Multifold stationary solutions of an auto-parametric non-linear 2DOF system

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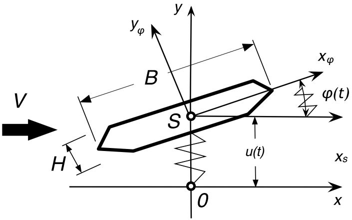
Outline

Description of the behaviour of a dynamical system, which represents a structure, is an important part of its design. When the prospective structure is subjected to complex ambient excitation, the correct understanding of the response character is necessary. For example, vibration of a slender prismatic body in an air cross-flow results from the aero-elastic interaction between the non-conservative and gyroscopic forces and effects emerging due to vortex shedding processes. A sufficient description of the post-critical behaviour is of fundamental importance for functionality and safety of any structure.

Motivation

Schematic 2DOF model of a bridge girder under wind loading



the 2DOF system response:

heave $u(t)$ (vertical direction)
pitch $\varphi(t)$ (rotation around point S)

the process is modelled by coupled generalized van der Pol equations.

Generalized van der Pol equation in the both components

With respect to experimental results, two limit cycles (stable and unstable) determine behaviour of the non-linear response of a bridge girder. In a theoretical model is such a behaviour made possible by the presence of fourth powers of u and φ in both damping terms:

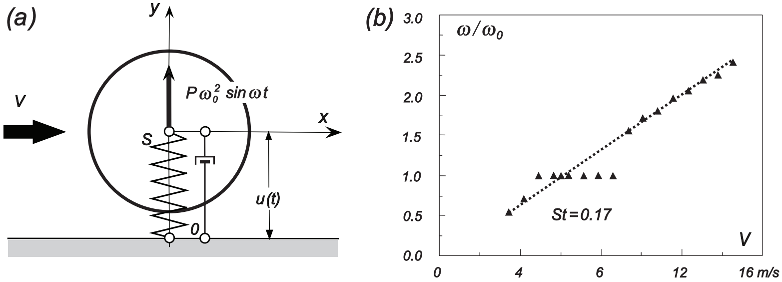
$$\begin{aligned} \ddot{u} + b_m(1 - \nu_u u^2 + \vartheta_u u^4)\dot{u} - hq \cdot \dot{\varphi} - p \cdot \varphi + \omega_u^2 \cdot u &= Q_u(t) \\ \ddot{\varphi} + b_I(1 - \nu_\varphi \varphi^2 + \vartheta_\varphi \varphi^4)\dot{\varphi} + gp \cdot u + q \cdot \dot{u} + \omega_\varphi^2 \cdot \varphi &= Q_\varphi(t) \end{aligned} \quad (1)$$

Motivation (SDOF excursion)

The generally used SDOF or 2DOF model of a structure in an air stream represent a reasonable compromise between complexity and ability to characterise the dynamic processes. The non-linear model introduces bi-quadratic damping terms into both coordinates in order to explain the concurrent effect of forced and self-excited vibrations. Both these types appear when the two vibration frequencies are close to each other and lead to a quasiperiodic response.

Generalized van der Pol equation —SDOF case

Figure: example usage of a SDOF model, and the lock-in effect modelled by the SDOF model.
(a) SDOF system outline, (b) lock-in domain as a function of the flow velocity



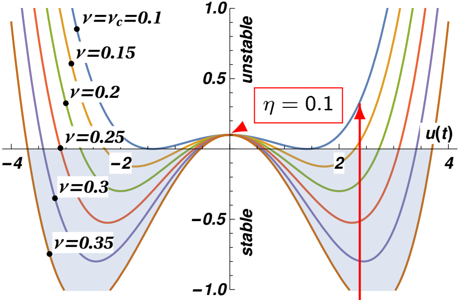
Motivation (SDOF excursion)

Nonlinear Damping in the Generalized van der Pol equation

damping term:

$$\eta\,(1 - \frac{\nu}{\eta}u^2 + \frac{\vartheta}{\eta}u^4) \quad (< 0)$$

- depends on amplitude
- stable response $\rightarrow$ negative
- i.e., necessary stability condition:
  $\nu_c > \sqrt{4\eta\vartheta^2}$ (but not sufficient)
- η scaling parameter
- ν stabilizing term
- ϑ destabilizing term



Value of nonlinear damping depending on the amplitude $u(t)$ for several values of the stabilizing quadratic term ν
$\eta \rightarrow 0.1, \vartheta \rightarrow 0.025, \omega_0 \rightarrow 1;$ $\nu_c = 0.1$

The choice of the indicated damping model causes an instability of the trivial solution and enables a stabilization of the system at a certain stable limit cycle, even though such displacement amplitudes can become unacceptable from the viewpoint of system reliability.

Mathematical Model

System of two generalized van der Pol equations

Behaviour of the 2DOF model of a bridge girder under wind loading when described by the generalized van der Pol equation in two coordinates ma be written as follows:

$$\begin{aligned} \ddot{u} + b_m(1 - \nu_u u^2 + \vartheta_u u^4)\dot{u} - hq \cdot \dot{\varphi} - p \cdot \varphi + \omega_u^2 \cdot u &= Q_u(t) \\ \ddot{\varphi} + b_l(1 - \nu_\varphi \varphi^2 + \vartheta_\varphi \varphi^4)\dot{\varphi} + gp \cdot u + q \cdot \dot{u} + \omega_\varphi^2 \cdot \varphi &= Q_\varphi(t) \end{aligned} \tag{2}$$

Symbols $\omega_u^2, \omega_\varphi^2$ stand for the eigen-frequencies in heaving or pitching modes, respectively; b_m, b_l denote damping parameters; q is a gyroscopic coefficient; p represents non-conservative force; g, h balance the physical dimensions.
In the damping terms, symbols ν_u, ν_φ characterize local destabilization (for $\vartheta_\bullet > 0$); $\vartheta_u, \vartheta_\varphi > 0$ stabilize displacement amplitudes within a stable limit cycle.

The model is suitable for description of the system in a resonant state (frequency locking). The excitation terms Q_u, Q_φ represent action of the vortex shedding. Both components u, φ are mutually connected by means of coefficients p, q.

Stationary response

When a harmonic external excitation is assumed:

$$Q_u = \Phi_u \cos(\Omega t)\,, \qquad Q_\varphi = \Phi_\varphi \cos(\Omega t) \tag{3}$$

the solution can be expected in the harmonic form

$$u \;=\; A_u \cos(\Omega t + \psi_u), \qquad \varphi \;=\; A_\varphi \cos(\Omega t + \psi_\varphi)\,. \tag{4}$$

The stationary solutions (4) has to fulfil a non-linear algebraic system consisting from four equations for two amplitudes A_u, A_φ, a relative phase shift $\Delta\psi = \psi_u - \psi_\varphi$ and the flutter frequency Ω:

$$\begin{aligned}
0 &= -b_m\Omega\left(1 - \frac{1}{4}\nu_u A_u^2 + \frac{1}{8}\vartheta_u A_u^4\right) A_u + \left(hq\Omega\cos\Delta\psi - p\sin\Delta\psi\right) A_\varphi - \Phi_u \sin\psi_u\,,\\
0 &= -b_l\Omega\left(1 - \frac{1}{4}\nu_\varphi A_\varphi^2 + \frac{1}{8}\vartheta_\varphi A_\varphi^4\right) A_\varphi - \left(q\Omega\sin\Delta\psi + gp\cos\Delta\psi\right) A_u - \Phi_\varphi \sin\psi_\varphi\,,\\
0 &= \left(\omega_u^2 - \Omega^2\right) A_u - \left(hq\Omega\sin\Delta\psi + p\cos\Delta\psi\right) A_\varphi - \Phi_u \cos\psi_u\,,\\
0 &= \left(\omega_\varphi^2 - \Omega^2\right) A_\varphi - \left(q\Omega\sin\Delta\psi - gp\cos\Delta\psi\right) A_u - \Phi_\varphi \cos\psi_\varphi\,.
\end{aligned} \tag{5}$$

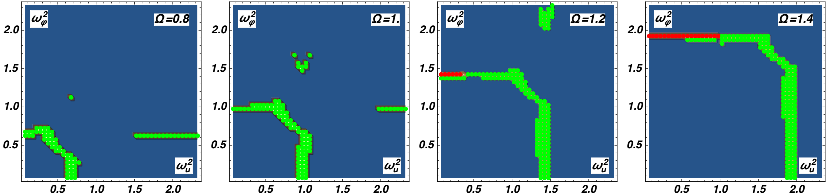
Numerical analysis

- The non-linear system Eq. (5) naturally possess none, one or more solutions for a given set of system parameters. Unfortunately, the solutions have to be identified numerically because the explicit solution is not feasible.
- A numerical solution procedure identifies only single value, which may be, but does not need to be, close to an initial guess. The modified Newton's method implemented in function `gsl_multiroot_fdfsolver_gnewton` from the GSL v2.5 was used.
- Equation (5) was repeatedly solved for fixed system parameters $g = h = 1, p = q = 0.2, b_{\{l,m\}} = 0.2, \nu_{\{u,\varphi\}} = 0.5, \vartheta_{\{u,\varphi\}} = 0.025$ and the excitation amplitude $\Phi_u = 0.5, \Phi_\varphi = 0$.
- Initial values:
  $A_{\{u,\varphi\}} \in (0.05, 2), \psi_{\{u,\varphi\}} \in (-\pi, \pi)$ ... (2 756 840 configurations)
  $\omega_u^2, \omega_\varphi^2 \in (0.01, 2.35), \Omega \in (0.1, 2)$ ... (83 942 frequency configurations).
  in total 2.314×10^{11} configurations
- The found solutions were normalized ($0 \leq A_{\{u,\varphi\}}$ and $0 \leq \psi_{\{u,\varphi\}} \leq 2\pi$) to exclude repeated cases within tolerance $||x_i - x_j|| < 10^{-6}$.
- The CESNET MetaCentrum NGI was used for the computation.

Results

- In total, 85 618 distinct solutions were found for 83 942 frequency configurations, from this number:

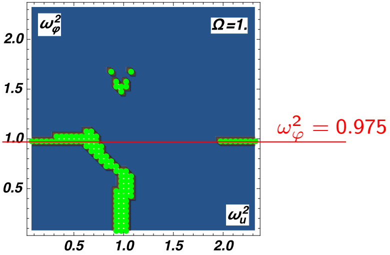
unique	threefold	fivefold
91%	8%	1%

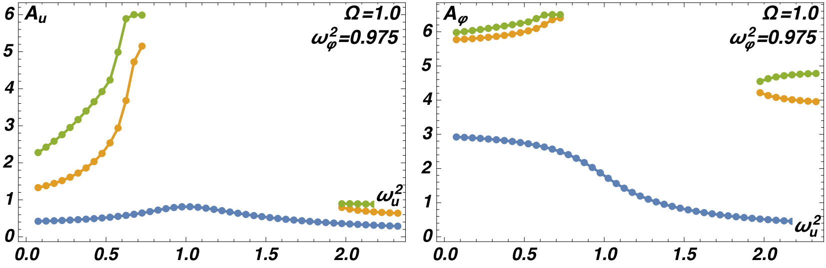


Number of solutions in the $\omega_u^2 \times \omega_\varphi^2$ plane: blue – unique, green dots – threefold, red dots – fivefold solutions

Results

- A detailed analysis of the case for $\Omega = 1$ and $\omega_\varphi^2 = 0.975$
- For selected eigen-frequencies $\omega_\varphi^2 = 0.975$ there are multiple stationary solution amplitudes $A_{\{u,\varphi\}}$.





Multiple solution branches for $\Omega = 1$ and $\omega_\varphi^2 = 0.975$.

Results

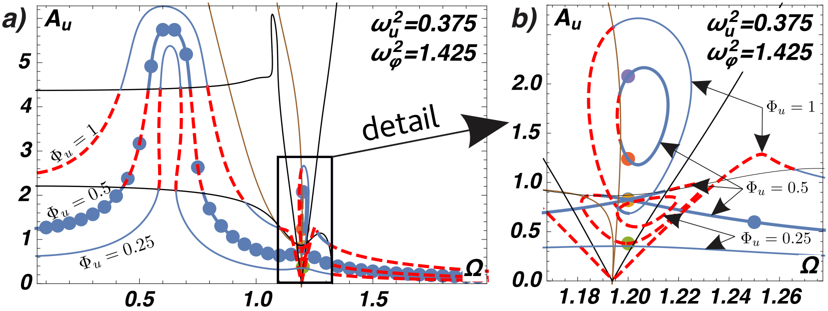
Numerical-analytical analysis using the continuation technique

- A numerical study brings a possibly huge set of discrete values, however, detailed information on their mutual connection is available only partially.
- The determined individual values of solutions can serve as good initial values for application of the numerical continuation technique.
- This technique allows for identification and categorization of the individual solution branches.

Results

An example of a fivefold solution — A_u component

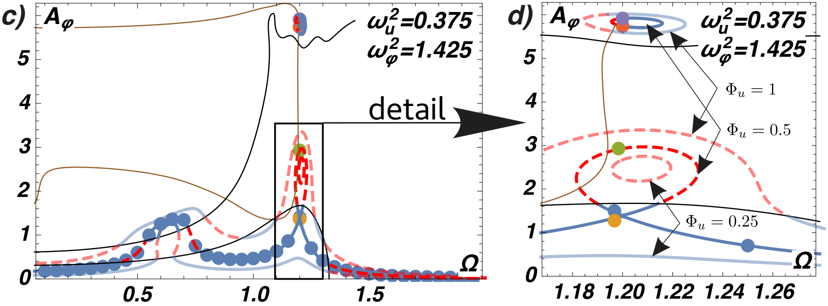
- Two solution branches for identified fivefold solution for $\omega_u^2 = 0.375, \omega_\varphi^2 = 1.425$ and excitation amplitude $\Phi_u = 0.5$.
- Curves corresponding to $\Phi_u = 0.25, \Phi_u = 1.0$ are also included.
- The plot a) presents a standard resonance peak of amplitude A_u around $\Omega \approx 0.6$ and the internal resonance zone around $\Omega \approx 1.2$, cf. the detailed plot in plot b).



Results

An example of a fivefold solution — A_φ component

- Two solution branches for identified fivefold solution for $\omega_u^2 = 0.375, \omega_\varphi^2 = 1.425$ and excitation amplitude $\Phi_u = 0.5$.
- Curves corresponding to $\Phi_u = 0.25, \Phi_u = 1.0$ are also included.
- The plot c) presents a standard resonance peak of amplitude A_φ around $\Omega \approx 0.6$ and the internal resonance zone around $\Omega \approx 1.2$, cf. the detailed plot in plot d).



Conclusions

- A system of coupled generalized van der Pol equations was analyzed.
- The presented methodology proved to be able to predict parameter areas where a multifold stationary response can be expected.
- The carefully conducted numerical analysis of a complicated non-linear systems provided results comparable to those obtained analytically.

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thank you for your attention