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Forced Vibration Analysis of Timoshenko Beam with Discontinuities by Means of Distributions

Jiri Sobotka

67201 Moravsky Krumlov, Pionyrska 479, Czech Republic

jiri.sobotka@cez.cz

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Abstract. The general equations of motion for forced vibration of Timoshenko beam have been used since they were derived assuming there were not any discontinuities in the shear force, the bending moment, the cross-section rotation and the deflection of the beam. However in practice, computing harmonic response of the beam, we encounter concentrated loading, concentrated supports, hinges connecting beam segments, concentrated masses or concentrated moments of inertia all of which may be situated between ends of the beam. The definition of the distributional derivative is chosen in order that all the cases causing jump discontinuities in the shear force or the bending moment or the cross-section rotation can be incorporated. As a result of this approach, more general equations of motion for forced vibration of Timoshenko beam implying all jump discontinuities mentioned are presented in this paper. An analytic closed-form solution to the system of equations is found with integration constants in the form of initial parameters. Making use of this approach, we can find exact expression for the harmonic steady-state response of the uniform beam without summing infinite series and without doing a modal analysis of the beam.

Introduction

Classical analytical method of calculating the harmonic steady-state response of the uniform beam is based on the following main steps [1, 2]. Firstly, we obtain a frequency equation for specific support conditions of the beam. Secondly, we solve the frequency equation for natural frequencies. Thirdly, we find orthogonal mode shapes corresponding to the natural frequencies of the beam. Finally, applying modal analysis, we express the response as a linear combination of the mode shapes by finding corresponding modal participation coefficients.

In this paper, a new analytical method is presented. Applying distributions, it is not necessary for natural frequencies, mode shapes and modal participation coefficients to be computed in analyzing the harmonic steady-state response of the uniform beam.

The Classical Equations of Motion for Forced Transverse Vibration of Timoshenko Beam

According to Timoshenko's theory, we can express simultaneous differential equations of motion for forced vibration of the beam without discontinuities in the shear force, in the bending moment, in the rotation of the cross section or in the transverse displacement of the centerline of the beam [2] as

$$\rho A(x)\frac{\partial^2 w}{\partial t^2} = kG\frac{\partial}{\partial x}\left[A(x)\left(\frac{\partial w}{\partial x}-\varphi\right)\right] + f(x,t) \ , \tag{1}$$

$$J(x)\frac{\partial^2 \varphi}{\partial t^2} = kGA(x)\left(\frac{\partial w}{\partial x}-\varphi\right) + E\frac{\partial}{\partial x}\left[J(x)\frac{\partial \varphi}{\partial x}\right] \ , \tag{2}$$

where $w(x, t)$ is the total transverse displacement of the beam centerline, $\varphi(x,t)$ is the rotation of the cross section assumed without warping, $A(x)$ is the cross-sectional area, $J(x)$ is the area moment of inertia of the cross section with respect to the centroid axis perpendicular to the plane of vibration of the beam, k is the shear correction factor , E is the modulus of elasticity (Young's modulus), G is the shear modulus of elasticity, and ρ is the density of the beam material.

The Model for Forced Vibration of Timoshenko Beam with Discontinuities

In practice, computing harmonic response of the beam, we encounter:

- concentrated loading,
- concentrated supports,
- hinges connecting beam segments,
- concentrated masses,
- concentrated mass moments of inertia

that may be situated between ends of the beam causing jump discontinuities in solution.

We can apply Schwartz theory of distributions [3] in order to express jump discontinuities in a quantity, which is to be differentiated.

The first-order distributional derivative of a function with a jump discontinuity contains a continuous part and a distributional one which is the product of a magnitude of the jump and the Dirac-delta distribution moved to the point of the jump as follows

$$\frac{dy}{dx} = \left\{\frac{dy}{dx}\right\} + \left[\lim_{x\to a+} y - \lim_{x\to a-} y\right] \delta(x-a)\,,$$

where $\left\{\frac{dy}{dx}\right\}$ is a classical derivative, $[\lim_{x\to a+} y - \lim_{x\to a-} y]$ is a finite magnitude of the jump at $x=a$, and $\delta(x-a)$ is Dirac distribution moved to the point of the jump.

The right-hand side of Eq. 3 is the distributional derivative of the shear force $Q(x,t)$ with $n_1+n_2+n_3$ jump discontinuities. The beam may be supported also between its ends at n_1 concentrated supports with reaction forces of r_i, may carry n_2 concentrated masses of m_i, and may also be subjected to n_3 concentrated transverse forces of f_i and to a distributed transverse loading of $f(x,t)$.

The right-hand side of Eq. 4 is the distributional derivative of the bending moment $M(x,t)$ with n_4+n_5 jump discontinuities. The beam may carry n_4 concentrated mass moments of inertia of j_i, and may be subjected to n_5 concentrated force pairs of s_i situated between ends of the beam.

The right-hand side of Eq. 5 is the distributional derivative of the cross-section rotation $\varphi(x,t)$ with n_6 jump discontinuities of ψ_i. The beam may contain n_6 internal hinges connecting beam segments.

The right-hand side of Eq. 6 is the classical derivative of the deflection $w(x,t)$ covering the effects of bending and shear deformations [2].

$$\frac{\partial Q}{\partial x} = \rho A(x)\frac{\partial^2 w}{\partial t^2} + \sum_{i=1}^{n_1} r_i\delta(x-\xi_i) + \sum_{i=1}^{n_2} m_i\frac{\partial^2 w(x=\eta_i)}{\partial t^2}\delta(x-\eta_i) - \sum_{i=1}^{n_3} f_i\delta(x-\zeta_i) - f(x,t)\,, \quad (3)$$

$$\frac{\partial M}{\partial x} = Q - \rho J(x)\frac{\partial^2 \varphi}{\partial t^2} - \sum_{i=1}^{n_4} j_i \frac{\partial^2 \varphi(x=\gamma_i)}{\partial t^2}\delta(x-\gamma_i) + \sum_{i=1}^{n_5} s_i \delta(x-\varepsilon_i) \ , \tag{4}$$

$$\frac{\partial \varphi}{\partial x} = -\frac{M}{EJ(x)} + \sum_{i=1}^{n_6} \psi_i \delta(x-\lambda_i) \ , \tag{5}$$

$$\frac{\partial w}{\partial x} = \varphi + \frac{Q}{kGA(x)} \ . \tag{6}$$

Forced Vibration Solution

Supposing a harmonic time variation of loading as

$$f(x,t) = f_a(x) cos\Omega(t) \, , \qquad s_i(t) = S_i \cos(\Omega t) \ , \qquad f_i(t) = F_i \cos(\Omega t)$$

and solution to equations (3) to (6) as

$$Q(x,t) = Q_a(x)\cos(\Omega t) \ , \qquad M(x,t) = M_a(x)\cos(\Omega t) \, ,$$

$$\varphi(x,t) = \varphi_a(x)\cos(\Omega t) \, , \qquad w(x,t) = w_a(x)\cos(\Omega t) \, ,$$

$$r_i(t) = R_i \cos(\Omega t) \ , \qquad \psi_i(t) = \Psi_i \cos(\Omega t)$$

where Ω is circular frequency of vibration, and denoting amplitudes of vibration at points with concentrated transverse inertia forces and bending moments as

$$W_i = \lim_{x\to\eta_i} w_a(x) \ , \Phi_i = \lim_{x\to\gamma_i} \varphi_a(x) \ , \tag{7}$$

we can derive system of ordinary differential equations (8) to (11) for unknown general amplitudes of the deflection, w_a, the rotation of the cross section, φ_a , the bending moment, M_a, and the shear force, Q_a , for a uniform beam as:

$$\frac{dQ_a}{dx} = -\rho A w_a(x)\Omega^2 + \sum_{i=1}^{n_1} R_i \delta(x-\xi_i) - \sum_{i=1}^{n_2} m_i W_i \Omega^2 \delta(x-\eta_i) - \sum_{i=1}^{n_3} F_i \delta(x-\zeta_i) - f_a(x) \ , \tag{8}$$

$$\frac{dM_a}{dx} = Q_a(x) + \rho J \varphi_a(x)\Omega^2 + \sum_{i=1}^{n_4} j_i \Phi_i \Omega^2 \delta(x-\gamma_i) + \sum_{i=1}^{n_5} S_i \delta(x-\varepsilon_i) \ , \tag{9}$$

$$\frac{d\varphi_a}{dx} = -\frac{M_a(x)}{EJ} + \sum_{i=1}^{n_6} \Psi_i \delta(x-\lambda_i) \ , \tag{10}$$

$$\frac{dw_a}{dx} = \varphi_a(x) + \frac{Q_a(x)}{kGA} \ . \tag{11}$$

General Amplitudes of a Uniform Timoshenko Beam

We can use the Laplace transform method to compute general solution to the system of Eqs. (8) to (11) with integration constants in the form of initial parameters. Appling this method, we obtain a system of linear algebraic equations, which may be expressed in matrix form as follows

$$\begin{pmatrix} -1 & p & -\rho J\Omega^2 & 0 \\ -\frac{1}{kGA} & 0 & -1 & p \\ 0 & \frac{1}{EJ} & p & 0 \\ p & 0 & 0 & \rho A\Omega^2 \end{pmatrix} \begin{pmatrix} L\{Q_a\} \\ L\{M_a\} \\ L\{\varphi_a\} \\ L\{w_a\} \end{pmatrix} = \begin{pmatrix} M_a(0) + \sum_{i=1}^{n_4} j_i \Phi_i \Omega^2 e^{-p\gamma_i} + \sum_{i=1}^{n_5} S_i e^{-p\varepsilon_i} \\ w_a(0) \\ \varphi_a(0) + \sum_{i=1}^{n_6} \Psi_i e^{-p\lambda_i} \\ Q_a(0) + \sum_{i=1}^{n_1} R_i e^{-p\xi_i} - \sum_{i=1}^{n_2} m_i W_i \Omega^2 e^{-p\eta_i} - \sum_{i=1}^{n_3} F_i e^{-p\zeta_i} - L\{f_a\} \end{pmatrix}, \tag{12}$$

or symbolically

$$\boldsymbol{Bl} = \boldsymbol{c} \ , \tag{13}$$

where L{} denotes the Laplace transform, p is a complex variable of the Laplace transform.

The matrix $\boldsymbol{B}$ of the system (12) is regular because its determinant is nonzero:

$$\det(\boldsymbol{B}) = det \begin{pmatrix} -1 & p & -\rho J\Omega^2 & 0 \\ -\frac{1}{kGA} & 0 & -1 & p \\ 0 & \frac{1}{EJ} & p & 0 \\ p & 0 & 0 & \rho A\Omega^2 \end{pmatrix} = p^4 + \left(\frac{1}{E} + \frac{1}{kG}\right)\rho\Omega^2 p^2 + \frac{\rho^2\Omega^4}{kGE} - \frac{A\rho\Omega^2}{EJ} \ . \tag{14}$$

Since the matrix $\boldsymbol{B}$ is regular, there exists its inverse matrix as follows

$$\boldsymbol{B}^{-1} = \frac{1}{\det(\boldsymbol{B})}\begin{pmatrix} \frac{\mathrm{A}\rho\Omega^2}{EJ} & -\frac{\mathrm{A}\rho\Omega^2(\rho\Omega^2+p^2E)}{E} & -pA\rho\Omega^2 & \frac{\mathrm{p}(\rho\Omega^2+p^2E)}{E} \\ \frac{\mathrm{p}(\rho\Omega^2+p^2kG)}{kG} & -pA\rho\Omega^2 & \frac{\rho\Omega^2(p^2kGJ+\mathrm{J}\rho\Omega^2-AkG)}{kG} & p^2 \\ -\frac{\rho\Omega^2+p^2kG}{EkGJ} & \frac{\mathrm{A}\rho\Omega^2}{EJ} & \frac{\mathrm{p}(\rho\Omega^2+p^2kG)}{kG} & -\frac{\mathrm{p}}{EJ} \\ -\frac{\mathrm{p}}{EJ} & \frac{\mathrm{p}(\rho\Omega^2+p^2E)}{E} & p^2 & \frac{EJp^2-AkG-J\rho\Omega^2}{AEkGJ} \end{pmatrix}.$$

The Laplace Transforms of the Unknown General Amplitudes

Multiplying the inverse matrix of $\boldsymbol{B}$ with the column vector at right-hand side of equation (12), we receive the Laplace transforms of unknown general amplitudes as follows

$$(L\{Q_a\} \quad L\{M_a\} \quad L\{\varphi_a\} \quad L\{w_a\})^T = \boldsymbol{B}^{-1}\boldsymbol{c} \ , \qquad (15)$$

thus

$$L\{Q_a\} = \frac{1}{\det(\boldsymbol{B})}[\frac{p(p^2E+\rho\Omega^2)}{E}Q_a(0) + \frac{\rho\mathrm{A}\Omega^2}{EJ}M_a(0) - p\rho\mathrm{A}\Omega^2\varphi_a(0) - \frac{\rho\mathrm{A}\Omega^2(p^2E+\rho\Omega^2)}{E}w_a(0) - \sum_{i=1}^{n_6} p\rho\mathrm{A}\Omega^2\Psi_i e^{-p\lambda_i} + \sum_{i=1}^{n_1}\frac{p(p^2E+\rho\Omega^2)}{E}R_i e^{-p\xi_i} - \sum_{i=1}^{n_2}\frac{p(p^2E+\rho\Omega^2)}{E}m_iW_i\Omega^2 e^{-p\eta_i} - \sum_{i=1}^{n_3}\frac{p(p^2E+\rho\Omega^2)}{E}F_i e^{-p\zeta_i} + \sum_{i=1}^{n_4}\frac{\rho\mathrm{A}\Omega^4}{EJ}j_i\Phi_i e^{-p\gamma_i} + \sum_{i=1}^{n_5}\frac{\rho\mathrm{A}\Omega^2}{EJ}S_i e^{-p\varepsilon_i} - \frac{p(p^2E+\rho\Omega^2)}{E}L\{f_a\}] \ , \qquad (16)$$

$$L\{M_a\} = \frac{1}{\det(\boldsymbol{B})}[p^2 Q_a(0) + \frac{p(p^2 kG + \rho\Omega^2)}{kG} M_a(0) + \rho\Omega^2 \frac{(p^2 JkG + J\rho\Omega^2 - AkG)}{kG} \varphi_a(0)$$
$$- p\rho A\Omega^2 w_a(0) + \sum_{i=1}^{n_6} \rho\Omega^2 \frac{(p^2 JkG + J\rho\Omega^2 - AkG)}{kG} \Psi_i e^{-p\lambda_i} + \sum_{i=1}^{n_1} p^2 R_i e^{-p\xi_i}$$
$$- \sum_{i=1}^{n_2} p^2 m_i W_i \Omega^2 e^{-p\eta_i} - \sum_{i=1}^{n_3} p^2 F_i e^{-p\zeta_i} + \sum_{i=1}^{n_4} \frac{p(p^2 kG + \rho\Omega^2)}{kG} j_i \Phi_i \Omega^2 e^{-p\gamma_i}$$
$$+ \sum_{i=1}^{n_5} \frac{p(p^2 kG + \rho\Omega^2)}{kG} S_i e^{-p\varepsilon_i}$$
$$- p^2 L\{f_a\}] \quad , \qquad (17)$$

$$L\{\varphi_a\} = \frac{1}{\det(\boldsymbol{B})}[-\frac{p}{EJ} Q_a(0) - \frac{(p^2 kG + \rho\Omega^2)}{EJkG} M_a(0) + \frac{p(p^2 kG + \rho\Omega^2)}{kG} \varphi_a(0) + \frac{\rho A\Omega^2}{EJ} w_a(0)$$
$$+ \sum_{i=1}^{n_6} \frac{p(p^2 kG + \rho\Omega^2)}{kG} \Psi_i e^{-p\lambda_i} - \sum_{i=1}^{n_1} \frac{p}{EJ} R_i e^{-p\xi_i} + \sum_{i=1}^{n_2} \frac{p}{EJ} m_i W_i \Omega^2 e^{-p\eta_i}$$
$$+ \sum_{i=1}^{n_3} \frac{p}{EJ} F_i e^{-p\zeta_i} - \sum_{i=1}^{n_4} \frac{(p^2 kG + \rho\Omega^2)}{EJkG} j_i \Phi_i \Omega^2 e^{-p\gamma_i}$$
$$- \sum_{i=1}^{n_5} \frac{(p^2 kG + \rho\Omega^2)}{EJkG} S_i e^{-p\varepsilon_i}$$
$$+ \frac{p}{EJ} L\{f_a\}] \quad , \qquad (18)$$

$$L\{w_a\} = \frac{1}{\det(\boldsymbol{B})}\Big[\frac{(EJp^2 - AkG + \mathrm{J}\rho\Omega^2)}{AEJkG}Q_a(0) - \frac{\mathrm{p}}{EJ}M_a(0) + p^2\varphi_a(0) + \frac{\mathrm{p}(p^2E + \rho\Omega^2)}{E}w_a(0)$$
$$+ \sum_{i=1}^{n_6} p^2\Psi_i e^{-p\lambda_i} + \sum_{i=1}^{n_1}\frac{(EJp^2 - AkG + \mathrm{J}\rho\Omega^2)}{AEJkG}R_i e^{-p\xi_i}$$
$$- \sum_{i=1}^{n_2}\frac{(EJp^2 - AkG + \mathrm{J}\rho\Omega^2)}{AEJkG}m_i W_i \Omega^2 e^{-p\eta_i} - \sum_{i=1}^{n_3}\frac{(EJp^2 - AkG + \mathrm{J}\rho\Omega^2)}{AEJkG}F_i e^{-p\zeta_i}$$
$$- \sum_{i=1}^{n_4}\frac{\mathrm{p}}{EJ}j_i\Phi_i\Omega^2 e^{-p\gamma_i} - \sum_{i=1}^{n_5}\frac{\mathrm{p}}{EJ}S_i e^{-p\varepsilon_i}$$
$$- \frac{(EJp^2 - AkG + \mathrm{J}\rho\Omega^2)}{AEJkG}L\{f_a\}\Big] \quad . \tag{19}$$

Partial Fraction Decompositions

Laplace transforms (16) to (19) of the general amplitudes contain rational functions with the denominator of $\det(\boldsymbol{B})$, which has the form of the quartic polynomial (14).

Before applying the inverse Laplace transform to find the general amplitudes, we have to convert the rational functions into partial fractions.

Performing partial fraction decompositions of these rational functions, we must distinguish among three different cases:

$$\Omega < \sqrt{\frac{AkG}{\mathrm{J}\rho}} \quad , \quad \Omega = \sqrt{\frac{AkG}{\mathrm{J}\rho}} \quad , \quad \Omega > \sqrt{\frac{AkG}{\mathrm{J}\rho}} \quad . \tag{20}$$

Firstly, assuming $< \sqrt{\frac{AkG}{\mathrm{J}\rho}}$, and introducing denotation:

$$\alpha^2 = -\frac{\left[\Omega\mathrm{J}\rho\mathrm{E} + \Omega\mathrm{J}\rho\mathrm{kG} - \sqrt{(\mathrm{J}\rho)^2(E - kG)^2\Omega^2 + 4EJ(kG)^2A\rho}\right]\Omega}{2EJkG} \quad , \quad \alpha^2 > 0 \tag{21}$$

$$\beta^2 = \frac{\left[\Omega\mathrm{J}\rho\mathrm{E} + \Omega\mathrm{J}\rho\mathrm{kG} + \sqrt{(\mathrm{J}\rho)^2(E - kG)^2\Omega^2 + 4EJ(kG)^2A\rho}\right]\Omega}{2EJkG} \quad , \tag{22}$$

the quartic polynomial of $\det(\boldsymbol{B})$, (14), may be factored as follows

$$\det(\boldsymbol{B}) = (p^2 - \alpha^2)(p^2 + \beta^2)\,. \tag{23}$$

Secondly, assuming $\Omega > \sqrt{\frac{AkG}{\mathrm{J}\rho}}$, and using denotation:

$$\gamma^2 = \frac{\left[\Omega\mathrm{J}\rho\mathrm{E} + \Omega\mathrm{J}\rho\mathrm{kG} - \sqrt{(\mathrm{J}\rho)^2(E - kG)^2\Omega^2 + 4EJ(kG)^2A\rho}\right]\Omega}{2EJkG} \quad , \quad \gamma^2 > 0 \tag{24}$$

the polynomial of $\det(\boldsymbol{B})$ may be factored as follows

$$\det(\boldsymbol{B}) = (p^2 + \gamma^2)(p^2 + \beta^2)\,. \tag{25}$$

Finally, assuming $\Omega = \sqrt{\frac{AkG}{J\rho}}$, and assigning denotation:

$$k^2 = \frac{A(E + kG)}{EJ}\ , \tag{26}$$

$\det(\boldsymbol{B})$ may be factored as follows

$$\det(\boldsymbol{B}) = p^2(p^2 + k^2)\,. \tag{27}$$

Now using symbolic programming approach via *Maple,* as an example, we can perform partial fraction decomposition of rational functions in (16) to (19), and compute their inverse Laplace transforms to find the general amplitudes of the shear force, the bending moment, the rotation of the cross section and the displacement of the beam centerline.

Conclusions

Equations (3) to (6) for forced vibration of the Timoshenko beam implying jump discontinuities in the shear force, the bending moment and the cross-section rotation have been presented.

The general closed-form solution to the system of equations (3) to (6) for harmonic steady-state response of the prismatic beam can be found using the Laplace transform method with integration constants in the form of initial parameters.

Having applied distributions, we can obtain exact expression for the harmonic steady-state response of the beam without summing infinite series and without doing a modal analysis.

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