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Using Harmonic Balance Method for Solving Frequency Response of Systems with Nonlinear Elastic Foundation

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- In practice, modeling of mechanical systems supported on an elastic foundation is a common problem in the construction industry, mining, rail transport, etc.

▲ Source: www.daibau.rs/cene/temelji_i_temeljna_ploca

▲ Source: www.justmovingaround.com/2016/07/06

▲ Source: www.rail-fastener.com

- A mathematical description of behavior of the elastic foundation usually causes the computational model of a mechanical system, although otherwise linear, to become nonlinear
- To obtain the frequency response of such system, a necessity arises to use one of the continuation methods
- Description of the frequency response then requires repeated solutions of the steady-state component of vibration response for varying excitation frequency
- However, direct integration of nonlinear motion equations includes a solution of the transient state as well, which in practice generally leads to a large number of integration steps and very long solution times
- A possible way-out is the application of the harmonic balance method, which allows for the determination of steady-state response component directly, provided it has a periodic or quasi-periodic time course

- Equation of motion of the system in time domain:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{p}_s + \mathbf{p}_h(t) - \mathbf{r}(\mathbf{x}, t); \qquad \mathbf{p}_h(t) = \mathbf{p}_{ha}\cos(\omega t + \psi)$$

- Where $\mathbf{M}$, $\mathbf{B}$, and $\mathbf{K}$ are a mass, damping and stiffness matrix, respectively, $\mathbf{x}$, $\dot{\mathbf{x}}$, and $\ddot{\mathbf{x}}$ are a displacement, velocity and acceleration vector, respectively, $\mathbf{p}_s$ is a vector of static loading, $\mathbf{p}_h$ is a vector of harmonic excitation, $\mathbf{r}$ is a vector of nonlinear foundation forces, $\mathbf{p}_{ha}$ is a vector of amplitudes of harmonic excitation, ω is an angular excitation frequency, ψ is a phase shift, and t is the time

- Equation of motion of the system in frequency domain:

$$\mathbf{x}(t) = \frac{\mathbf{c}_0}{2} + \sum_{k=1}^{N_F} [\mathbf{c}_k\cos(k\omega t) + \mathbf{s}_k\sin(k\omega t)];$$

Solution is assumed in the form of the turncated Fourier series.

$$\mathbf{A}(\omega)\mathbf{u} = \mathbf{q} - \mathbf{b}(\mathbf{u}); \qquad \mathbf{A}(\omega) = \mathrm{diag}\left(\mathbf{K}, \mathbf{A}_1, \dots, \mathbf{A}_k, \dots, \mathbf{A}_{N_F}\right); \qquad \mathbf{A}_j = \begin{bmatrix} \mathbf{K} - (k\omega)^2\mathbf{M} & k\omega\mathbf{B} \\ -k\omega\mathbf{B} & \mathbf{K} - (k\omega)^2\mathbf{M} \end{bmatrix};$$

$$\mathbf{u} = [\mathbf{c}_0 \quad \mathbf{c}_1 \quad \mathbf{s}_1 \quad \cdots \quad \mathbf{c}_k \quad \mathbf{s}_k \quad \cdots \quad \mathbf{c}_{N_F} \quad \mathbf{s}_{N_F}]^\mathrm{T}; \qquad \mathbf{q} = [\mathbf{p}_s \quad \sin(\psi)\mathbf{p}_{ha} \quad \cos(\psi)\mathbf{p}_{ha} \quad \mathbf{0} \quad \cdots \quad \mathbf{0}]^\mathrm{T}$$

- Where $\mathbf{A}$ is a dynamic stiffness matrix, vector $\mathbf{u}$ contains vectors of the Fourier coefficients $\mathbf{c}_0$, $\mathbf{c}_k$, and $\mathbf{s}_k$ of absolute, cosine and sine terms, respectively, $\mathbf{q}$ and $\mathbf{b}$ are a vector of linear and nonlinear forces in frequency domain, respectively, and N_F is the number of harmonic terms of the turncated Fourier series

- Obtaining actual iteration of nonlinear forces using the alternating frequency-time scheme:

$$\mathbf{u} \xrightarrow[\mathbf{x}=\mathbf{T}(\omega)\mathbf{u}]{\mathrm{DFT}^{-1}} \mathbf{x} \rightarrow \mathbf{r}(\mathbf{x},t) \xrightarrow[\mathbf{b}=\mathbf{T}^{+}(\omega)\mathbf{r}]{\mathrm{DFT}} \mathbf{b}(\mathbf{u})$$

- Where $\mathbf{T}$ is a linear operator of the inverse Fourier transform and $\mathbf{T}^{+}$, being the Moore-Penrose pseudoinverse of $\mathbf{T}$, is a linear operator of the direct Fourier transform

- Linear operator of the inverse Fourier transform (DFT) has the form:

$$\mathbf{T}(\omega) = [0{,}5 \cdot \mathbf{1} \quad \mathbf{t}_{c,1} \quad \mathbf{t}_{s,1} \quad \cdots \quad \mathbf{t}_{c,k} \quad \mathbf{t}_{s,k} \quad \cdots \quad \mathbf{t}_{c,N_F} \quad \mathbf{t}_{s,N_F}];$$

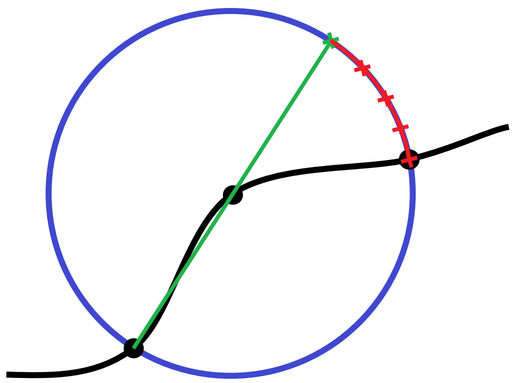
$$\mathbf{t}_{c,k} = [\cos(k\omega t_1) \quad \cdots \quad \cos(k\omega t_m) \quad \cdots \quad \cos(k\omega t_N)]^{\mathrm{T}}; \quad \mathbf{t}_{s,k} = [\sin(k\omega t_1) \quad \cdots \quad \sin(k\omega t_m) \quad \cdots \quad \sin(k\omega t_N)]^{\mathrm{T}}$$

- Where $\mathbf{1}$ is a vector of ones and N is a number of collocation points over a period of vibration

- Calculation of initial guess (predictor) from arc length a and two last solutions $\mathbf{u}^{i-2}, \omega^{i-2}$ and $\mathbf{u}^{i-1}, \omega^{i-1}$:

$$\mathbf{u}_0^i = \mathbf{u}^{i-1} + a\frac{\mathbf{u}^{i-1} - \mathbf{u}^{i-2}}{\sqrt{(\mathbf{u}^{i-1} - \mathbf{u}^{i-2})^{\mathrm{T}}(\mathbf{u}^{i-1} - \mathbf{u}^{i-2}) + (\omega^{i-1} - \omega^{i-2})^2}};$$

$$\omega_0^i = \omega^{i-1} + a\frac{\omega^{i-1} - \omega^{i-2}}{\sqrt{(\mathbf{u}^{i-1} - \mathbf{u}^{i-2})^{\mathrm{T}}(\mathbf{u}^{i-1} - \mathbf{u}^{i-2}) + (\omega^{i-1} - \omega^{i-2})^2}}$$



- Corrections of displacements $\delta\mathbf{u}_{\mathrm{j}}^{\mathrm{i}}$ and excitation frequency $\delta\omega_{\mathrm{j}}^{\mathrm{i}}$ are in j-th iteration coupled by equation:

$$\delta\mathbf{u}_j^i = \mathbf{x}_1 - \delta\omega_j^i\mathbf{x}_2 \quad (1)$$

- Where vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are obtained from solution of a set of equations:

$$\left[\mathbf{A}(\omega^{i-1}) - \frac{\partial\mathbf{b}(\mathbf{u}^{i-1})}{\partial\mathbf{u}}\right]\mathbf{x}_1 = \mathbf{A}(\omega^{i-1})\mathbf{u}^{i-1} - \mathbf{b}(\mathbf{u}^{i-1}) - \mathbf{q}; \quad \left[\mathbf{A}(\omega^{i-1}) - \frac{\partial\mathbf{b}(\mathbf{u}^{i-1})}{\partial\mathbf{u}}\right]\mathbf{x}_2 = \left[\frac{\partial\mathbf{A}(\omega^{i-1})}{\partial\omega}\right]\mathbf{u}^{i-1}$$

- Here, $\partial\mathbf{b}(\mathbf{u}^{i-1})/\partial\mathbf{u}$ is a matrix of partial derivatives of the vector of nonlinear forces in frequency domain $\mathbf{b}$ with respect to displacement vector $\mathbf{u}$ and $\partial\mathbf{A}(\omega^{i-1})/\partial\omega$ is a vector of partial derivatives of the dynamic stiffness matrix $\mathbf{A}$ with respect to angular excitation frequency ω

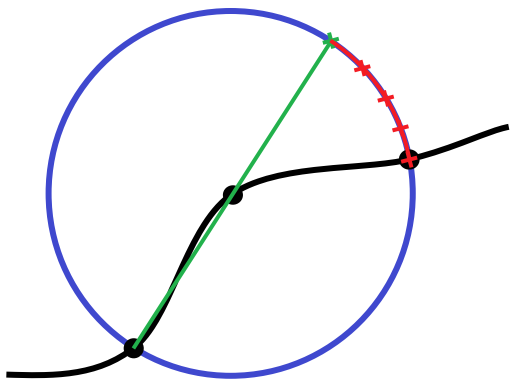
- Correction of excitation frequency $\delta\omega_j^i$ is one of the roots $\delta\omega_1$ and $\delta\omega_2$ of quadratic equation:

$$a_1\delta\omega^2 + a_2\delta\omega + a_3 = 0;$$

$$a_1 = \mathbf{x}_2{}^{\mathrm{T}}\mathbf{x}_2 + 1;$$

$$a_2 = 2\left[\left(\mathbf{u}^{i-1} - \mathbf{u}_j^i - \mathbf{x}_1\right)^{\mathrm{T}}\mathbf{x}_2 + \omega_j^i - \omega^{i-1}\right];$$

$$a_3 = \left(\mathbf{u}_j^i - \mathbf{u}^{i-1} + \mathbf{x}_1\right)^{\mathrm{T}}\left(\mathbf{u}_j^i - \mathbf{u}^{i-1} + \mathbf{x}_1\right) + \left(\omega_j^i - \omega^{i-1}\right)^2 - a^2$$



- The root that represents forward sense of continuation can be identified from the larger of the two products:

$$\left(\mathbf{u}_j^i + \mathbf{x}_1 - \delta\omega_1\mathbf{x}_2 - \mathbf{u}^{i-1}\right)^{\mathrm{T}}\left(\mathbf{u}_j^i - \mathbf{u}^{i-1}\right); \quad \left(\mathbf{u}_j^i + \mathbf{x}_1 - \delta\omega_2\mathbf{x}_2 - \mathbf{u}^{i-1}\right)^{\mathrm{T}}\left(\mathbf{u}_j^i - \mathbf{u}^{i-1}\right)$$

- Finally, after calculating the correction of displacements $\boldsymbol{\delta}\mathbf{u}_j^i$ from equation **(1)** (see previous slide), all that remains in current iteration is to update the vector of displacements and angular excitation frequency:

$$\mathbf{u}_{j+1}^i = \mathbf{u}_j^i + \delta\mathbf{u}_j^i; \quad \omega_{j+1}^i = \omega_j^i + \delta\omega_j^i$$

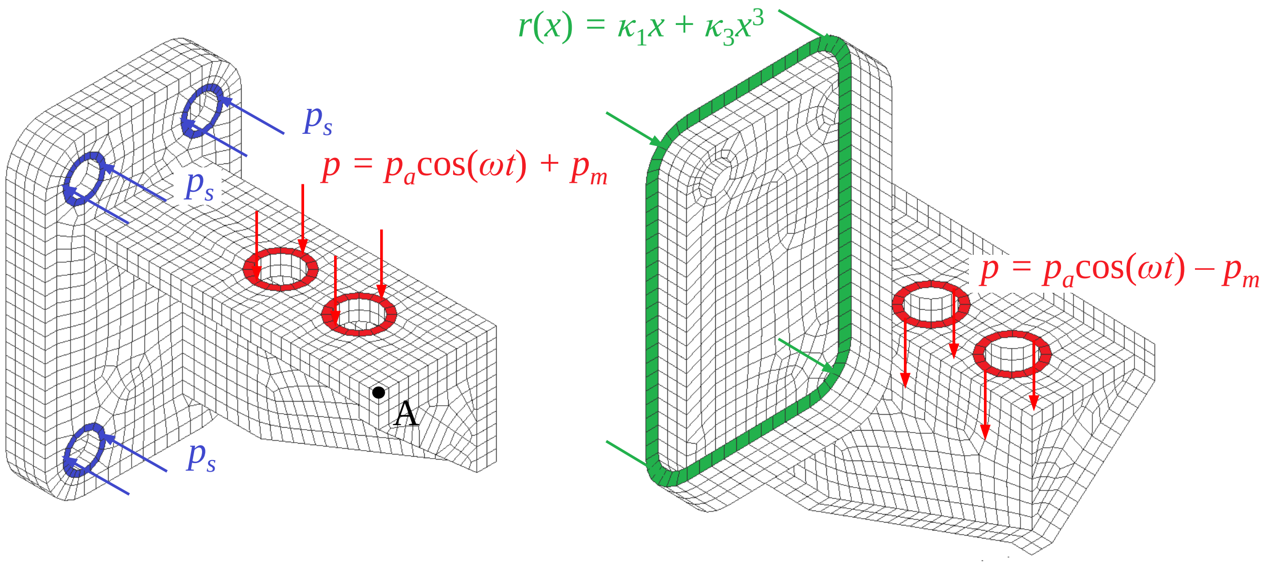
- The iterative process ends when suitable convergence criteria are met

- A console, discretized by 3D finite elements, excited by a pulsating pressure load and mounted on a bilateral nonlinear elastic support, objective is to calculate frequency response of the system

Physical parameters of the test problem:

Parameter	Symbol	Value	Unit
Young's modulus	E	$69 \cdot 10^{10}$	Pa
Poisson's number	μ	0.33	–
Mass density	ρ	2 700	kg/m^3
Mass proportional damping coefficient	α	100	1/s
Stiffness proportional damping coefficient	β	$5 \cdot 10^{-5}$	s
Constant pressure	p_s	$5 \cdot 10^6$	Pa
Amplitude of distributed excitation pressure	p_a	$1 \cdot 10^6$	Pa
Mean value of distributed excitation pressure	p_m	$5 \cdot 10^6$	Pa
Linear stiffness coefficient of foundation	κ_1	$1 \cdot 10^9$	N/m^3
Cubic stiffness coefficient of foundation	κ_3	$5 \cdot 10^{15}$	N/m^5

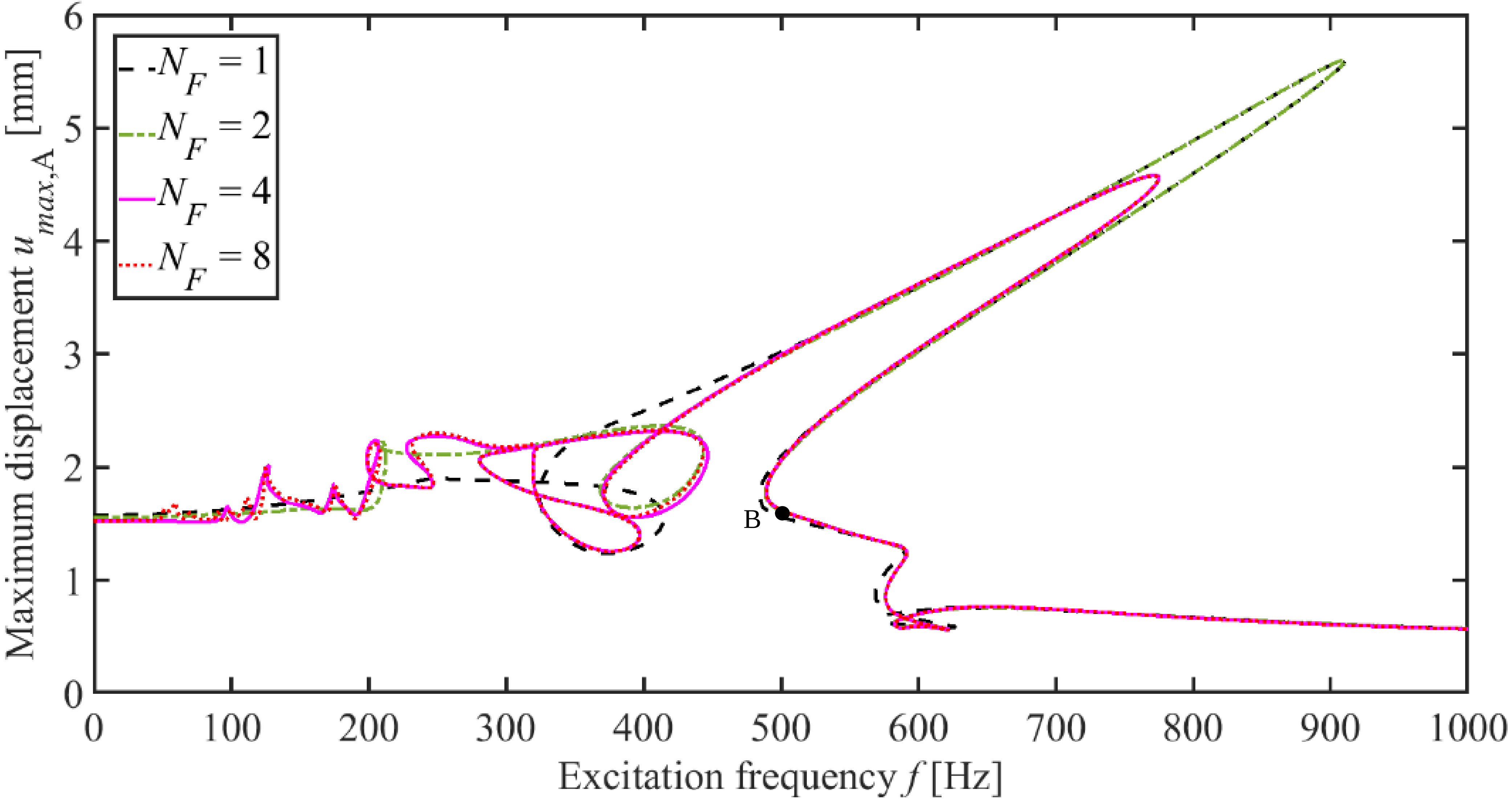
Finite element model of the test problem:



Basic finite element mesh parameters:

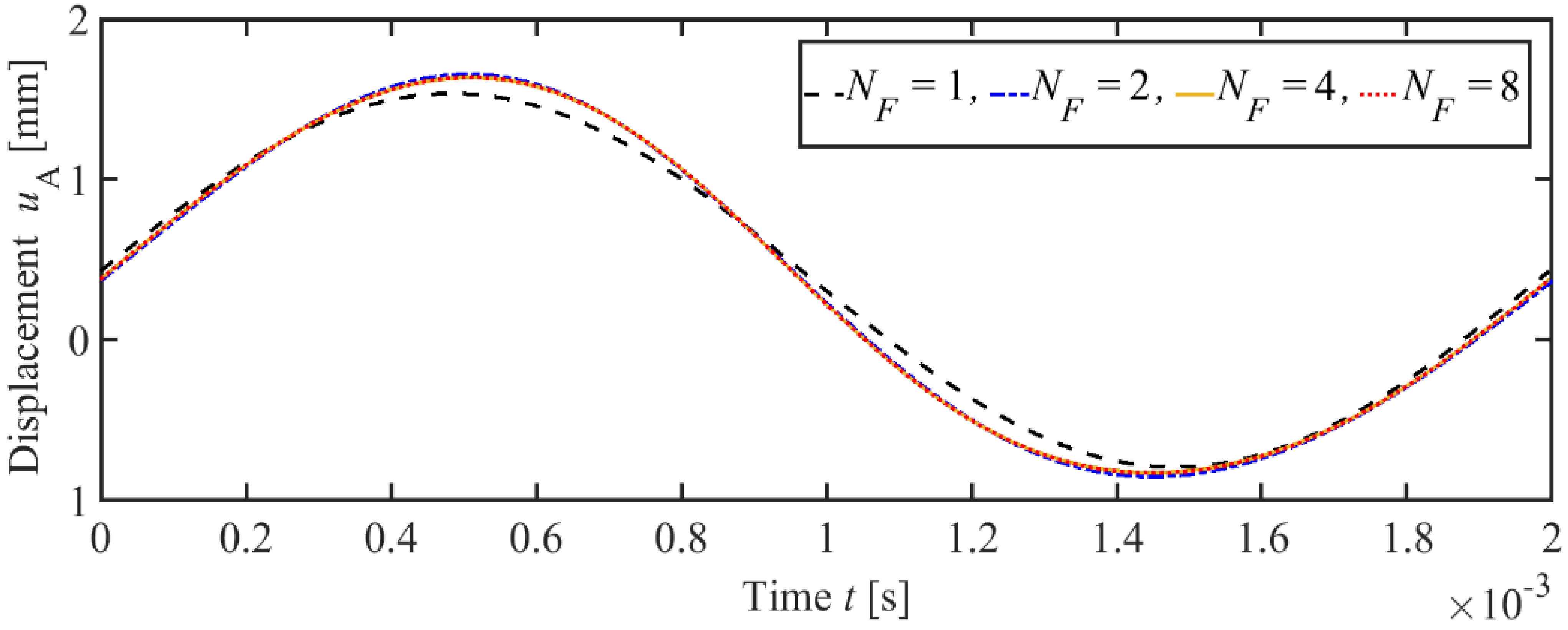
Number of elements(hexahedral)	2 232
Number of nodes	3 965
Number of degrees of freedom	11 895

- Frequency response of the test problem at node A (see previous slide):

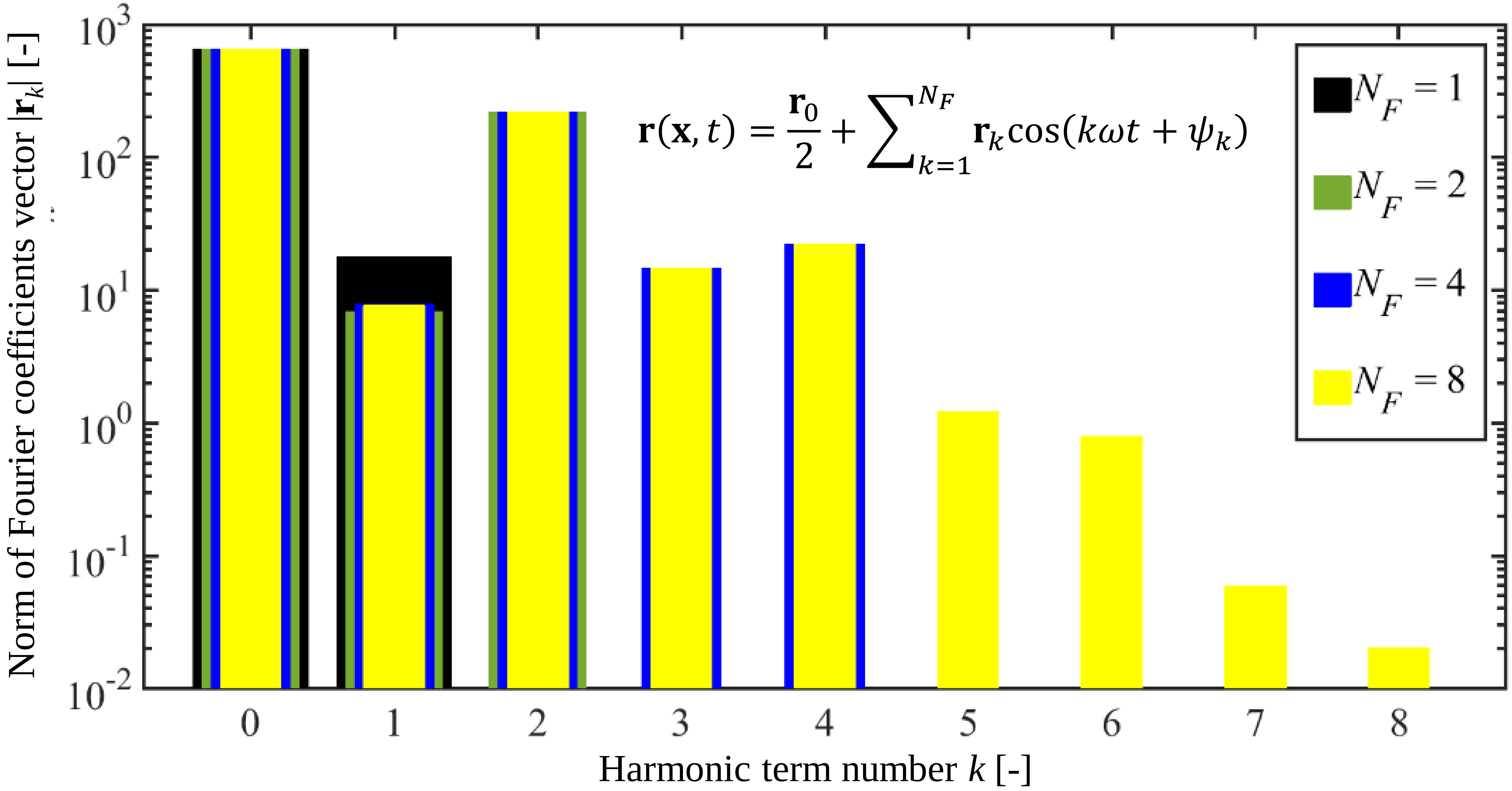


Results of the test problem

- Vibration response for excitation frequency $f = 500\ \mathrm{Hz}$ (at point B, see previous slide):



- Frequency spectrum of the foundation forces for excitation frequency $f = 500$ Hz:



- The frequency response of a console, discretized by three-dimensional finite elements, excited by a pulsating pressure load and mounted on a bilateral nonlinear elastic support, was investigated

- The arc length continuation method was used to calculate the frequency response
- The harmonic balance method was used to solve the steady-state vibration response at each increment of continuation

- For 4 harmonic terms of the Fourier series, no significant change in the shape of the frequency response curve was observed for the excitation frequency values from 300 Hz above, compared to the case with 8 harmonic terms

- The frequency spectrum of nonlinear reaction forces of the elastic foundation was also investigated

- Already for 8 harmonic members of the Fourier series, a significantly decreasing trend of the superharmonic components of the reaction forces has been observed from the 4th term above

Thank You for your attention.

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