}$ as

$$\mathbf{p} = \operatorname{grad}\phi + \operatorname{curl}\boldsymbol{\psi} \quad \text{with the condition} \quad \operatorname{div}\boldsymbol{\psi} = \mathbf{0} \tag{1}$$

The proof of the Helmholtz decomposition of a vector field on arbitrary domains and with prescribed non-homogeneous boundary conditions can be found in [6]. Here, formulæ for the scalar potential ϕ and the vector potential $\boldsymbol{\psi}$ have been also derived.

For the following text, we denote expressions $\mathbf{p}_1 = \operatorname{grad}\phi$ and $\mathbf{p}_2 = \operatorname{curl}\boldsymbol{\psi}$. The main properties of the Helmholtz decomposition of the vector field $\mathbf{p}$ decomposed by Eq. 1 are following

$$\operatorname{div}\mathbf{p} = \operatorname{div}\mathbf{p}_1, \quad \operatorname{curl}\mathbf{p} = \operatorname{curl}\mathbf{p}_2, \quad \operatorname{curl}\mathbf{p}_1 = \mathbf{0}, \quad \operatorname{div}\mathbf{p}_2 = 0 \tag{2}$$

These expressions serve for derivation of a decomposition of equations of motion in solids.

The equations of motion in solids (the law of balance of linear momentum) [5] defined on a domain $\Omega \subset \mathbb{R}^3$ take the vector form as

$$\operatorname{div} \boldsymbol{\sigma} + \mathbf{b} = \rho \ddot{\mathbf{u}} \tag{3}$$

where we assume that Dirichlet and Neumann boundary conditions of the problem are defined. Here, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\mathbf{b}$ is the volume intensity vector, $\ddot{\mathbf{u}}(\mathbf{x}, t)$ is the acceleration vector field, $\mathbf{x} \in \Omega$ is the position vector, t is the time and ρ is the mass density. Each vector field, taking place at the equations of motion Eq. 3 (i.e. $\operatorname{div} \boldsymbol{\sigma}$, $\mathbf{b}$ and $\ddot{\mathbf{u}}$), can be decomposed by the Helmholtz decomposition into two parts: irrotational (marked with the subscript L) and rotational (marked with the subscript S). We suppose that boundary conditions for $\ddot{\mathbf{u}}$ and $\operatorname{div} \boldsymbol{\sigma}$ should be also respected. Then, we have got

$$(\operatorname{div} \boldsymbol{\sigma})_L + (\operatorname{div} \boldsymbol{\sigma})_S + \mathbf{b}_L + \mathbf{b}_S = \rho \ddot{\mathbf{u}}_L + \rho \ddot{\mathbf{u}}_S \tag{4}$$

If the operators div and the operator curl are applied on Eq. 4 and with respect to Eq. 2, it yields

$$\operatorname{div}\big[(\operatorname{div}\boldsymbol{\sigma})_L + \mathbf{b}_L - \rho\ddot{\mathbf{u}}_L\big] = 0, \qquad \operatorname{curl}\big[(\operatorname{div}\boldsymbol{\sigma})_S + \mathbf{b}_S - \rho\ddot{\mathbf{u}}_S\big] = \mathbf{0} \tag{5}$$

The two last equations are vanished only if it is valid

$$(\operatorname{div} \boldsymbol{\sigma})_L + \mathbf{b}_L = \rho \ddot{\mathbf{u}}_L, \qquad (\operatorname{div} \boldsymbol{\sigma})_S + \mathbf{b}_S = \rho \ddot{\mathbf{u}}_S \tag{6}$$

These two last equations are the sought decomposed system of the equations of motion in solids.

In the case of linear elastodynamic theory of an isotropic homogeneous body and neglecting of body forces, the equation Eq. 6left describes propagation of longitudinal waves and the equation Eq. 6right controls propagation of shear waves [5].

Summary

We presented a technique for decomposition of equations of motion in solids into longitudinal and shear components. These partitioned equations of motion serve for accurate numerical modelling of wave propagation problems. In [2, 3], a specific explicit time integration method using the partitioned equations of motion in finite element computation has been published. Further, the discrete longitudinal and shear operators as counterparts of the operators div and curl, which are able to guarantee the decomposition of an arbitrary discretized vector field by the linear finite element method as the sum of the $curl$-zero (longitudinal) and $divergence$-zero (shear) components, have been found in [2, 3].

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