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STABILITY OF A BAR INFLUENCED BY SMALL AND LARGE IMPERFECTIONS

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Abstract: *The geometrical and physical imperfections of systems can drastically reduce their critical loading. These imperfections are usually of stochastic character and, therefore, they act as random parametric perturbations of coefficients of corresponding differential equations. In this paper, the imperfections are introduced as multidimensional statistics on the set of a large number of realizations of the same system. As far as the amount of information is small or the imperfections themselves cannot be considered small, the convex analysis is preferable. The paper compares results obtained by both stochastic and convex analyses for hyper-prism and demonstrates when each of them is more convenient to be used. Besides of the hyper-prism, the possibilities and properties of other modifications of convex method are considered, especially those based on the definition of imperfection zone marked as a centric hyper-ellipsoid or as an eccentric hyper-ellipsoid. The analytical background was brought up to the level when only a few configurations of imperfections are sufficient to be evaluated numerically. These configurations are obtained by means of the convex analysis as points of extreme critical loading using the Lagrange method of constrained extremes.*

Keywords: Convex domain method, System stability, Hyper-prism and hyper-ellipsoid domains.

1. Introduction

The geometric imperfection of thin-walled systems can be divided into three groups. Large imperfections usually do not have a random character and their influence can be respected by a certain representative deterministic shape. In contrast, small imperfections are typically stochastic in nature and their influence is necessary to be examined using methods of stochastic mechanics. The original work on this topic was probably published by Jacquot (1972). This was followed by a number of other papers discussing this problem from other viewpoints, see, e.g., (Elishakoff, 1978).

An extensive area of medium imperfections is between these two categories. They have mostly also a random character and thus their influence can be studied by means of stochastic methods. It reveals, however, that permissible loads, resulting from a probabilistic model, are not significantly higher than those following from the convex analysis of the Lagrangian type, see, e. g., papers by Kirkpatrick and Holmes (1989), Elishakoff and Ben-Haim (1990), Lindberg (1992) or Volmir (1963). However, the main goal of the convex method reveals in the case when only a limited set of stochastic information about investigated imperfections is available.

2. Variants of the convex method

Shape imperfections are usually described using some generalized coordinates. The aim is to best describe the real shape of the system using as few parameters as possible. The generalized coordinates should be chosen in such a way that they are stochastically either fully independent, or have a non-zero correlation with only a minimal number of other parameters.

The basic idea of the convex method is to limit the length of a radius vector whose components are parameters describing the state of imperfections in a certain convex finite-dimensional hyper-body in the space of imperfection parameters. Let us call this domain as a control body. Dimension of the corresponding

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hyper-space equals the number of random parameters describing an imperfect shape of the investigated system. In this hyper-body (including its boundary) is then looked for the minimal critical load value in the specified sense.

The critical load P_c is a function of parameters V_{0k}, which can be formally written in the form:

$$P_c = F(\mathbf{V}_0) \quad ; \quad \mathbf{V}_0 = |V_{01}, ..., V_{0m}|^T \tag{1}$$

Function $F(\boldsymbol{V}_0)$ is implicit and nonlinear. It represents usually a certain numerical process. If we searching its absolute minimum on a mentioned body, it means to find constrained extremes on its boundary and free extremes within this domain, to evaluate possible singular points and, finally, to choose the absolute minimum among all these cases.

The chosen shape has a major influence on the following procedure of solution and on final obtained results. Although this shape is not a priori limited, practical reasons lead to employment of different variants of hyper-prism or hyper-ellipsoid.

3. Hyper-prism comparison of stochastic and convex methods

The body including boundary is described in the simplest case by a system of inequalities:

$$\mathcal{R}_h\{V_{0k} : |V_{0k}| \leq d_k\}, \quad d_k \geq 0, \quad k = 1, ..., m. \tag{2}$$

Each of m parameters is described separately. Each constraint can be modified independently. Each parameter d_k can be a function of additional quantities, for instance of the time. The method, however, has some shortcomings. Because the procedure minimizes directly the components of the radius-vector and not the radius-vector itself, the total level of imperfection is not very transparent. Also, the domain boundary has not a continuous normal vector.

Using Eqs. (1) and (2), one can construct an auxiliary function:

$$G(\mathbf{V}_0) = F(\mathbf{V}_0) = + \sum_{k=1}^{m} \lambda_k \cdot (V_{0k}^2 - d_k^2) \tag{3}$$

where λ_k are Lagrangian multipliers. The differential system can be deduced from Eq. (2):

$$\frac{\partial F(\mathbf{V}_0)}{\partial V_{0k}} + 2\,\lambda_k \cdot V_{0k} = 0, \quad V_{0k}^2 - d_k^2 = 0, \quad k = 1, .., m. \tag{4}$$

The system Eq. (4) is a nonlinear system for unknowns V_{ik}, λ_k. With respect to the fact that the body (2) has not the smooth boundary, considering $m = 4$, candidates can be selected among these points:

$$\begin{array}{lllll}
a) & V_{0k} = \pm d_k\,; & k = 1, ..., 4 & & \\
b) & V_{0i} = \pm d_i\,; & V_{0j} = \pm d_j\,; & V_{0k} = \pm d_k\,; & \partial F(\mathbf{V}_0)/\partial V_{0l} = 0 \\
c) & V_{0i} = \pm d_i\,; & V_{0j} = \pm d_j\,; & \partial F(\mathbf{V}_0)/\partial V_{0k} = 0; & \partial F(\mathbf{V}_0)/\partial V_{0l} = 0 \\
d) & V_{0i} = \pm d_i\,; & \partial F(\mathbf{V}_0)/\partial V_{0j} = 0; & \partial F(\mathbf{V}_0)/\partial V_{0k} = 0; & \partial F(\mathbf{V}_0)/\partial V_{0l} = 0 \\
e) & \partial F(\mathbf{V}_0)/\partial V_{0i} = 0; & \partial F(\mathbf{V}_0)/\partial V_{0j} = 0; & \partial F(\mathbf{V}_0)/\partial V_{0k} = 0; & \partial F(\mathbf{V}_0)/\partial V_{0l} = 0
\end{array} \tag{5}$$

Indices $i, j, k, l = 1, \ldots, 4$ in cases b) - e). In each combination they are different numbers from each other. If any of derivations $\partial F(\mathbf{V}_0)/\partial V_{0k} = 0$ in any point of the body Eq. (2) gets infinite value, then it is necessary to treat this singular point separately, as it is beyond influence of Eqs. (5).

Such points are all cases where there is one or more coordinates $V_{0k} = 0$. In other words, every point given by Eq. (5), where it holds $\partial F(\mathbf{V}_0)/\partial V_{0k} = 0$, is replaced by conditions $V_{0k} = 0$. Therefore, it is sufficient to consider the whole task in points (5) and select the minimum, which represents the critical load. Practically, that means we have identified a finite number of initial imperfection configurations and, subsequently, such setting of initial imperfections which leads to the minimum P_c.

Stochastic approach considers parameters V_{0k} to be random numbers on a set of the system realizations. In order to reach results comparable with those obtained using the convex method on the hyper-prism, one considers individual V_{0k} to be stochastically independent. It means that each V_{0k} can be described separately

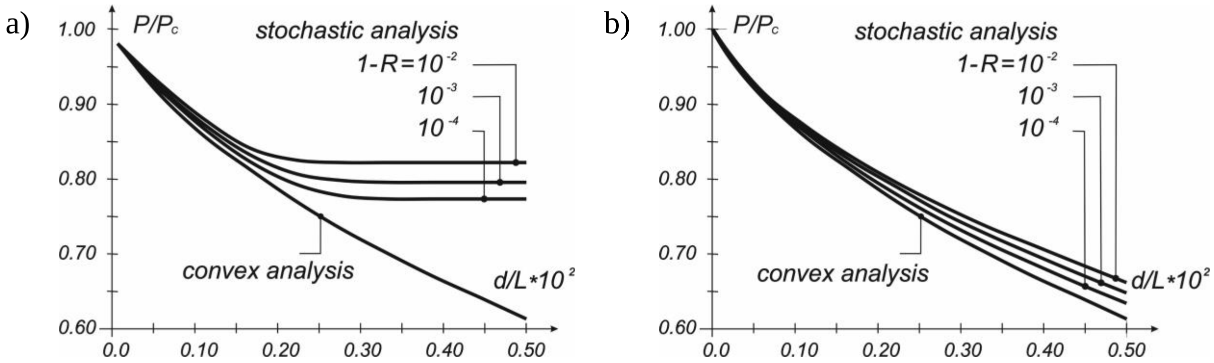

Fig. 1: Dependence of the critical load P_c on initial conditions setting using stochastic and convex analysis: a) $\frac{D}{L} = 0.001$; b) $D/L = 0.01$.

by means of PDF $p(V_{0k})$ in one variable V_{0k}. The joint PDF is a product of all these functions. In practice, the marginal PDF approach the normal distribution with truncated tails, see, e.g., (Náprstek, 2019). So that, it can be written:

$$p(V_{0k}) = \left\langle \begin{array}{lll} N_k \cdot exp(-V_{0k}^2/D_k^2) & ; & |V_{0k}| \leq d_k \\ 0 & ; & |V_{0k}| > d_k \end{array} \right. \tag{6}$$

where: D_k – variance of V_{0k} (in Gaussian meaning);
d_k – parameters limiting tails of PDF - identical with Eq. (2);
N_k – normalization constant:

$$N_k = \left(2D_k \cdot \text{erf}(\frac{d_k}{D_k})\right)^{-1} \quad ; \quad \text{erf}(x) = \int_0^x e^{-\xi^2} d\,\xi \tag{7}$$

Let us compare critical loads of an elastic prismatic bar resting on an elastic layer obtained using both methods:

$$EJ\frac{d^4v(x)}{dx^4} + P\frac{d^2v(x)}{dx^2} + r_1v(x) - r_3v^3(x) = -P\frac{d^2v_0(x)}{dx^2} \tag{8}$$

$$v(x) = \frac{d^2v(x)}{dx^2} = 0 \quad \text{for} \quad x = 0, L \tag{9}$$

where: $v_0(x)$ – initial deflection of the bar axis from a straight line;
r_1, r_3 – positive constants characterizing linear and nonlinear parts of the layer stiffness;
P – axial force ($P > 0$ represents pressure).

Initial deflection (imperfection) $v_0(x)$ and deflection $v(x)$ can be written in a form of truncated trigonometric expansions:

$$v_0(x) = \sum_{k=1}^{m} V_{0k} \cdot \sin(k\pi\frac{x}{L}) \quad ; \quad v(x) = \sum_{k=1}^{m} V_k \cdot \sin(k\pi\frac{x}{L}) \tag{10}$$

Random variables V_{0k} in Eq. (10) correspond to imperfection parameters in the meaning of this section. If we consider the effect of V_{0k} on $v_0(x)$ or $v(x)$ in the meaning of a particular norm, reference can be made to the basic theory of Fourier series. The necessary conditions for convergence of Fourier series imply that d_k should converge faster than k^{-2} for $k \to \infty$. In fact it means that the influence of imperfections, starting at a certain $k = m + 1$, has to be lower than the specified norm of the displacement increment. Although these conditions will be mostly met in practice and, moreover, higher modes do not emerge in the non-linear formulation, a necessity to limit the index k to a certain value is theoretically inconvenient. On the other hand, the introduction of a finite number of modes allows a more transparent analysis of the problem.

Let us now consider the numerical results summarized in Fig. 1, see (Volmir, 1963). The graph compares critical loads obtained by the stochastic analysis at the reliability level $1 - R = 10^{-2}, 10^{-3}$, 10^{-4} and critical loads obtained by convex analysis. The first four members of the expansion Eq. (10) are taken into account in both cases. All constraints d_k used in both types of analysis are identical and are equal to d. Stochastic analysis was performed by means of Monte Carlo simulations, every time performing

10^5 simulation samples. Relatively small imperfections V_{0k} are considered in case a), $D/L = 0.001$, the same value is used for $k = 1, \ldots, 4$. In case b), $D/L = 0.01$ is considered. From the pictures we can see that in case of small variance is, starting roughly $d/L = 0.002$, the difference between the stochastic and convex analysis results considerable and not too dependent on the level of reliability. In addition, the critical load for $d/L > 0.002$ at $D/L = 0.001$ almost does not change. In case b), i.e. $D/L = 0.01$, the differences between results of both methods are rather small, albeit critical loads obtained by means of convex analysis are slightly more conservative. Differences between the two methods arise from the fact that the convex method only perceives the extreme settings of imperfections given by Eq. (2) and does not reflect the feasibility of the particular setting of initial conditions, which may occur with the specified probability.

4. Centric hyper-ellipsoids

Many inconveniences can be avoided using a hyper-ellipsoid instead of a hyper-prism. Let us define:

$$\mathcal{R}_{Ce}\,\{\mathbf{V}_0 :\ \mathbf{V}_0^T\,\mathbf{F}\,\mathbf{V}_0\ \leq\ r^2\} \tag{11}$$

$\mathbf{V}_0$ – vector of Fourier coefficients, see Eqs. (2) and (10); $\mathbf{F}$ – square positive definite matrix of a quadratic form, r^2 – constant characterizing a size of the body.

The shape and orientation of principal axes of the body Eq. (11) can be sensitively influenced by means of selection of a matrix $\mathbf{F}$. For instance, introduced $\mathbf{F}$ as a unit matrix:

$$\mathbf{F}\ =\ \mathbf{I} \tag{12}$$

The diagonal matrix Eq. (12) can be modified in various ways. For example, individual diagonal elements can correspond to the squared reciprocal value of the respective member variance. After another modification we can select function $g_q(k)$:

$$g_q(k)\ =\ \left\langle \begin{array}{lll} 1 & ; & k < k_c \\ (k_c/k)^q & ; & k \geq k_c \end{array} \right. \tag{13}$$

where a constant q is chosen usually between 1 and 2. Matrix $\mathbf{F}$ is selected to be diagonal once again:

$$\mathrm{F}\ =\ \mathrm{diag}|g_q^{-2}(1), ..., g_q^{-2}(m)| \tag{14}$$

Whether the matrix $\mathbf{F}$ is chosen in the form of Eqs. (12) or (14), it is necessary first to establish a link between the parameter r and the maximum total initial deviations. The initial deviation $v_0(x)$ can be written in the form of a scalar product:

$$v_0(x)\ =\ \mathbf{V}_0^T \cdot \boldsymbol{\varphi}(x) \tag{15}$$

where $\boldsymbol{\varphi}(x)$ are generalized orthonormal coordinates. In case of Fourier generalized coordinates, $\boldsymbol{\varphi}(x)$ is a trigonometric sequence similar, for example, to the series Eq. (10). The parameter r, i.e. the ellipsoid size, is to be chosen so that the largest value of initial imperfections for all possible vector selection $\mathbf{V}_0$ equals Δ. It means that the following should apply:

$$\Delta\ =\ \max_x\{\max_{\mathbf{V}_0 \in R_{Ce}}\ \{\mathbf{V}_0^T\boldsymbol{\varphi}(x)\}\} \tag{16}$$

The extreme is searched for using Lagrangian method as a next step. With respect to the diagonal character of $\mathbf{F}$, application of the Minkowski inequality, integration on the interval $x \in (0; L)$ and some additional modifications, the upper estimate of the maximal deflection can be deduced:

$$v_{max}(P)\ \leq\ \Delta\ \sum_{k=1}^{m} (\|\varphi_k(x,P)\|^2/F_{kk})^{1/2} \cdot \left(\sum_{k=1}^{m}\ 1/F_{kk}\right)^{-1/2} \tag{17}$$

It reveals that the maximal deflection can be assessed by means of a weighted mean of norms of functions $\varphi_k(x, P)$.

5. Eccentric hyper-ellipsoid

Let us consider a model based on an eccentric hyper-ellipsoid:

$$\mathcal{R}_{Ee}\,\{\mathbf{V}_0 :\ (\mathbf{V}_0 - \mathbf{S}_0)^T\mathbf{F}\,(\mathbf{V}_0 - \mathbf{S}_0)\ \leq\ r^2\} \tag{18}$$

$\mathbf{S}_0$ – shift of the hyper-ellipsoid center; nominal values of Fourier coefficients of initial imperfections.

The shift $\mathbf{S}_0$ provides additional degrees of freedom to admit non-zero initial imperfections in Fourier series coefficients. Vector $\mathbf{S}_0$ can be defined as follows:

$$\mathbf{S}_0 \;=\; |S_{01}, ..., S_{0m}|^T \;=\; |g_q(1), ..., g_q(m)|^T \tag{19}$$

Let us select parameter r, which assess the ellipsoid size, and let us try to put it into a relation with the largest initial displacement of the system. The maximal initial displacement on a set of realizations as a function of space coordinates can expressed as follows:

$$v_{0max} \;=\; \max_{\mathbf{V}_0 \epsilon R_{Ee}} \{\mathbf{V}_0^T \boldsymbol{\varphi}(x)\} \tag{20}$$

We look for an extreme of a linear function which, again, emerges on the boundary of the domain. Hence the Lagrangian procedure will be used. The auxiliary constraint follows from the shape of the body Eq. (18):

$$(\mathbf{V}_0 - \mathbf{S}_0)^T \mathbf{F}\, (\mathbf{V}_0 - \mathbf{S}_0) \;\leq\; r^2 \tag{21}$$

Provided **F** is selected in the diagonal form, one obtains:

$$v_{0max}(x) \;=\; \mathbf{S}_0^T \cdot \boldsymbol{\varphi}(x) \;+\; r\sqrt{\sum_{k=1}^{m} \frac{1}{F_{kk}}} \tag{22}$$

The parameter r should be chosen so that Δ equals the largest value, which $v_{0max}(x)$ takes in the interval $x \in (0; L)$. It means that r is selected so that it holds:

$$\Delta = \max_x\{v_{0max}(x)\}, \quad \Longrightarrow \quad r = \Big(\Delta \;-\; \max_x\{\mathbf{S}_0^T \cdot \boldsymbol{\varphi}(x)\}\Big) \cdot \Big(\sum_{k=1}^{m} \frac{1}{F_{kk}}\Big)^{-1/2} \tag{23}$$

To find the maximum displacement at the ellipsoid boundary, the Lagrange method will be used again. We are looking for a maximum of $\mathbf{V}_0^{\boldsymbol{T}} \boldsymbol{\varphi}(x, P)$ respecting auxiliary conditions. If **F** is diagonal, we obtain after several steps an estimate of the maxima:

$$v_{max}(P) = \max_x\{\mathbf{S}_0^T \cdot \boldsymbol{\varphi}(x, P)\} + \Big(\Delta - \max_x\{\mathbf{S}_0^T \cdot \boldsymbol{\varphi}(x)\}\Big) \cdot \left(\sum_{k=1}^{m} (\|\varphi_k(x, P)\|^2 / F_{kk})^{1/2}\right) \cdot \left(\sum_{k=1}^{m} 1/F_{kk}\right)^{1/2} \tag{24}$$

Let us remember that $\mathbf{S}_0$ is the vector of Fourier coefficients of the nominal imperfections. For this reason is $\mathbf{S}_0^{\boldsymbol{T}} \boldsymbol{\varphi}(x)$ a nominal displacement in the point x and $\max_x(\mathbf{S}_0^{\boldsymbol{T}} \boldsymbol{\varphi}(x))$ is the maximal displacement in the whole interval $x \in (0, L)$.

Let us compare the most important results using the centric and eccentric ellipsoids. Expression Eq. (24) fulfills Eq. (17) for $\mathbf{S}_0 = 0$, as it can be expected, because $\mathcal{R}_{Ee}\{\cdot\}$ reduces onto $\mathcal{R}_{Ce}\{\cdot\}$ when $\mathbf{S}_0 = 0$. Both expressions have a form of weighted mean values, where the weight coefficients are diagonal elements of the matrix $\mathbf{F}^{-1}$. It means that sensitivity of both expressions to an upper value of m, index k and to a selection of a particular form of the matrix **F** is roughly the same. However, numerical results indicate that the centric ellipsoid leads to more conservative results than the eccentric one. It follows from the fact that a summation in the numerator in Eq. (22) is multiplied by Δ, while in Eq. (24) is the corresponding factor $\left(\Delta - \max_x(\mathbf{S}_0^{\boldsymbol{T}} \boldsymbol{\varphi}(x))\right)$.

6. Conclusion

Determination of a critical load of a bar by means of the convex method was outlined. Analytical and numerical results using three types of a control body, namely hyper-prism, centric hyper-ellipsoid and eccentric hyper-ellipsoid were shown. It reveals that the most conservative results follow from an analysis performed on the hyper-prism. More favorable results can be obtained using centric and the most favorable eccentric ellipsoid. Indeed, it seems that the eccentric ellipsoid respects better the natural character of imperfections.

In a particular case is the control body selection highly dependent on the extent and structure of the information available regarding initial imperfections. Provided information extent is very limited, it is necessary to select a proper control body of the hyper-prism. Although probabilistic analysis gives always more favorable results, it cannot be used always. The cause may be again a lack of information which makes impossible application of a probabilistic analysis. Another cause may be relatively large imperfections, when the difference in results obtained by both convex as well as probabilistic methods is rather small.

The convex method makes it possible to give a very clear overview concerning a contribution of particular components of imperfections to reduce the critical load with respect to nominal state. Requirements to limit imperfections can be relatively easy to define using few parameters with obvious geometric interpretations. The significant advantage of the convex method is also much less effort necessary to carry out an analysis in a particular case compared to any variant of stochastic analysis.

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References

Elishakoff, I. (1978) Axial impact buckling of a column with random initial imperfection. ASME Jour. Appl. Mech., vol. 45, pp. 361-365.

Elishakoff, I., Ben-Haim, Y. (1990) Dynamics of a thin cylindrical shell under impact with limited deterministic information on initial imperfections. Jour. Struct. Safety, vol. 8, pp. 103-112.

Jacquot (1972) Nonstationary random column buckling problem. ASCE Jour. Eng. Mech., vol. 98, pp.1173-1182.

Kirkpatrick, S. W., Holmes, B. S. (1989) Effect of initial imperfections on dynamic buckling of shells. ASCE Jour. Eng. Mech., vol. 115, pp. 1075-1093.

Lindberg, H. E. (1992) An evaluation of convex modelling for multimode dynamic buckling. ASME Jour. Appl. Mech., vol. 59, pp. 929-936.

Náprstek, J. (1996) Response of a simple systems with a parametric stochastic uncertainty. In: Proc. Engineering Mechanics'96 (Kratochvíl et al., eds.), IMTMB TU Brno, Svratka, pp. 7-32.

Náprstek, J. (1996) Strongly non-linear stochastic response of a system with random initial imperfections. In: 7th ASCE EMD/STD J. S. Conference on Probabilistic Mechanics and Structural Reliability, Worcester P.I. (Mass.), pp. 740-743.

Náprstek, J., Fischer, C. (2019) Response probability density of a system with cross-correlated parametric and additive input noises. Applied and Computational Mechanics, 13, 1, pp. 39-52.

Volmir, A. S. (1963) Stability of elastic systems (in Russian). Gosizdat. fiz-mat. lit., Moscow.