Global topology stiffness optimization of frame structures by moment-sum-of-squares hierarchy

M. Tyburec[†], J. Zeman, M. Kružík, D. Henrion

[†]Department of Mechanics, Faculty of Civil Engineering, Czech Technical University in Prague

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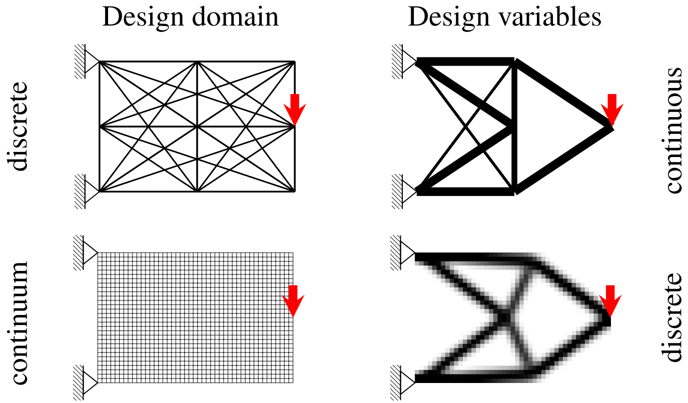
Introduction to discrete topology optimization

- **Aim**: Most efficient distribution of mass
- Discretized continuum (FEM), optimizing (vanishing) properties of elements



- Efficiency measure: **compliance** $c = \sum_{j=1}^{n_{\mathrm{lc}}} \omega_j \mathbf{f}_j^{\mathrm{T}} \mathbf{u}_j$
- $\mathbf{u}_j$ solves elastic **equilibrium** of j-th load case, $\mathbf{K}_j(\mathbf{a})\mathbf{u}_j = \mathbf{f}_j$
- Limited material **volume** $\overline{V}$
- Physical **admissibility** of the design variables, cross-sections $\mathbf{a}$

$$
\begin{aligned}
\min_{\mathbf{a},\mathbf{u}_j} \quad & \sum_{j=1}^{n_{\mathrm{lc}}} \omega_j \mathbf{f}_j(\mathbf{a})^{\mathrm{T}} \mathbf{u}_j \\
\text{s.t.} \quad & \mathbf{K}_j(\mathbf{a})\mathbf{u}_j = \mathbf{f}_j(\mathbf{a}),\ \forall j \in \{1 \dots n_{\mathrm{lc}}\}, \\
& V(\mathbf{a}) \leq \overline{V}, \\
& \mathbf{a} \geq \mathbf{0}.
\end{aligned}
$$

- **Non-convex** because of polynomial compliance and equilibrium equation
- $\mathbf{K}_j(\mathbf{a})$ may be singular
- $\mathbf{K}_j(\mathbf{a})$ **linear** in $\mathbf{a}$ (bending stiffness neglected):
  - **Convex** truss topology optimization/variable thickness sheet optimization (linear SDP)
- For $\mathbf{K}_j(\mathbf{a})$ higher degree **polynomial** function of $\mathbf{a}$
  - General convex reformulation not known
  - Only local optimization approaches have been developed so far, e.g., optimality criteria, nested approach with MMA, etc.
  - Quality of local optima not assessed

Nonlinear semidefinite programming formulation

Nonlinear semidefinite programming formulation

$$
\begin{aligned}
\min_{\mathbf{a},\mathbf{u}_j} \quad & \sum_{j=1}^{n_{\mathrm{lc}}} \omega_j \mathbf{f}_j(\mathbf{a})^{\mathrm{T}} \mathbf{u}_j \\
\text{s.t.} \quad & \mathbf{K}_j(\mathbf{a})\mathbf{u}_j = \mathbf{f}_j(\mathbf{a}),\ \forall j \in \{1 \dots n_{\mathrm{lc}}\}, \\
& \boldsymbol{\ell}^{\mathrm{T}} \mathbf{a} \le \overline{V}, \\
& \mathbf{a} \ge \mathbf{0}.
\end{aligned}
$$

- Avoid non-convexity in some cases and reduce the number of variables
- Assume that $\mathbf{f}_j(\mathbf{a}) \in \mathrm{Im}\left(\mathbf{K}_j(\mathbf{a})\right)$ and $\mathbf{K}_j(\mathbf{1}) \succ 0$
- Elimination of displacement variables $\mathbf{u}_j$ using Moore-Penrose pseudo-inverse†

Nonlinear semidefinite programming formulation

$$
\begin{aligned}
\min_{\mathbf{a},\mathbf{u}_j} \quad & \sum_{j=1}^{n_{\mathrm{lc}}} \omega_j \mathbf{f}_j(\mathbf{a})^{\mathrm{T}} \left[\mathbf{K}_j(\mathbf{a})\right]^{\dagger} \mathbf{f}_j(\mathbf{a}) \\
\text{s.t.} \quad & \mathbf{f}_j \in \mathrm{Im}(\mathbf{K}_j(\mathbf{a})),\ \forall j \in \{1 \ldots n_{\mathrm{lc}}\}, \\
& \boldsymbol{\ell}^{\mathrm{T}}\mathbf{a} \leq \overline{V}, \\
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- Avoid non-convexity in some cases and reduce the number of variables
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Nonlinear semidefinite programming formulation

$$
\begin{aligned}
\min_{\mathbf{a},\mathbf{c}} \quad & \boldsymbol{\omega}^{\mathrm{T}}\mathbf{c} \\
\text{s.t.} \quad & c_j - \mathbf{f}_j(\mathbf{a})^{\mathrm{T}}\mathbf{K}_j(\mathbf{a})^{\dagger}\mathbf{f}_j(\mathbf{a}) = 0, \ \forall j \in \{1 \ldots n_{\mathrm{lc}}\}, \\
& \mathbf{f}_j(\mathbf{a}) \in \mathrm{Im}(\mathbf{K}_j(\mathbf{a})), \forall j \in \{1 \ldots n_{\mathrm{lc}}\}, \\
& \boldsymbol{\ell}^{\mathrm{T}}\mathbf{a} \leq \overline{V}, \\
& \mathbf{a} \geq \mathbf{0}
\end{aligned}
$$

- Introducing **slack** variables (compliances) $\mathbf{c}$
- c_j is bounded from below by 0 since $[\mathbf{K}_j(\mathbf{a})]^{\dagger} \succeq \mathbf{0}$
- We can use generalized Schur complement lemma because

$$
\mathbf{f}_j(\mathbf{a}) \in \mathrm{Im}(\mathbf{K}_j(\mathbf{a})) \leftrightarrow \left[\mathbf{I} - \mathbf{K}_j(\mathbf{a})\mathbf{K}_j(\mathbf{a})^{\dagger}\right]\mathbf{f}_j(\mathbf{a}) = \mathbf{0}
$$

Nonlinear semidefinite programming formulation

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\begin{aligned}
\min_{\mathbf{a},\mathbf{c}} \quad & \boldsymbol{\omega}^{\mathrm{T}}\mathbf{c} \\
\text{s.t.} \quad & c_j - \mathbf{f}_j(\mathbf{a})^{\mathrm{T}}\mathbf{K}_j(\mathbf{a})^{\dagger}\mathbf{f}_j(\mathbf{a}) \geq 0,\ \forall j \in \{1 \ldots n_{\mathrm{lc}}\}, \\
& \mathbf{f}_j(\mathbf{a}) \in \mathrm{Im}(\mathbf{K}_j(\mathbf{a})), \forall j \in \{1 \ldots n_{\mathrm{lc}}\}, \\
& \boldsymbol{\ell}^{\mathrm{T}}\mathbf{a} \leq \overline{V}, \\
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$$

Nonlinear semidefinite programming formulation

$$\min_{\mathbf{a},\mathbf{c}} \quad \boldsymbol{\omega}^{\mathrm{T}}\mathbf{c}$$

$$\text{s.t.} \quad \begin{pmatrix} c_j & -\mathbf{f}_j(\mathbf{a})^{\mathrm{T}} \\ -\mathbf{f}_j(\mathbf{a}) & \mathbf{K}_j(\mathbf{a}) \end{pmatrix} \succeq 0, \ \forall j \in \{1 \ldots n_{\mathrm{lc}}\}$$

$$\boldsymbol{\ell}^{\mathrm{T}}\mathbf{a} \leq \overline{V},$$

$$\mathbf{a} \geq \mathbf{0}$$

- Introducing slack variables (compliances) $\mathbf{c}$
- c_j is bounded from below by 0 since $[\mathbf{K}_j(\mathbf{a})]^{\dagger} \succeq \mathbf{0}$
- We can use **generalized Schur complement** lemma because

$$\mathbf{f}_j(\mathbf{a}) \in \operatorname{Im}(\mathbf{K}_j(\mathbf{a})) \leftrightarrow \left[\mathbf{I} - \mathbf{K}_j(\mathbf{a})\mathbf{K}_j(\mathbf{a})^{\dagger}\right] \mathbf{f}_j(\mathbf{a}) = \mathbf{0}$$

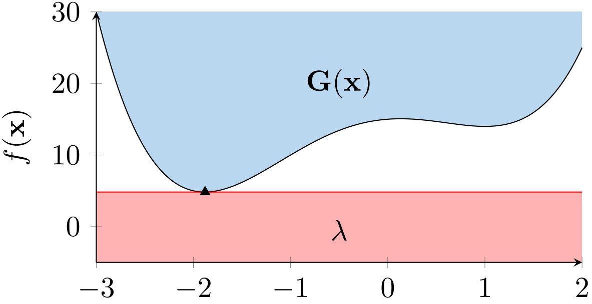
Moment-sum-of-squares hierarchy

- Consider (a non-convex, $\mathcal{NP}$-hard) optimization problem

$$f^* = \inf f(\mathbf{x}), \quad \text{s.t. } \mathbf{G}(\mathbf{x}) \succeq 0$$

- This is equivalent to an **infinite**-dimensional **convex** problem

$$\begin{aligned} f^* &= \sup_{\lambda} \lambda, \quad \text{s.t. } \forall \mathbf{x} \in \mathcal{K}(\mathbf{G}) : f(\mathbf{x}) - \lambda \geq 0 \\ &= \sup_{\lambda} (f - \lambda), \quad \text{s.t. } (f - \lambda) \in C_k(\mathcal{K}(\mathbf{G})) \end{aligned}$$



Assumption 1 Assume that there exist SOS polynomials $\mathbf{x} \mapsto p_0(\mathbf{x})$ and $\mathbf{x} \mapsto \mathbf{R}(\mathbf{x})$ such that the superlevel set $\{\mathbf{x} \in \mathbb{R}^n : p_0(\mathbf{x}) + \langle \mathbf{R}(\mathbf{x}), \mathbf{G}(\mathbf{x}) \rangle\}$ is compact.

- If Assumption 1 holds, then the dual problem is equivalent to an **infinite-dimensional** generalized problem of **moments**
- We consider a **sequence** of finite-dimensional **truncations** of increasing size

$$
\begin{aligned}
f^{(r)} = \min_{\mathbf{y}} \; & \mathbf{q}^{\mathrm{T}} \mathbf{y} \\
\text{s.t. } & \mathbf{M}_k(\mathbf{y}) \succeq 0, \\
& \mathbf{M}_{k-d}(\mathbf{G}(\mathbf{x})\mathbf{y}) \succeq 0
\end{aligned}
$$

- $\mathbf{y}$ are moments associated with the polynomial space basis
- The truncations are **linear SDPs**, hence **convex**
- Finite-dimensional relaxations: $\forall r : f^{(r)} \leq f^*$
- **Monotonically convergent** (larger portion of the infinite-dimensional SDP is considered with increasing r): $f^{(r)} \uparrow f^*$ as $r \to \infty$
- **Sufficient condition of global optimality** from the theory of moments — rank flatness of the moment matrices
- Numerical procedure for **extraction** of (the) globally-optimal solutions

Back to our nonlinear SDP

Moment-sum-of-squares hierarchy

- To satisfy Assumption 1 we need a **compact feasible set**
- **Bounds** on the design variables
  - $0 \leq a_i \leq \overline{a}_i$, where $\overline{a}_i = \overline{V}/\ell_i$
  - $0 \leq c_j \leq \overline{c}/\omega_j$, where $\overline{c} = \sum_{j=1}^{n_{\mathrm{lc}}} \omega_j \mathbf{f}_j(\hat{\mathbf{a}})^{\mathrm{T}} \mathbf{K}_j(\hat{\mathbf{a}})^{-1} \mathbf{f}_j(\hat{\mathbf{a}})$
- Improve numerical solution robustness by **scaling** to the $[-1, 1]$ domains
  - $a_i = 0.5(a_{\mathrm{s},i} + 1)\overline{a}_i$
  - $c_j = 1/(2\omega_j)(c_{\mathrm{s},j} + 1)\overline{c}$
- Add **bound** constraints
  - $a_{\mathrm{s},i}^2 \leq 1$ and $c_{\mathrm{s},j}^2 \leq 1$
  - $a_{\mathrm{s},i}^4 \leq 1$ and $c_{\mathrm{s},j}^4 \leq 1$ (tighter than $\bullet^2$)
- Assumption 1 is then **satisfied**

$$\min_{\mathbf{a}_\mathrm{s},\mathbf{c}_\mathrm{s}} \sum_{j=1}^{n_\mathrm{lc}} \left[0.5\left(c_{\mathrm{s},j}+1\right)\overline{c}\right]$$

$$\text{s.t.} \begin{pmatrix} \frac{1}{2\omega_j}\left(c_{\mathrm{s},j}+1\right)\overline{c} & -\mathbf{f}_j(\mathbf{a}_\mathrm{s})^\mathrm{T} \\ -\mathbf{f}_j(\mathbf{a}_\mathrm{s}) & \mathbf{K}_j(\mathbf{a}_\mathrm{s}) \end{pmatrix} \succeq 0,\ \forall j \in \{1\dots n_\mathrm{lc}\},$$

$$2 - n_\mathrm{e} - \mathbf{1}^\mathrm{T}\mathbf{a}_\mathrm{s} \geq 0,$$

$$1 - \boldsymbol{\omega}^\mathrm{T}\mathbf{c}_\mathrm{s} \geq 0,$$

bound constraints

- We can now build a monotonic sequence of lower bounds and eventually extract optimum

- But a simple correction to the solution of relaxations also provides us with **feasible upper bounds**!
  - Let $\tilde{\mathbf{a}}$ and $\mathbf{c}_{\mathrm{H}}$ be the cross-section areas and compliances constructed from the first-order moments
  - $\tilde{\mathbf{a}}$ satisfies $\boldsymbol{\ell}^{\mathrm{T}}\mathbf{a} \le \overline{V}$, $\mathbf{a} \ge \mathbf{0}$, as well as the bound constraints
  - The pair $\tilde{\mathbf{a}}$ and $\mathbf{c}_{\mathrm{H}}$ does not satisfy the PMI (equilibrium) in general
  - We can increase the compliances to $\tilde{\mathbf{c}}$ to satisfy the equilibrium, where

$$\tilde{c}_j = \mathbf{f}_j(\tilde{\mathbf{a}})^{\mathrm{T}} \mathbf{K}_j(\tilde{\mathbf{a}})^{\dagger} \mathbf{f}_j(\tilde{\mathbf{a}})$$

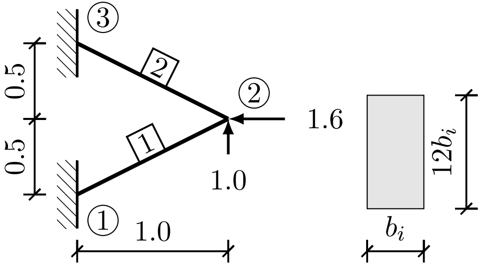
  - This can be done because $\mathbf{f}_j(\tilde{\mathbf{a}}) \in \mathrm{Im}(\mathbf{K}_j(\tilde{\mathbf{a}}))$, which follows from $I_i(y_{a_i^2}, y_{a_i^3}) > 0 \implies \tilde{a}_i > 0$

Theorem $\boldsymbol{\omega}^{\mathrm{T}}\tilde{\mathbf{c}} - f^{(r)} \leq \varepsilon$ is a sufficient condition of global ε-optimality.

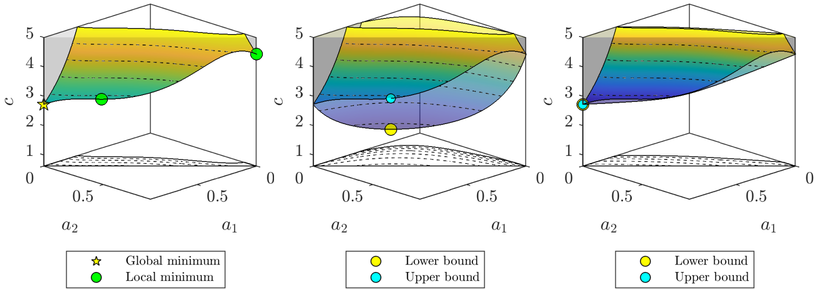
- Because Assumption 1 is **independent of** the objective function, $\mathcal{K}(\mathbf{G})^{(r)} \uparrow \mathrm{conv}(\mathcal{K}(\mathbf{G}))$
- For $r \to \infty$ optimization of $f(\mathbf{x})$ over $\mathcal{K}(\mathbf{G})$ is equivalent to optimization of $f(\mathbf{x})$ over $\mathrm{conv}(\mathcal{K}(\mathbf{G}))$

Theorem For optimization problems possessing a unique global optimum it holds that $\boldsymbol{\omega}^{\mathrm{T}}\tilde{\mathbf{c}} - f^{(r)} = 0$ as $r \to \infty$.

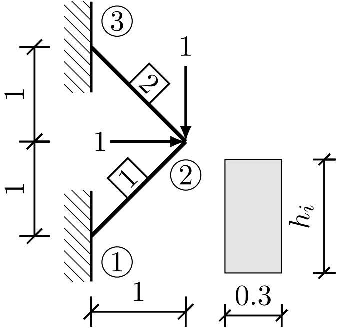
Examples



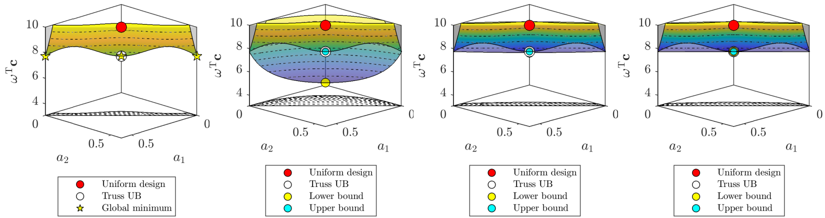
- $\overline{V} = 1$
- Three local optima
- Local optimization techniques usually fail to attain the global optimum

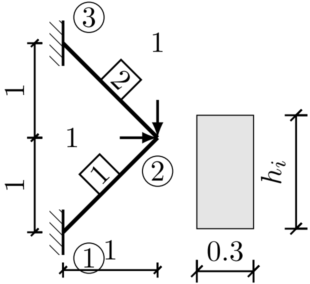


Example 2

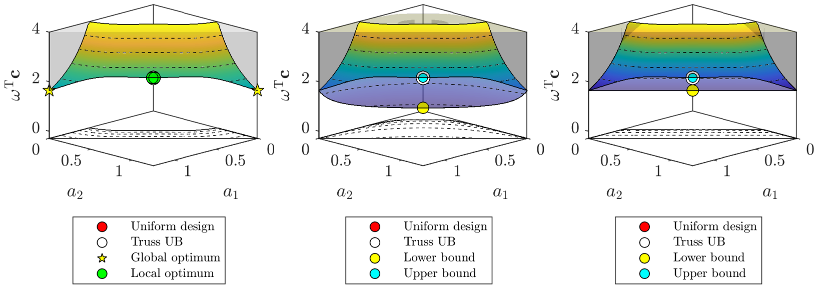


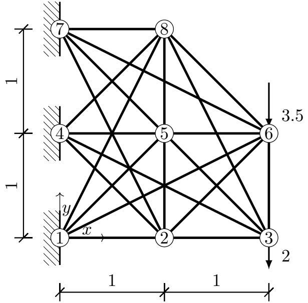
- $\overline{V} = 0.816597322$
- Three global optima
- All are extracted in the 4-th relaxation
- Sufficient condition of global optimality luckily satisfied

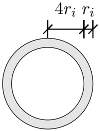




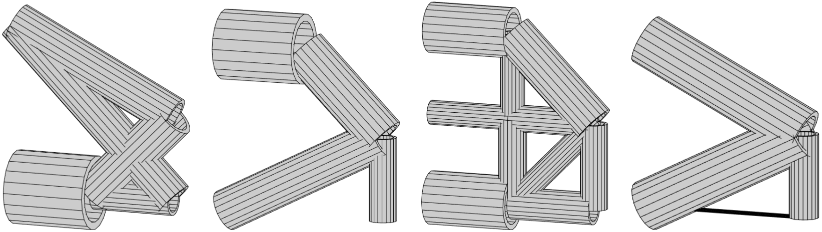
- $\overline{V} = 1.776314871$
- Irreducible optimality gap
- Convergence only certified by the rank condition







- $c_{\mathrm{MMA}} = 1698$
- $c_{\mathrm{NSDP}} = 1741$
- $c^{(1)} = 3276$
- $c^{(2)*} = 1669$



Conclusions

- Global solution to discrete topology optimization problems
- We have shown examples of topology optimization of frames, but the procedure also applies to the thickness optimization of shells
- Simple sufficient condition of global ε-optimality
- For unique optimum, $\varepsilon = 0$ for $r \to \infty$
- Finite (and small) r is usually required

M. Tyburec, J. Zeman, M. Kružík, and D. Henrion, *Global optimality in minimum compliance topology optimization of frames and shells by moment-sum-of-squares hierarchy*, 2020. arXiv: `2009.12560 [math.OC]`.

Thank you for your kind attention.

marek.tyburec@fsv.cvut.cz

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