National Conference with International Participation
ENGINEERING MECHANICS 2003
Svratka, Czech Republic, May 12 – 15, 2003
paper no. **144**

3D MATHEMATICAL MODEL OF SPHERICAL PARTICLE SALTATORY MOVEMENT IN OPEN CHANNEL WITH ROUGH BED

N. Lukerchenko*, Z. Chára*, P. Vlasák*

Summary: *One of the main modes of bed load transport is saltation when particles hop up from the bed and follow ballistic-like trajectories. Mathematical models of saltation are mostly two-dimensional although the particle motion is actually three- dimensional. The aim of the present study is development of the 3D mathematical model of solid particle saltatory motion over rough bed in an open channel. The motion of the particle is described by the equation of motion of particle mass center and by equation of the particle rotation around particle center. Gravitational force, buoyancy force, drag force, Basset history force, Magnus force and forces due to added mass and fluid acceleration are taken into account together with the mean velocity profile for turbulent flow in open channel with rough boundary. The model allows calculating 3D trajectories of the particle, statistical characteristics of saltation, such as mean length and height of the trajectory, longitudinal velocity and also the lateral dispersion of the particles.*

1. Introduction

Sediment transport rate can be evaluated using numerical models for particle saltation in particle loaded open channel flow (Wiberg & Smith, 1989; Sekine & Kikkawa, 1992; Sekine & Parker, 1992). Determination of kinematic parameters of particle saltatory movement, such as mean longitudinal velocity of the particle, length and height of the particle trajectory is important for developing bed load transport models (Ashida & Michiue, 1972; Bagnold, 1973; Bridge & Dominic, 1984; Bridge & Bennet, 1992).

The numerical simulation of trajectories of the particles transported by water has been developed by several researchers (e.g., Reizes, 1978; Murthy & Hooshiari, 1982; Van Rijn, 1984; Wiberg & Smith, 1985; Nino & Garsia, 1994; Lee et al., 2000; Loukertchenko at al., 2002). As a rule, such models are based on following assumptions: Lagrangian equations of particle motion, spherical shape of the saltating particles, fluid flow and the particle motion are two-dimensional and the effect of turbulent velocity fluctuations on the mean particle motion is not considered. However, assumption of 2D-particle-motion certainly decreases number of parameters that could be calculated, in particular a lateral dispersion of the particles (Nino & Garsia, 1998; Chara & Vlasak, 2000).

* Nikolai Lukerchenko, CSc., Ing. Zdenek Chara, CSc., Ing. Pavel Vlasak, DrSc.:
Institute of Hydrodynamics ASCR, Pod Patankou 30/5, 166 12 Prague 6, Czech Republic;
Phone: +420-233325629, Fax: +420-233324361, E-mail: vlasak@ih.cas.cz

The aim of this paper is to develop 3D mathematical model of solid particle motion over rough bed in an open channel flow.

2. System of equations of 3D motion of a spherical particle

Let us consider a saltation process of a single solid spherical particle the flow in an open channel with rough bed. If particle concentration in fluid is sufficiently small then a single particle motion can represent motion of all particles. If concentration of the particles in fluid is sufficiently small then a single particle motion can represent motion of all particles. Influence of the particles on the fluid flow as well as the effect of velocity fluctuations of turbulent fluid on the mean particle motion can be also neglected.

This assumption can be valid for saltatory process, which is described as the unsuspended transport of particles over a granular bed of the channel in the form of consecutive hops, Bagnold, 1973. No upward impulses affect the particles except impulses resulting from successive contacts between the particles and the bed. This definition excludes the effect of turbulence as the mechanism that supports particle saltation. Moreover, Francis, 1973, demonstrated, by increasing the viscosity of the fluid in a flume under the same flow conditions, that saltation has the same essential character both in laminar and turbulent flow.

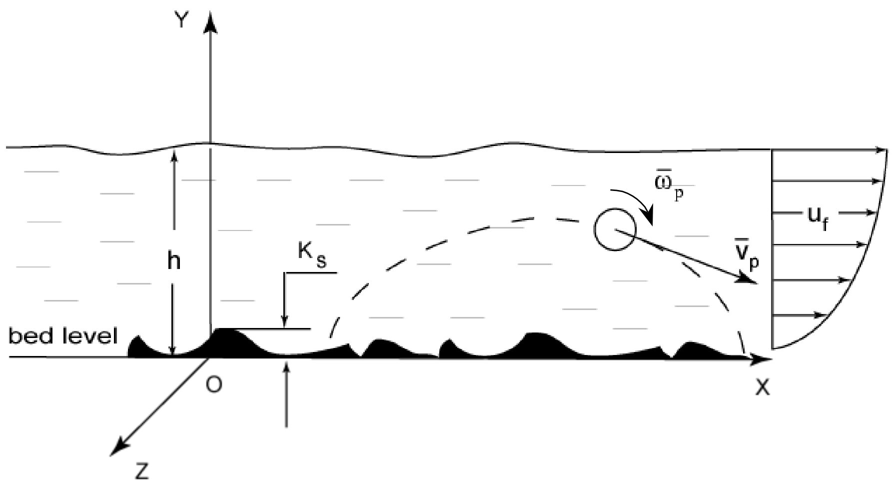

Fig. 1. Particle in fluid flow over rough bed

Let us consider a particle of diameter d_p and density ρ_p moving in the fluid of density ρ and kinematic viscosity ν in the channel with bed roughness k_s and fluid depth h, as it is illustrated in Fig. 1. Using the Mei et al., 1991, form of the governing equations (introduced by Maxey & Riley, 1983) for the motion of a small spherical particle in an unbounded fluid, Nino & Garsia, 1994, proposed equations for 2D mean trajectory of the saltating particle in a turbulent boundary layer. Extending these equations to the 3D model and taking into account also particle rotation, the system of the governing dimensionless equations for particle saltatory motion over rough bed can be written as

$$\frac{d\bar{r}_p}{dt} = \bar{v}_p \tag{1}$$

$$\frac{d\bar{v}_p}{dt} = a\left(\bar{F}_D + \bar{F}_\mathrm{m} + \bar{F}_g + \bar{F}_B + \bar{F}_M\right) \tag{2}$$

$$\frac{d\bar{S}_p}{dt} = \overline{M} \tag{3}$$

$$\bar{F}_D = -3/4\,\rho\, C_D \left|\bar{v}_R\right| \bar{v}_R \qquad \text{for } \mathrm{Re}_p >> 1 \tag{4a}$$

$$\bar{F}_D = -18\rho\nu\bar{v}_R \qquad \text{for}\,\mathrm{Re}_p << 1 \tag{4b}$$

$$\bar{F}_m = \rho\, C_m\left(\bar{v}_p \cdot \nabla\right)\bar{v}_f \tag{5}$$

$$\bar{F}_g = \frac{\rho}{\tau_*}\frac{\bar{g}}{g} \tag{6}$$

$$\bar{F}_B = -9\rho\left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \int_0^t \frac{d\bar{v}_R}{d\tau}\frac{d\tau}{(t-\tau)^{\frac{1}{2}}} \tag{7}$$

$$\bar{F}_M = \rho\, C_M\left(\left(\bar{S}_p - \frac{1}{2} rot\bar{v}_f\right)\times \bar{v}_R\right) \tag{8}$$

$$\overline{M} = -\frac{\mathrm{C_S}}{I_p(1+R)}\left(\bar{S}_p - \frac{1}{2}rot\bar{v}_f\right)\left|\bar{S}_p - \frac{1}{2}rot\bar{v}_f\right|; \qquad \text{for}\,\mathrm{Re}_\omega >> 1 \tag{9a}$$

$$\overline{M} = -\frac{\pi\left(\nu/d_p u_*\right)}{I_\mathrm{p}(1+R)}\left(\bar{S}_p - \frac{1}{2}rot\bar{v}_f\right); \qquad \text{for } \mathrm{Re}_\omega << 1 \tag{9b}$$

$$C_D = \frac{24}{\mathrm{Re}_p}\left(1 + 0.15\left(\mathrm{Re}_p\right)^{1/2} + 0.017\mathrm{Re}_p\right) - \frac{0.208}{1 + 10^4\,\mathrm{Re}_p^{-0.5}} \tag{10}$$

where

$\bar{v}_R = \bar{v}_p - \bar{v}_f$ - particle relative velocity;

$\bar{v}_f = \left(v_{fx}, 0, 0\right)$ - fluid velocity;

$\bar{g} = \left(g\sin\theta_b, -g\cos\theta_b, 0\right)$;

$\mathrm{Re}_p = \left|\bar{v}'_R\right| d_p / \nu$ - particle Reynolds number and

$\mathrm{Re}_\omega = \left|\bar{\omega}_p\right| d_p^2 / \nu$ - particle rotating Reynolds number.

The system of equations (1) - (10) has been made dimensionless by considering the following scaling equations:

$$\bar{r}_p' = (x_p', y_p', z_p') = d_p(x_p, y_p, z_p) = d_p\bar{r}_p; \quad \bar{v}_p' = (v_{px}', v_{py}', v_{pz}') = u_*(v_{px}, v_{py}, v_{pz}) = u_*\bar{v}_p;$$

$$t' = (d_p / u_*)t; \quad v_{fx}' = u_* v_{fx}; \quad \bar{v}_R' = u_*\bar{v}_R; \quad \bar{\omega}_p = (u_* / d_p)\bar{S}_p; \quad I_p' = I_p \rho_p d_p^5; \quad \rho = 1;$$

$$\tau_* = \frac{u_*^2}{gRd_p}; \quad a = \frac{1}{1+R+C_m}; \; R = \rho_p/\rho - 1;$$

where upper asterisk represents dimensional variables.

The expression (10) for drag coefficient C_D as function of Re_p was proposed by Yen, 1992. The value of Magnus force coefficient $C_M = 3/4$ was proposed by Rubinov & Keller, 1961, for $\mathrm{Re}_p << 1$ and $\mathrm{Re}_\omega << 1$. For $\mathrm{Re}_p >> 1$ and $\mathrm{Re}_\omega >> 1$ the value of this coefficient $C_M = 2$ was derived by Goldshtik & Sorokin, 1968.

A proper value of the drag-rotating coefficient C_S must be obtained experimentally. For a numerical model this coefficient can be used as calibrating.

The flow velocity distribution is affected by the roughness Reynolds number $\mathrm{Re}_s = u_* k_s/\nu$ and the velocity distribution can be described by the logarithmic law

$$v_{fx}'(y') = \frac{1}{k}\ln\left(\frac{y'}{y_0'}\right) \tag{11}$$

where $k = 0.4$ is the Karman constant; $y_0' = 0.11(\nu/u_*) + 0.033k_s$ is the elevation where fluid velocity equals zero (Nikuradse, 1933). The equation (11) closures the set of equations for calculation the three dimensional particle saltation movements.

3. Conclusions

- 3D mathematical model of spherical particle saltatory movement in an open channel with rough bed taking into account a particle rotation was developed.
- The model allows calculating 3D trajectories of the particle, the statistical characteristics of saltation, such as mean length, height and longitudinal velocity of the particle saltation.
- The lateral turbulent dispersion of the particles can be evaluated, too.

4. Acknowledgement

The authors gratefully acknowledge the support of the Grant Agency of Academy of Sciences of the Czech Republic under the project number A 2060201.

5. Notation

a - dimensionless parameter;
C_D - drag coefficient;
C_m - added mass coefficient;

C_M - Magnus force coefficient;
C_S - drag rotating coefficient;
d_p - particle diameter;
$\overline{F}_B$ - Basset history force per unit volume;
$\overline{F}_D$ - drag force per unit volume;
$\overline{F}_g$ - submerged gravitational force per unit volume;
$\overline{F}_m$ - force due to added mass per unit volume;
$\overline{F}_M$ - Magnus force per unit volume;
g - gravitational acceleration;
I_p - dimensionless particle momentum of inertia;
$k = 0.4$ - Karman constant;
k_s - bed roughness;
$\overline{M}$ - momentum of force acting on a rotating particle in fluid;
R - particle submerged specific density;
$\mathrm{Re}_s = u_* k_s / \nu$ - roughness Reynolds number;
$\mathrm{Re}_p = \left|\overline{v}_R^{'}\right| d_p / \nu$ - particle Reynolds number;
$\mathrm{Re}_\omega = \left|\overline{\omega}_p\right| d_p^2 / \nu$ - particle rotating Reynolds number.
$\overline{r}_p = \left(x_p, y_p, z_p\right)$ - dimensionless radius-vector of the particle center of mass;
$\overline{r}_p^{'} = (x_p^{'}, y_p^{'}, z_p^{'})$ - radius-vector of the particle center of mass;
$\overline{S}_p = \left(S_{px}, S_{py}, S_{pz}\right)$ - dimensionless angular particle velocity vector;
t - dimensionless time;
$t^{'}$ - time;
u_* - fluid shear velocity;
$\overline{v}_f$ - fluid velocity;
$\overline{v}_p^{'} = (v_{px}^{'}, v_{py}^{'}, v_{pz}^{'})$ - vector of translational particle velocity;
$\overline{v}_p = \left(v_{px}, v_{\mathrm{py}}, v_{pz}\right)$ - vector of dimensionless translational particle velocity;
$\overline{v}_R = \overline{v}_p - \overline{v}_f$ - vector of dimensionless particle relative velocity;
x - dimensionless coordinate in the longitudinal direction;
y - dimensionless coordinate in the normal direction;
z - dimensionless coordinate in the transversal direction;
$x^{'}$ - coordinate in the longitudinal direction;
$y^{'}$ - coordinate in the normal direction;
$y_0^{'}$ - elevation of point, where fluid velocity equals zero;
$z^{'}$ - coordinate in the transversal direction;
θ_b - angle of the bed inclination;
μ - dynamic viscosity;
ν - kinematic viscosity;
ρ - fluid density;

ρ_p - particle density;

τ_* - dimensionless shear stress (Shields' parameter);

$\overline{\omega}_p$ - vector of particle angular velocity.

6. References

Abbott, J.E. & Francis, J.R.D. (1977) Saltation and suspension trajectories of solid grains in a water stream, *Philos. Trans. R. Soc. London A, 284,* pp. 225-254.

Ashida, K. & Michiue, M. (1972) Study on hydraulic resistence and bedload transport rate in alluvial streams, *Proc. Jpn. Soc. Civ. Eng., 206*, pp. 59-69.

Bagnold, R.A. (1973) The nature of saltation and of 'bed-load' transport in water, *Proc. R. Soc. London A, 332,* pp. 473-504.

Bridge, J.S. & Bennett, S.J. (1992) A model for the entrainment and transport of sediment grains of mixed sizes, shapes, and densities, *Water Resour. Res., 28(2),* pp. 337-363.

Bridge, J.S. & Dominic, D.F. (1984) Bed load grain velocities and sediment transport rates, *Water Resour. Res., 20(4),* pp. 476-490.

Chara, Z. & Vlasak, P. (2000) Lateral dispersion of solid particles in water flow. *Int. Conf. Engineering Mechanics,* pp. 161-166.

Francis, J.R.D. (1973) Experiments on the motion of solitary grains along the bed of a water stream, *Proc. R. Soc. London A, 332,* pp. 443-471.

Goldshtik, M.A. & Sorokin V.N. (1968) On motion of a particle in vortex chamber. *J. Appl. Math. Eng. Phys.,6,* pp. 149-152 (in Russian).

Lee, H.Y., Chen, Y.H., You, J.Y. & Lin, Y.T. (2000) Investigations of continuous bed load saltating process, *J. Hudraul. Eng., ASCE, 126(9),* pp.691-700.

Lukerchenko, N., Chara, Z. & Vlasak P. (2002) Mathematical modelling of spherical particle saltation over rough bed in a two-dimensional open channel flow*, Proc., 15th Int. Conf. on Mathematical methods in engineering and technology, Tambov, Russia, Vol. 1,* pp. 36 – 39.

Maxey, M.R. & Riley, J.J. (1983) Equation of motion of a small rigid sphere in a nonuniform flow, *Phys. Fluids, 26(4),* pp. 883-889.

Mei, R., Adrian R.J. & Hanratty T.J. (1991) Particle dispersion in isotropic turbulence under Stokes drag and Basset force with gravitational settling, *J. Fluid Mech., 225,* pp. 481-495.

Murphy, P.J. & Hooshiari, H. (1982) Saltation in water dynamics, *J. Hydraul. Eng., ASCE, 108(HY11),* pp. 1251-1267.

Nikuradse, J. (1933) Stromungsgesetze in rauben rohren. *Ver. Deut. Ing*., Forschugsheft 361 (in German).

Nino, Y. & Garsia, M. (1994) Gravel saltation. 2. Modeling, *Water Resour. Res., 30(6),* pp. 1915-1924.

Nino, Y. & Garsia, M. (1998) Experiments on saltation of sand in water. *J. Hydraul. Eng. ASCE, 124(10),* pp. 1014-1025.

Reizes, J.A. (1978) Numerical study of continuous saltation, *J. Hydraul. Eng., ASCE, 104(HY9),* pp. 1303-1321.

Rubinov, S. & Keller, J. (1961) The transverse force on a spinning sphere moving in a viscous fluid, *J. Fluid Mech., 11,* pp. 447-459.

Sekine, M. & Kikkawa, H. (1992) Mechanics of saltating grains, II, *J. Hydraul. Eng., ASCE, 118(4),* pp. 536-558.

Sekine, M. & Parker, G. (1992) Bed-load transport on transverse slope, I, *J. Hydraul. Eng., ASCE, 118(4),* pp. 513-535.

Van Rijn, L.C. (1984) Sediment transport, I, Bed load transport, *J. Hydraul. Eng., ASCE., 110(10),* pp. 1431-1456.

Wiberg, P.L. & Smith, J.D. A theoretical model for saltating grains in water, J. Geophys. Res., 1985, 90(C4), 7341-7354.

Wiberg, P.L. & Smith J.D. (1989) Model for calculating bedload transport of sediment, *J. Hydraul. Eng., ASCE, 115(1),* pp. 101-123.

Yen, B.C. (1992) Sediment fall velocity in oscillating flow, *Water Resour. Environ. Eng. Res. Rep. 11, Dep. of Civ. Eng., Univ. of Va.,* Charlottesville.