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On the Finite Element Method Application for Approximation of Free-Surface Flows with Surface Tension

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Abstract: This paper focuses on the mathematical modelling and the numerical approximation of the flow of two immiscible incompressible fluids. The surface tension effects are taken into account and mixed boundary conditions are used. The weak formulation is introduced, discretized in time, and the finite element method is applied. The free surface motion is treated with the aid of the level set method. The numerical results are shown.

Introduction

The mathematical modelling of two-phase flows with the consideration of the free surface motion influenced by the surface tension is addressed in various scientific as well as technical applications. Such a problem is important both from the mathematical modelling point of view and also from the technical practice. Particularly, its numerical approximation is very challenging task, see among others [1], [2] or [3]. The approximation of the surface tension or the contact angle naturally can play a key role here.

In this paper, we consider the two-dimensional flow of two immiscible fluids, the problem is mathematically described and the variational formulation is introduced. For the discretization the finite element(FE) method is used. The free surface motion is realized using the level set method, cf. [7] or [5]. In the case of high surface tension, a modification of the standard FE method is required to avoid the spurious currents, see [6] or [1]. For the verification of the implemented method a benchmark problem is solved, cf. [3].

Mathematical description

Let us consider the computational domain $\Omega \subset \mathbb{R}^2$ with the Lipschitz continuous boundary $\partial\Omega$ with its mutually disjoint parts Γ_W, Γ_S, Γ_O. The domain is occupied at time t by two immiscible fluids, i.e. $\Omega = \Omega^A_{(t)} \cup \Omega^B_{(t)}$. Let us denote the Heaviside function $H(x,t)$ as $H(x,t) = 1$ for $x \in \Omega^A_{(t)}$, $H(x,t) = 0$ for $x \in \Omega^B_{(t)} \cup \hat{\Gamma}_t$. The density and the viscosity functions are then defined by $\rho(x,t) = \rho^A H(x,t) + (1 - H(x,t))\rho^B$ and $\mu(x,t) = \mu^A H(x,t) + (1 - H(x,t))\mu^B$, respectively. Then the twophase flow is governed by the equation

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla\cdot\boldsymbol{\sigma} = \rho\boldsymbol{f} + \gamma\kappa\boldsymbol{n}\delta_{\hat{\Gamma}_t}, \tag{1}$$

which in the weak form reads

$$\int_\Omega \rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right)\cdot\boldsymbol{v} + \boldsymbol{\sigma}\cdot(\nabla\boldsymbol{v})\,\mathrm{d}x = \int_{\hat{\Gamma}_t}\gamma\kappa\boldsymbol{n}\cdot\boldsymbol{v}\,\mathrm{d}S + \int_\Omega \rho\boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x, \tag{2}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor given by $\boldsymbol{\sigma} = -pI + \mu(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u})$.

In order to treat the surface tension term, the Laplace-Beltrami operator $\triangle_\Gamma = \nabla_\Gamma \cdot \nabla_\Gamma$ is used, where ∇_Γ denotes the tangent derivative with respect to the surface. Now, using the relation $\kappa \boldsymbol{n} = \triangle_\Gamma \boldsymbol{x}$ and applying the integration by parts on $\hat{\Gamma}_t$ we get

$$\int_{\hat{\Gamma}_t} \gamma \kappa \boldsymbol{n} \cdot \boldsymbol{v} \,\mathrm{d}S = -\int_{\hat{\Gamma}_t} \gamma (\nabla_\Gamma \boldsymbol{x}) \cdot (\nabla_\Gamma \boldsymbol{v}) \,\mathrm{d}S, \tag{3}$$

where for the sake of simplicity it was assumed that $\hat{\Gamma}_t$ is a closed curve.

Furthermore, to treat the motion of the free surface $\hat{\Gamma}_t$ the *level set* method is applied. First, the initial condition for the level set function $\phi = \phi(x,t)$ is prescribed by $\phi(x,0) = \text{dist}(x, \hat{\Gamma}_0) > 0$ for $x \in \Omega^A_{(0)}$, $\phi(x,0) = -\text{dist}(x, \hat{\Gamma}_0) < 0$ for $x \in \Omega^B_{(0)}$, and $\phi(x,0) = 0$ for $x \in \hat{\Gamma}_0$. The motion of the interface $\hat{\Gamma}_t$ is then realized by forcing the function ϕ to solve the equation

$$\frac{\partial \phi}{\partial t} + \boldsymbol{u} \cdot \nabla \phi = 0. \tag{4}$$

Numerical approximation

The presented problem is approximated in time using the backward difference formula and then the problem is spatially discretized by the FE subspaces defined over an admissible triangulation $\mathcal{T}_h$. For the approximation the well-known Taylor-Hood FE are used. In order to treat the discontinuity, the extended finite element method XFEM is used. The XFEM enlarges the original FE pressure space using the localization of an enrichment function, which in this case is the discontinuous Heaviside function.

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