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INVERSE MASS MATRIX FOR HIGHER-ORDER FINITE ELEMENT METHOD IN LINEAR FREE-VIBRATION PROBLEMS

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Abstract: *In the paper, we present a direct inverse mass matrix in the higher-order finite element method for solid mechanics. The direct inverse mass matrix is sparse, has the same structure as the consistent mass matrix and preserves the total mass. The core of derivation of the semi-discrete mixed form is based on the Hamilton's principle of least action. The cardinal issue is finding the relationship between discretized velocities and discretized linear momentum. Finally, the simple formula for the direct inverse mass matrix is presented as well as the choice of density-weighted dual shape functions for linear momentum with respect to the displacement shape function with a choice of the lumping mass method for obtaining the correct and positive definitive velocity-linear momentum operator. The application of Dirichlet boundary conditions into the direct inverse mass matrix for a floating system is achieved using the projection operator. The suggested methodology is tested on a free-vibration problem of heterogeneous bar for different orders of shape functions.*

Keywords: Higher-order Finite element method, Direct inverse mass matrix, Consistent and lumped mass matrix, Free vibration problem, Heterogeneous bar.

1. Introduction

The direct inversion of consistent-like mass matrix is a promising tool for accurate and efficient modeling in dynamic problems of solids based on the finite element method. It is known that the inversion of the consistent mass matrix in the finite element method is fully filled, therefore the sparse variant obtained by the direct assembling is desirable. The direct inverse mass matrix can be looked at as a good approximation of the classical inversion of the mass matrix that has a favorable frequency spectrum and is preserving the total mass of bodies.

The penalized three-field formulation for the direct inverse (reciprocal) mass matrix (RMM) has been derived in work (Tkachuk and Bischoff, 2015). This approach uses the dual shape functions including the Dirichlet boundary conditions. The direct inverse mass matrix has been revised and re-derived for a floating system, subsequently, with the application of the Dirichlet boundary conditions via a projection operator for the linear finite element method (González et. al., 2018) and for the isogeometric analysis (González et. al., 2019). The application of the Dirichlet boundary conditions is based on the localized Lagrange multipliers (Park et. al., 2000). In this contribution, we present the extension of the mentioned approach for the higher-order finite element method with a suitable mass lumping and its application into free vibration problems.

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2. Direct inversion of mass matrix in finite element method

In this section, we present the theoretical description of a strong and weak form of constrained elasto-dynamics and the formula for the direct inverse mass matrix.

2.1. Strong form and Hamilton's principle for constrained elastodynamics

Let $\Omega \subset R^3$, be an open, bounded domain with a piecewise smooth boundary $\Gamma = \partial\Omega$. The strong formulation of dynamic problem for a body occupying the domain with the boundary Γ is formulated as

$$div\,\boldsymbol{\sigma} + \boldsymbol{b} = \dot{\boldsymbol{p}} on\ \Omega\ \times [0,T] \tag{1}$$

$$\boldsymbol{u} = \boldsymbol{u}_b on\ \Gamma_b\ \times [0,T] \tag{2}$$

$$\boldsymbol{\sigma}\ \cdot\ \boldsymbol{n} = \boldsymbol{t} on\ \Gamma_t\ \times [0,T] \tag{3}$$

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}^0, \dot{\boldsymbol{u}}|_{t=0} = \dot{\boldsymbol{u}}^0\ in\ \Omega \tag{4}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{b}$ is the volume force per volume, the linear momentum $\boldsymbol{p}$ is connected to the velocity field $\dot{\boldsymbol{u}}$ and the mass density ρ as $\boldsymbol{p} = \rho\,\dot{\boldsymbol{u}}$, $\boldsymbol{u}$ is the displacement field, $\boldsymbol{u}_b$ is the prescribed displacement on Γ_b (Dirichlet boundary conditions), $\boldsymbol{n}$ is the outward normal defined on Γ, $\boldsymbol{t}$ is the traction vector defined on Γ_t (Neumann boundary conditions), $\boldsymbol{u}^0$ a $\dot{\boldsymbol{u}}^0$ are the initial conditions for displacement and velocity fields. We assume a linear elastic constitutive relationship via the Hooke's law $\boldsymbol{\sigma} = \boldsymbol{D}\,\boldsymbol{\epsilon}$, where $\boldsymbol{D}$ is the fourth-order elastic tensor, $\boldsymbol{\epsilon}$ is the small infinitesimal strain tensor defined as the symmetrical part of the displacement gradient $\boldsymbol{\varepsilon} = sym\ (grad\ \boldsymbol{u})$**.**

The strong form defined in Eqs. (1) - (4) can be reformulated in the sense of the Hamilton's principle of least action in the mixed variational form as follows (González, et. al., 2018)

$$\delta H(\boldsymbol{u},\boldsymbol{p},\boldsymbol{l}) = \int_{t_1}^{t_2}\left\{\int_\Omega\ \delta\boldsymbol{p}\cdot\left(\dot{\boldsymbol{u}} - \frac{1}{\rho}\,\boldsymbol{p}\right)\,d\Omega + \int_\Omega\,(\delta\boldsymbol{u}\cdot\dot{\boldsymbol{p}} + \delta\boldsymbol{\varepsilon}{:}\,\boldsymbol{\sigma})\,d\Omega + \int_\Omega\ \delta\boldsymbol{u}\cdot\boldsymbol{b}\,d\Omega + \int_{\Gamma_b}\delta\boldsymbol{u}\cdot\boldsymbol{l}\,d\Gamma + \int_{\Gamma_b}\delta\boldsymbol{l}\cdot(\boldsymbol{u} - \boldsymbol{u}_b)\,d\Gamma\right\}dt\quad = 0 \tag{5}$$

where $\boldsymbol{l}$ is the field of Lagrange multipliers. The stationary solution of (5) will be employed as the framework for semi-discrete systems of the problem Eqs. (1) - (4).

2.2. Semi-discrete system

We assume discretization of the displacement, linear momentum and Lagrange multiplier fields via independent shape functions as

$$\boldsymbol{u}(\boldsymbol{\xi}) = \boldsymbol{N}_u(\boldsymbol{\xi})\mathbf{u}, \boldsymbol{p}(\boldsymbol{\xi}) = \rho(\boldsymbol{\xi})\boldsymbol{N}_p(\boldsymbol{\xi})\mathbf{p}, \boldsymbol{l}(\boldsymbol{\xi}) = \boldsymbol{N}_\lambda(\boldsymbol{\xi})\boldsymbol{\lambda} \tag{6}$$

Where $\mathbf{u}, \mathbf{p}, \boldsymbol{\lambda}$ are the vector of nodal values of displacement, linear momentum and Lagrange multipliers. After applying the spatial discretization, we can specify the semi-discrete system in the following form

$$\mathbf{A}\,\dot{\mathbf{p}} + \mathbf{B}\,\boldsymbol{\lambda} = \mathbf{f} - \mathbf{K}\mathbf{u}\quad \text{Equilibrium equation} \tag{7}$$

$$\mathbf{A}^{\mathrm{T}}\,\dot{\mathbf{u}} - \mathbf{C}\,\mathbf{p} = \mathbf{0}\quad \text{Momentum equation} \tag{8}$$

$$\mathbf{B}^{\mathrm{T}}\mathbf{u} - \mathbf{L}_{\boldsymbol{b}}\mathbf{u}_{\boldsymbol{b}} = \mathbf{0}\quad \text{Boundary constraints} \tag{9}$$

$$-\mathbf{L}_b^{\mathrm{T}}\,\boldsymbol{\lambda} = \mathbf{0}\quad \text{Action and reaction principle} \tag{10}$$

where matrices are defined by the integration over the domain of a finite element

$$\mathbf{A}_e = \int_{\Omega_e}\rho\boldsymbol{N}_u^{\mathrm{T}}\,\boldsymbol{N}_p\,d\Omega, \mathbf{C}_e = \int_{\Omega_e}\rho\boldsymbol{N}_p^{\mathrm{T}}\,\boldsymbol{N}_p\,d\Omega \tag{11}$$

$$\mathbf{B}_e = \int_{\Gamma_{be}}\boldsymbol{N}_u^{\mathrm{T}}\,\boldsymbol{N}_\lambda\,d\Gamma\ , \mathbf{L}_{be} = \int_{\Gamma_{be}}\boldsymbol{N}_\lambda^{\mathrm{T}}\,\boldsymbol{N}_u\,d\Gamma \tag{12}$$

All matrices are then assembled into the global matrices that can be used for solving the problems.

2.3. Reciprocal mass matrix, dual shape functions and application of boundary conditions

After elimination of the linear momentum $\mathbf{p}$ from Eq. (8) and substitution into Eq. (7) we have the equilibrium equation in the form

$$\mathbf{A}\,\mathbf{C}^{-1}\,\mathbf{A}^{\mathrm{T}}\,\ddot{\mathbf{u}} + \mathbf{B}\,\boldsymbol{\lambda} = \mathbf{f} - \mathbf{K}\mathbf{u} \tag{13}$$

We can determine the mass matrix related to inertia forces as

$$\mathbf{M} = \mathbf{A}\,\mathbf{C}^{-1}\,\mathbf{A}^{\mathrm{T}} \tag{14}$$

and its inversion yields the reciprocal mass matrix in the final form

$$\mathbf{M}^{-1} = \mathbf{A}^{-\mathrm{T}}\,\mathbf{C}\,\mathbf{A}^{-1} \tag{15}$$

In the following, we suggest an optimal choice of the matrices $\mathbf{A}$ and $\mathbf{C}$. The shape functions related to the discretization of the linear momentum $\boldsymbol{N}_p$ are chosen as the dual density-weighted function with respect to the displacement shape function as follows:

$$\boldsymbol{N}_p(\boldsymbol{\xi}) = \rho(\boldsymbol{\xi})\boldsymbol{N}_u(\boldsymbol{\xi})\mathbf{M}_e^{-1}\mathbf{M}_e^{L}, \tag{16}$$

where the consistent mass matrix is given by

$$\mathbf{M}_e = \int_{\Omega_e} \rho \boldsymbol{N}_u^{\mathrm{T}}\,\boldsymbol{N}_u\,d\Omega \tag{17}$$

and $\mathbf{M}_e^{L}$ is the lumped mass matrix. After the substitution of (16) and (17) into (11), we have

$$\mathbf{A}_e = \mathbf{M}_e^{L}, \mathbf{C}_e = \mathbf{A}_{\boldsymbol{e}}^{\mathrm{T}}\,\mathbf{M}_e^{-1}\,\mathbf{A}_e \tag{18}$$

Now we have the diagonal version of the matrix $\mathbf{A}$ and the evaluation of the reciprocal mass matrix (15) is simple due to the inversion of $\mathbf{A}$. The projection matrix for the higher-order FEM needs to respect the higher-order shape functions. The HRZ lumping scheme (Hilton, et. al., 1976) for obtaining lumped (diagonal) version of the matrix $\mathbf{A}$ is a suitable method for the higher-order FEM.

Note, the elemental mass matrix can be evaluated via averaging of the lumped and consistent mass matrix

$$\mathbf{M}_e = (1-\beta)\mathbf{M}_e^{L} + \;\beta\,\mathbf{M}_{\mathrm{e}}^{C}, \tag{19}$$

where $\beta \epsilon [0,1]$ is the averaging parameter.

We apply the Dirichlet homogeneous boundary conditions prescribed on Γ_b via the projection $\boldsymbol{P}$ on the reciprocal mass matrix which can be obtained by eliminating of $\boldsymbol{u}_b$ and $\boldsymbol{\lambda}$ from the system (7) - (10). The reciprocal mass matrix after applying the Dirichlet boundary conditions takes the form

$$\mathbf{M}_{\boldsymbol{b}}^{-1} = \boldsymbol{P}\,\mathbf{M}^{-1}, \boldsymbol{P} = \mathbf{I} - \mathbf{M}^{-1}\boldsymbol{B}\,[\mathbf{B}^{\mathbf{T}}\,\mathbf{M}^{-1}\,\mathbf{B}]^{-\mathbf{1}}\,\mathbf{B}^{\mathrm{T}} \tag{20}$$

3. Free vibration numerical test

We test the mentioned direct inverse mass matrix in a free vibration problem of heterogeneous bars discretized using shape functions with the orders $p = 1, 2, 3$ and 5. The free vibration problem in FEM based on the direct inverse mass matrix is an eigen-value problem in the form

$$(\mathbf{M}_{\boldsymbol{b}}^{-1}\mathbf{K}_b - \omega^2\mathbf{I})\,\Phi = \mathbf{0} \tag{21}$$

where $\mathbf{K}_b$ and $\mathbf{M}_b$ are the stiffness and mass matrices for the constrained dynamic system.

In the numerical test, we assume a bimaterial bar made of steel and aluminium. The lengths of the parts are $L_1 = 5$ m and $L_2 = 5$ m, cross-sections $A_1 = 10\cdot10^{-4}$ m^2, $A_2 = 5\cdot10^{-4}$ m^2, mass densities $\rho_1 = 2700$ kg/m^3, $\rho_2 = 7850$ kg/m^3, elastic Young's moduli $E_1 = 69\cdot10^9$ Pa, $E_2 = 210\cdot10^9$ Pa, number of elements for each bar part is 50.

The total mass for the bar can be computed as the total mass $m = A_1\,L_1\,\rho_1 + A_2\,L_2\,\rho_2 = 33.125\;kg$.

In Fig. 1, one can see the eigen-frequency spectrum for the standard mass matrix and the reciprocal version with the consistent, averaged and lumped matrix $\mathbf{C}$ for linear and quadratic FEM. In Fig. 2, one can see the eigen-frequency spectrum for shape functions of order $p = 3$ and $p = 5$. Based on the results, the direct inverse mass matrix gives excellent results for lower frequency range for all matrices and orders. For the lumped C, the direct inverse mass matrix produces the same results as the classical approach, which is correct. Also the total mass of the bar is preserved in the case of all orders p.

4. Conclusions

We have presented the method of direct inverse mass matrix in higher-order FEM and its application to free vibration problems. The lower frequency spectrum exhibits a very nice agreement with the classical mass matrices. The suggested methodology conserves the total mass. In future, we plan to test the quality of the direct inverse mass matrix in complex dynamic tests and real applications.

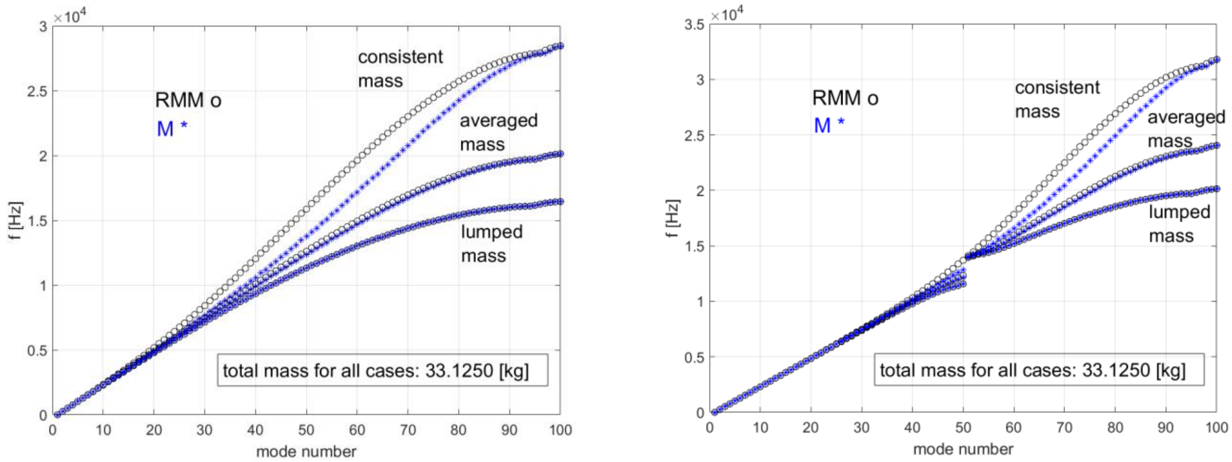

Fig. 1: Frequency spectrum of heterogeneous bar for different mass matrices, linear FEM (on the left) and quadratic FEM (on the right). RMM marks the reciprocal (direct inverse) mass matrix.

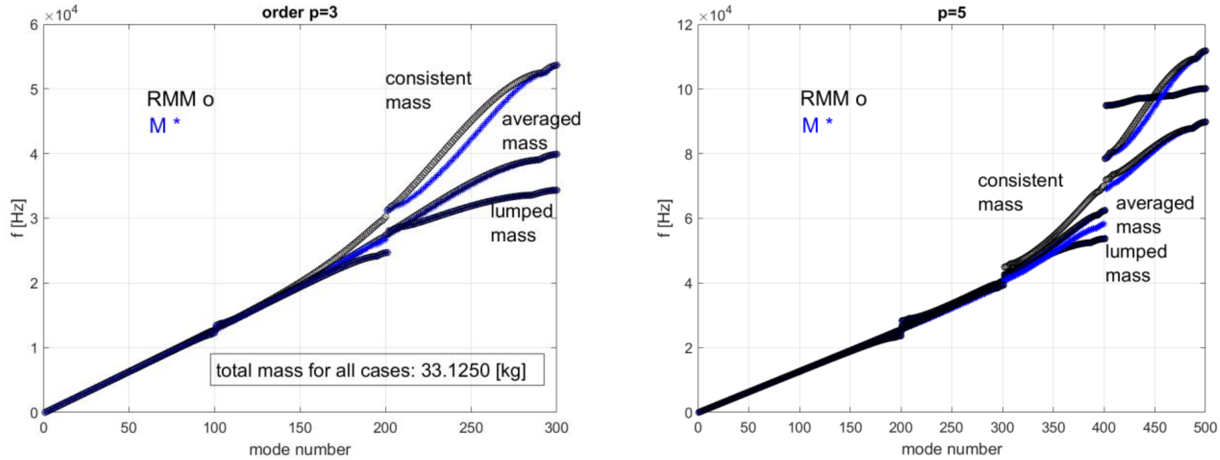

Fig. 2: Frequency spectrum of heterogeneous bar with order p = 3 and p = 5.

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