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FETI DOMAIN DECOMPOSITION METHOD APPLIED TO SOLUTION OF CONTACT PROBLEM WITH LARGE DISPLACEMENTS

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Summary: *The paper deals with application of the FETI (Finite Element Tearing and Interconnecting) method to finite element solution of contact problems while taking into account another nonlinearity, namely the large displacements and rotations. We show, in terms of numerical experiments (i) performance of the algorithms stemming from the FETI method, in particular its numerical and parallel scalabilities and optimality of the dual penalty, and (ii) solution of a Hertzian contact problem, i.e. achieved accuracy of the numerical solution by comparison with the analytical one, and the convergence rate.*

1 Introduction

Solution to contact problems between solid bodies in general poses difficulties to the finite element solvers. In spite of the fact that the forces generated by contact are formally of the same form as the boundary conditions introduced by externally applied surface tractions, we do not generally know either the distributions of the contact tractions throughout the areas currently in contact or shapes and magnitudes of these areas until we have run the problem. Their evaluation have to be part of the solution, which implies that the contact problems are inherently strongly nonlinear and an iterative approach has to be invoked. There exist many method to model the contact boundary conditions, see for example Dobiáš (1997). Let us recall herein the Lagrange multiplier method, the Penalty method, the Perturbed Lagrangian method and the Augmented Lagrangian method as the most commonly used in practice.

One of new methods which can successfully be applied to solution to contact problems is the FETI (Finite Element Tearing and Interconnecting) method. This method is based

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on decomposition of a spatial domain into non-overlapping sub-domains that interact with each other in terms of the Lagrange multipliers. This method is one of the most successful algorithms for parallel solution of problems described by elliptic partial differential equations. The FETI methodology turns the contact problem into the quadratic programming problem with equality constraints and non-negativity constraints.

The authors of this paper developed in close cooperation a code that can be run in the framework of our in-house general purpose finite element computational system PMD (Package for Machine Design).

The contact non-linearities often occur along with other non-linearities. In this paper we deals with large displacements and finite rotations, i.e. with the geometric non-linearities.

2 Outline of theory

In this section the FETI method and our approach to the solution to non-linearities are succinctly described.

2.1 FETI method for variational inequalities

The FETI method proposed by Farhat & Roux (1992) turned out to be one of the most successful algorithms for parallel solution of problems described by elliptic partial differential equations. The FETI method is based on decomposition of a spatial domain into non-overlapping sub-domains that are "glued" by Lagrange multipliers. Theoretical results and experimental evidence by Farhat et al. (1994) and Mandel & Tezaur (1996) were presented to establish scalability of the variants of the FETI algorithm including those using the so called natural coarse grid. Let us recall that an algorithm is called scalable for a given class of problems if the cost of solution of the discretized problem is proportional to the number of nodal parameters and the speed up due to the parallel implementation is proportional to the number of processors. The FETI method was adapted also to the solution of variational inequalities with theoretical and experimental results showing that it is possible to preserve the scalability of FETI even for solution of these more complex problems (Dureisseix & Farhat, 2001, and Dostál et al., 2000).

We can describe the basic ideas on a model problem of Fig. 1. The left membrane

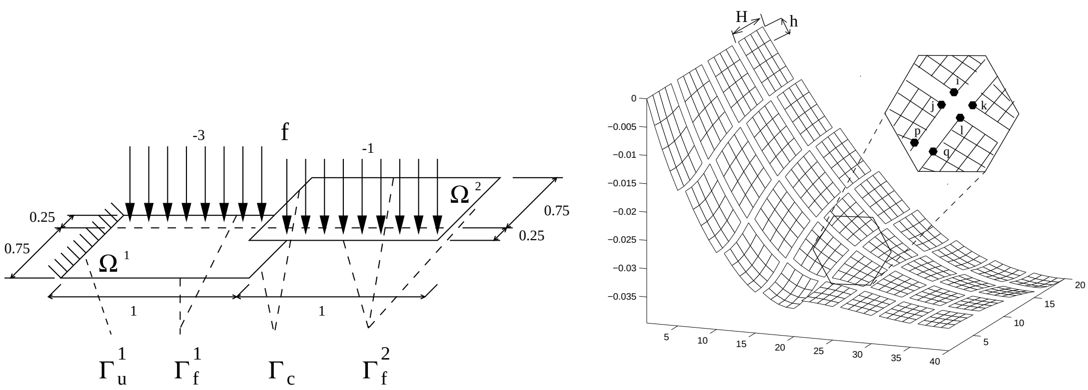

Figure 1: Model problem and its solution

is fixed on the left edge, the right is floating as in Fig. 1 and the left edge of the right membrane is not allowed to penetrate below the right edge of the left membrane.

Using the finite element discretization, we get the discretized version of our model problem with the auxiliary domain decomposition that reads

$$\min \frac{1}{2}u^\top Au - f^\top u \quad \text{s.t.} \quad B^I u \leq 0 \quad \text{and} \quad B^E u = 0. \tag{1}$$

In (1), A denotes a positive semidefinite stiffness matrix, the full rank matrices B^I and B^E describe the discretised inequality and gluing conditions, respectively, and f represents the discrete analog of forces.

The FETI methodology (Dostál et al. 2000) turns this variational inequality eliminating the primal variables into the quadratic programming problem with equality constraints and non-negativity bound - this constraints is a considerable complication as it is necessary to identify the active constraints in the solution

$$\min \frac{1}{2}\lambda^\top BA^\dagger B^\top \lambda - \lambda^\top BA^\dagger f \quad \text{s.t.} \quad \lambda_I \geq 0 \quad \text{and} \quad R^\top (f - B^\top \lambda) = 0, \tag{2}$$

where $A^\dagger$ denotes a generalized inverse that satisfies $AA^\dagger A = A$, and R denotes the full rank matrix whose columns span the kernel of A. We shall choose R so that its entries belong to $\{0, 1\}$ and each column corresponds to some floating auxiliary sub-domain Ω^{ij} with the nonzero entries in the positions corresponding to the indices of nodes belonging to Ω^{ij}.

Even though problem (2) is much more suitable for computations than (1), further improvement may be achieved by adapting some simple observations and the results of Farhat et al. (1994) and Mandel & Tezaur (1996). Let us denote

$$F = BA^\dagger B^\top, \quad \widetilde{G} = R^\top B^\top, \quad \widetilde{e} = R^\top f, \quad \widetilde{d} = BA^\dagger f,$$

and let $\widetilde{\lambda}$ solve $\widetilde{G}\widetilde{\lambda} = \widetilde{e}$, so that we can transform the problem (2) to minimisation on the subset of the vector space by looking for the solution in the form $\lambda = \mu + \widetilde{\lambda}$. Since

$$\frac{1}{2}\lambda^\top F\lambda - \lambda^\top \widetilde{d} = \frac{1}{2}\mu^\top F\mu - \mu^\top (\widetilde{d} - F\widetilde{\lambda}) + \frac{1}{2}\widetilde{\lambda}^\top F\widetilde{\lambda} - \widetilde{\lambda}^\top \widetilde{d},$$

problem (2) is, after returning to the old notation, equivalent to

$$\min \quad \frac{1}{2}\lambda^\top F\lambda - \lambda^\top d \quad \text{s.t} \quad G\lambda = 0 \quad \text{and} \quad \lambda^I \geq -\widetilde{\lambda}^I \tag{3}$$

with $d = \widetilde{d} - F\widetilde{\lambda}$. The same procedure may be applied to contact problems of elasticity.

This problem turns out to be a suitable starting point for development of efficient algorithms for variational inequalities, as we can use the analysis of the FETI method by Mandel & Tezaur (1996).

2.2 Contact and geometrical non-linearities

While the FETI method is directly applicable to the solution to contact problems with small displacements, linearly elastic, and frictionless, any other non-linearity necessitates introduction of additional iteration loop as a consequence of geometrically or materially nonlinear behaviour of structural system. In our case the non-linearity we take into account, apart from the contact, is the one caused by large displacements and finite rotations. To this end we use the total Lagrangian formulation which includes all kinetic non-linear effects. As a strain measure we make use of the Green–Lagrange tensor and as a stress measure the second Piola–Kirchhoff tensor which is work–conjugate with the previously mentioned strain tensor, see e.g. Bathe (1996).

In the above mentioned additional iteration loop, we evaluate, in terms of algorithm stemming from the FETI method, nodal displacements and nodal forces due to contact (i.e. the Lagrangian multipliers) in the current deformed geometry. Then the nodal point forces corresponding to element stresses are evaluated. The residual of this forces minus the external loading and the contact forces represents a new right hand side for another iteration cycle.

3 Numerical experiments

To illustrate the performance of the algorithms, in particular its numerical and parallel scalabilities and optimality of the dual penalty, we have implemented algorithms in C exploiting the package PETSc, see Balay et al. The experiments were run on the Lomond 18-processor Sun HPC 6500 Ultra SPARC-II based SMP system with 400 MHz, 18 GB of shared memory, 90 GB disc space, nominal peak performance 14.4 GFlops, 16 kB level 1 and 8 MB level 2 cache in EPCC Edinburgh and on the SGI Origin 38000 with shared memory, 128 processors R12000, 400 MHz, 48128 MB RAM, 500 GB disk space, DLT 7000 stacker 70 GB,net 3x ATM 155 Mb/sec, FDDI 1Gb/sec in Linz.

The FETI method was recently combined with the penalty method (Dostál & Horák) to obtain a theoretically supported scalable algorithm for solution of coercive and semicoercive variational inequalities (this penalty method is optimal in the sense that a given bound on the relative error of violation of the equality constraints may be achieved with the value of the penalty parameter independent of the discretisation parameter). Fig.2a illustrates that the algorithm presented enjoys high parallel scalability (for problem with $h = 1/512, H = 1/8$, primal dimension 540800, dual dimension 14975 and 53 iterations in 73.6 sec). Fig.2b indicates that algorithm enjoys numerical scalability (the number of the conjugate gradient iterations for a given ratio H/h varies very moderately with changing dimension of the problem).

We used also the augmented Lagrangian algorithm proposed by Dostál, Friedlander, Santos and Gomes (Dostál et al., 2000) which generates approximations of the Lagrange multipliers in the outer loop while the bound constrained quadratic programming problems are solved by efficient algorithms in the inner loop. The results for the largest problems are in Table 1. Numerical experiments with the model variational inequality discretised by up to more than eight million of nodal variables indicate that the algorithm may be efficient and are in good agreement with the theory. Though we have restricted our attention to the model problem, all the reasoning may also be exploited to solution

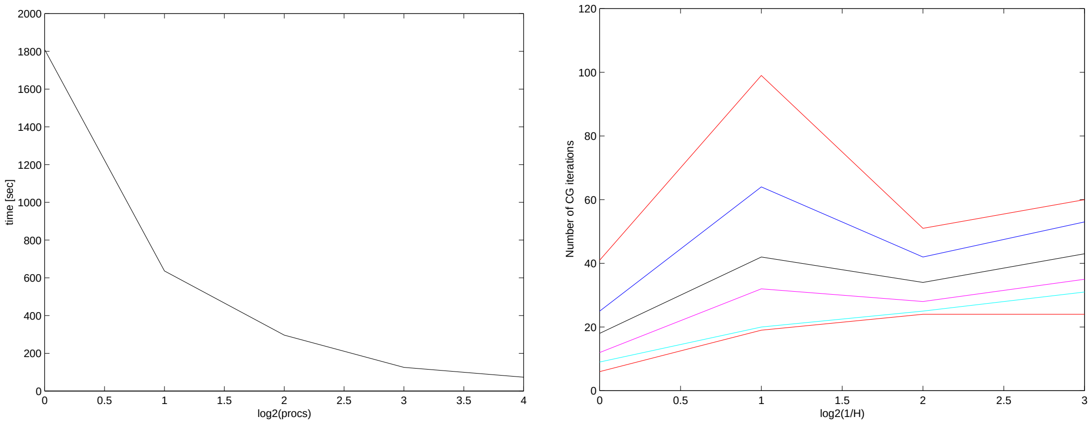

Figure 2: Parallel and numerical scalability

of contact problems of elasticity including those with Coulombian friction (Dostál et al., 2002).

Table 1: Large problems by augmented Lagrangians using SGI Origin

h	H	prim. dim.	dual. dim.	num. of subdom.	procs	cg. iter.	time [sec]
1/1024	1/8	2130048	29823	128	32	47	167
1/2048	1/8	8454272	59519	128	64	65	1281

The algorithms stemming from the FETI method were also implemented in our in-house general purpose finite element computational system PMD (Package for Machine Design).

We tested them against solution to a classic Hertzian problem. It consists in contact of two cylindrical bodies with their axes lying parallel to each other while they are pressed against each other by a force perpendicular to the axes. They make contact over a strip lying parallel to the axes.

The Hertz theory of elastic frictionless contact yields analytical formulae for distribution of stresses within bodies, contact pressure on contacting surfaces, values of displacement of some points, etc. (Johnson, 1985).

The radius of the first cylinder is $R = 1000$mm and the radius of the second body is infinite, which means that the body is a half-space. The material properties of both bodies were the same and were as follows: Young's modulus was $E = 2.0 \times 10^{11}$ Pa and Poisson's ratio $\nu = 0.3$. The mesh was modelled with 1098 elements and 2070 nodes.

The results are shown in Fig.3. It shows distribution of σ_{zz} evaluated by the penalty method and the Lagrangian multiplier method, and for comparison the analytical solution given by McEwen's formula is also plotted. The stresses σ_{zz} are those in the direction perpendicular to the axis of the cylindrical body and surface of the half-space body.

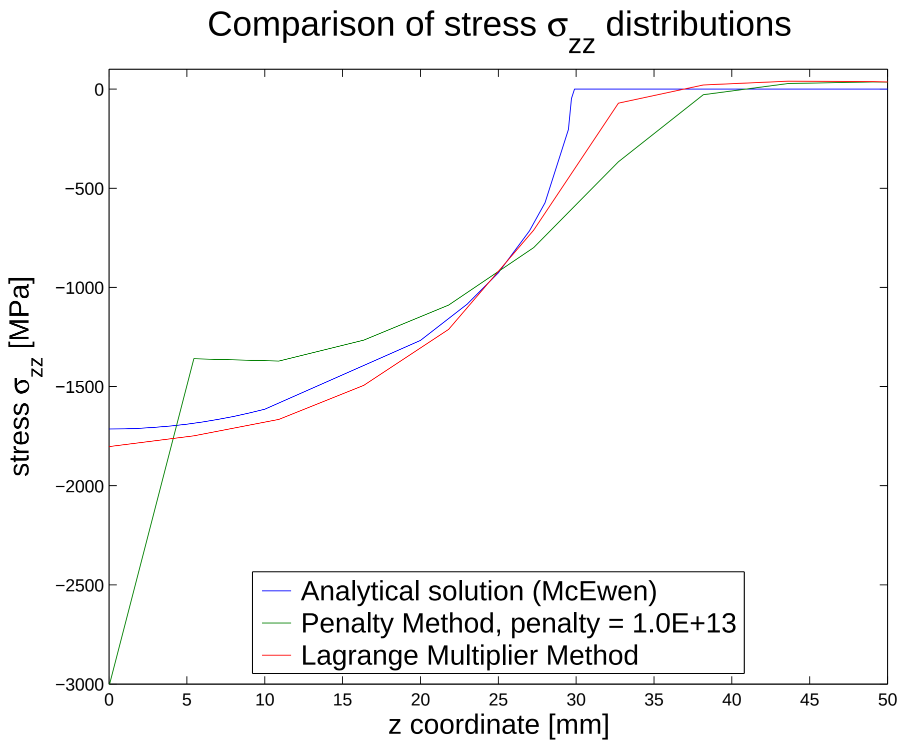

Figure 3: Comparison of analytical and numerical solutions: Distribution of σ_{zz}

4 Conclusions

The numerical experiments shown that the FETI method is very suitable for analysis of large scale problems on high performance supercomputers because the algorithms stemming from it can be successfully implemented in parallel due to the nature the FETI method and they show good scalability.

In addition, the idea that the individual sub-domains, into which the body is divided, interact with each other via the Lagrangian multipliers or forces, can also be applied to solution of contact problem. The reached accuracy is very good and we are able to solve semicoercive problems.

5 Acknowledgements

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