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DEFORMATION ANALYSIS OF AN INFLATED CYLINDRICAL MEMBRANE OF COMPOSITE WITH RUBBER MATRIX REINFORCED BY TEXTILE CORDS

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Summary: *We present the orthotropic hyperelastic material model for numerical simulation of the loading of the cylindrical membrane. The coefficients of strain energy function of the hyperelastic orthotropic material are fitted to the experimental results by the nonlinear least squares method. The components of the deformation gradient are determined from measured displacements of the grid points drawn on the cylindrical surface of the spring. The stress tensor is calculated from the membrane theory. The deformed shape of the spring surface is measured from the photographic records. The strain energy function is expressed in terms of tensorial invariants with regard to the assumed material symmetry. The deformation of air-spring is calculated by solving the system of five first-order ordinary differential equations with the material constitutive law and proper boundary conditions.*

1. Introduction

The main purpose of authors is the numerical simulation of inflation of the composite cylindrical membrane made of rubber matrix reinforced by textile cords. Some orthotropic and transversely hyperelastic constitutive models appropriate for such type of material can be found in literature. Most of them are represented by strain energy function in the form of a polynomial, exponential or logarithmic [3, 4, 10] function of strain invariants regarding the assumed material orthotropy. However the development of the constitutive theory of anisotropic elastic or viscoelastic materials at finite strains is still far to be complete and the publications in this field are sparse. The constitutive equations of the transversally isotropic material in the nonlinear stress and deformation domain are presented in the papers of Holzapfel and coll. [3], Bonet and Burton [2] and Verron [11].

We use the consistent constitutive model of direction dependent hyperelastic material presented in papers of Ogden, Holzapfel, Gasser and coll. [3, 4] applied by authors to the problem of the mechanical response of arterial walls and of fiber reinforced composites at finite strains. The deformation field is generally determined by the finite element method. However, we use the method of the numerical integration of the system of the ordinary differential equations of problem described by Green and Adkins [9] and recently for

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isotropic membrane by Guo [4]. We incorporated into this procedure our own orthotropic material law. This method appeared to be quite promising and we presume to use it for the inverse identification of material parameters. Details on the experimental setup and the experiment evaluations can be found in the previous papers of authors [5-8].

2. Deformation of cylindrical orthotropic membrane

We determine the main geometric features of the inflated membrane in according with the derivation in [1, 4]. The thin cylindrical membrane of air-spring at Fig. 1 has the initial radius of mid-surface R, and length 2L. Its initial wall thickness H is assumed to be uniform. The undeformed profile of membrane is described by polar coordinate system, (X, Φ, R). The cylindrical membrane is inflated by the internal pressure.

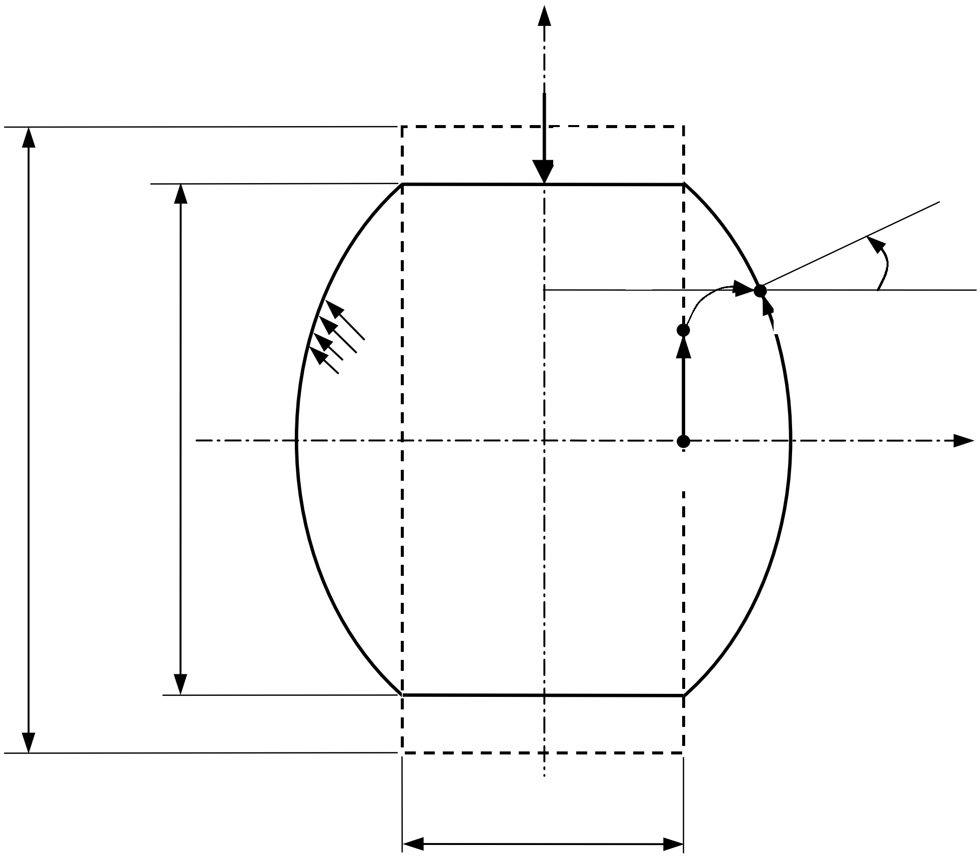

Fig. 1 Undeformed and deformed profile of cylindrical orthotropic membrane

The deformed cylindrical membrane is referred to the polar coordinate system (x, ϕ, r). A material particle moves during the deformation from the position in the undeformed profile, C(X, Φ, R) to the deformed profile, c(x, ϕ, r), along its quasi-equilibrium path. We assume the axisymmetric deformation, $\phi \equiv \Phi$. The principal stretch in axial and circumferential directions, principal curvatures and geometric relations are

$$\lambda_1 = \frac{ds}{dS},\ \lambda_2 = \frac{r}{R},\ \frac{dr}{ds} = -sin\theta,\ \frac{dx}{ds} = cos\theta,\ \kappa_1 = \frac{d\theta}{ds},\ \kappa_2 = \frac{cos\theta}{r} \tag{1}$$

where s is the arc length measured from pole (x = 0) to the particle c(x, ϕ, r) along the meridian of the deformed profile. S is the length corresponding to s in the undeformed profile. We introduce an auxiliary variable θ, the angle of the tangent line. The radius r and the thickness h of the membrane are with respect to the deformed configuration. The radial stretch λ_3 is determined from the incompressibility constraint

$$\lambda_1 \lambda_2 \lambda_3 = 1 \tag{2}$$

then $$h = \frac{H}{\lambda_1 \lambda_2} \tag{3}$$

where R and H are the radius and the thickness in the undeformed configuration.

3. Constitutive equations

The Cauchy stresses are defined as the partial derivatives of strain energy function Ψ with respect to the deformation [3, 4]. We have the following expressions:

$$\begin{aligned} \sigma_1 - \sigma_3 &= \lambda_1 \frac{\partial \Psi(\lambda_1, \lambda_2)}{\partial \lambda_1}, \\ \sigma_2 - \sigma_3 &= \lambda_2 \frac{\partial \Psi(\lambda_1, \lambda_2)}{\partial \lambda_2}, \end{aligned} \tag{4}$$

If we express Ψ as a function of the three principal stretches $\Psi = \Psi(\lambda_1, \lambda_2, \lambda_3) - p^*(J-1)$, with the indeterminate Lagrange multiplier p^*, we can express Cauchy stresses [3] as

$$\sigma_a = -p^* + \lambda_a \frac{\partial \Psi}{\partial \lambda_a}, \quad a = 1, 2, 3 \tag{5}$$

We assume the isochoric deformation and we neglect the dissipation due to irreversible effects. The energy stored in the fibers is assumed in the form of an exponential function. The free energy function in two dimensional problem can be supposed in the form [1]

$$\Psi(\lambda_1, \lambda_2) = \sum_{i=1}^{3} \frac{\mu_i}{\alpha_i} \left(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_1^{-\alpha_i} \lambda_2^{-\alpha_i} - 3 \right) + \frac{k_1}{k_2} \left\{ \exp[k_2 (\lambda_2^2 \cos^2 \alpha + \lambda_1^2 \sin^2 \alpha - 1)^2] - 1 \right\}, \tag{6}$$

where λ_1 and λ_2 are the axial and circumferential stretches respectively, and α is the angle of the two families of reinforcing fibers. We suppose the reinforcing fibers are double-helically arranged in the matrix material symmetrically to the circumferential direction. The angle α of fibers is supposed to be 48.8°. The parameters μ_i and α_i of Ogden's model of rubber [3] are

μ_1 = 630 kPa, μ_2 = 1.2 kPa, μ_3 = -10 kPa, α_1 = 1.3, α_2 = 5, α_3 = -2.

The stress-like parameter k_1 and the non-dimensional parameter k_2 are determined from the experimental results and from the 2D cylindrical membrane approximation.

4. Identification of material parameters

The theory of nonlinear membranes has been presented by Green and Adkins [9] and applied to various inflated structures [4]. The quasi-static equilibrium equations of problem are

$$\frac{d}{ds}\left(T_1 r\right) = T_2 \frac{dr}{ds}, \qquad \kappa_1 T_1 + \kappa_2 T_2 = p, \tag{7}$$

where p is the inner pressure, T_1 and T_2 are the stress resultant forces per unit length in the meridional and circumferential directions. The stress resultant forces in the deformed configuration are

$$T_1 = h\sigma_1, \quad T_2 = h\sigma_2, \tag{8}$$

where Cauchy stresses σ_1 and σ_2 are given by (5).

We effectuated several series of experiments of inflation of cylindrical air-spring with the variable axial force F and the inner pressure. The Cauchy stress is determined from the equilibrium in Fig. 2

$$\sigma_1 = \frac{p\pi r^2 - F}{2\pi r h \cos\theta} \tag{9}$$

Substituting r = λ_2R and (3) into (9) we obtain σ_1 as

$$\sigma_1 = A\frac{1}{\cos\theta}\lambda_1\left(\lambda_2^2 - B\right) \tag{10}$$

where $$A = \frac{pR}{2H}, \; B = \frac{F}{\pi p R^2} \tag{11}$$

We can deduce σ_2 from equilibrium equation $(7)_2$. We assume σ_3 = - p after the theory of inflated membrane.

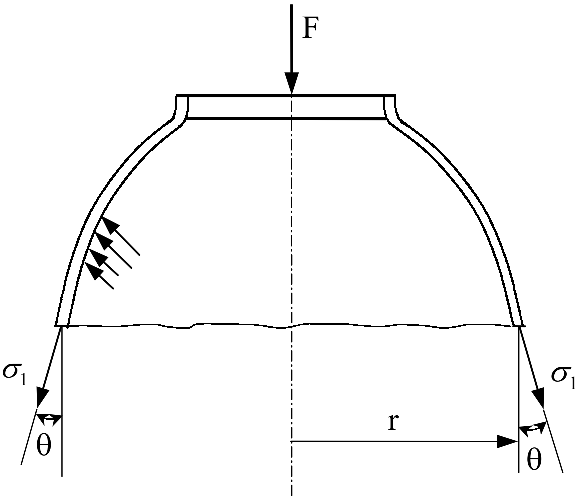

Fig. 2

After the substitutions into equations (4) we obtain set of the nonlinear equations for the two variables k_1 and k_2

$$\sum_{i=1}^{3}\mu_i\left[\lambda_1^{\alpha_i}-\left(\lambda_1\lambda_2\right)^{-\alpha_i}\right]+4k_1\exp(k_2m^2)m\lambda_1^2\sin^2\alpha=C+p$$

$$\sum_{i=1}^{3}\mu_i\left[\lambda_2^{\alpha_i}-\left(\lambda_1\lambda_2\right)^{-\alpha_i}\right]+4k_1\exp(k_2m^2)m\lambda_2^2\cos^2\alpha=p\left(D+1\right)-C\frac{\kappa_1}{\kappa_2} \tag{12}$$

$$\sum_{i=1}^{3}\mu_i\left[\lambda_2^{\alpha_i}-\lambda_1^{\alpha_i}\right]+4k_1\exp(k_2m^2)m\left(\lambda_2^2\cos^2\alpha-\lambda_1^2\sin^2\alpha\right)=pD-C\left(1+\frac{\kappa_1}{\kappa_2}\right)$$

where $C=\frac{p}{2H}\frac{1}{\cos\theta}\lambda_1\left(\lambda_2^2R-\frac{F}{\pi Rp}\right)$, $\mathrm{D}=\frac{pR}{H\cos\theta}\lambda_1\lambda_2^2$, $\mathrm{m}=\lambda_2^2\cos^2\alpha+\lambda_1^2\sin^2\alpha-1$

The experimentally measured values of λ_1 and λ_2 in several points of the central part of our cylindrical membrane were substituted into the equations (12). Taking the logarithm of (12) we will get a set of linear equations for the variables $\ln k_1$ and k_2. The resulting overdetermined system of linear equations was solved in Matlab. The parameters were k_1=4.187e+04 kPa and k_2 = -23.775. The function of the Helmholtz energy potential for these parameters is convex.

5. Determination of deformation of cylindrical membrane

After the substitution of (1), (3), (5) and (8) into equations of equilibrium (7) we get after some simplifications the system of five ordinary differential equations for the principal stretches λ_1 and λ_2, the tangent angle θ, the coordinate x in the deformed configuration and the inner pressure p with respect to the coordinate X of the undeformed configuration

$$\frac{d\lambda_1}{dX}=\frac{1}{\frac{A}{\cos\theta}\left(\lambda_2^2-B\right)-M}\left[\frac{\lambda_1\sin\theta}{R}\left(N-A\frac{2}{\cos\theta}\lambda_1\lambda_2\right)+A\frac{\sin\theta}{\cos^2\theta}\lambda_1\left(\lambda_2^2-B\right)Q\right]$$

$$\frac{d\lambda_2}{dX}=-\lambda_1 R^{-1}.sin\theta$$

$$\begin{aligned}\frac{d\theta}{dX}=\frac{1}{R}\frac{\cos\theta}{A\left(\lambda_2^2-B\right)}\Bigg\{\frac{\cos\theta}{\lambda_2}\Bigg[-p+\sum_{i=1}^{3}\mu_i\left(\lambda_2^{\alpha_i}-\left(\lambda_1\lambda_2\right)^{-\alpha_i}\right)+\\ +4k_1\exp(k_2m^2)m\lambda_2^2\cos^2\alpha\Bigg]-2A\lambda_1\lambda_2\Bigg\}\end{aligned} \tag{13}$$

$$\frac{dx}{dX}=\lambda_1.cos\,\theta\,,$$

$$\frac{dp}{dX}=0\,,$$

where

$$\begin{aligned}Q=\frac{1}{R}\frac{\cos\theta}{A\left(\lambda_2^2-B\right)}\Bigg\{\frac{\cos\theta}{\lambda_2}\Bigg[-p+\sum_{i=1}^{3}\mu_i\left(\lambda_2^{\alpha_i}-\left(\lambda_1\lambda_2\right)^{-\alpha_i}\right)+\\ +4k_1\exp(k_2m^2)m\lambda_2^2\cos^2\alpha\Bigg]-2A\lambda_1\lambda_2\Bigg\}\end{aligned} \tag{14}$$

$$\begin{aligned}M=\frac{1}{\lambda_1}\sum_{i=1}^{3}\mu_i\alpha_i\left[\lambda_1^{\alpha_i}+\left(\lambda_1\lambda_2\right)^{-\alpha_i}\right]+\\ +8k_1\exp(k_2m^2)\lambda_1\sin^2\alpha\left[\lambda_1^2\sin^2\alpha\left(2k_2m^2+1\right)+m\right]\end{aligned} \tag{15}$$

$$N=\frac{1}{\lambda_2}\sum_{i=1}^{3}\mu_i\alpha_i\left(\lambda_1\lambda_2\right)^{-\alpha_i}+8k_1\exp(k_2m^2)\lambda_1^2\lambda_2\sin^2\alpha\cos^2\alpha\left(2k_2m^2+1\right) \tag{16}$$

We solved the set of differential equations (13) by the shooting method in Matlab with the boundary condition for λ_1^0 and λ_2^0 determined from the experiments. The results are at the Fig. 3 where calculated stretches and deformed profile of membrane is compared with experimental one.

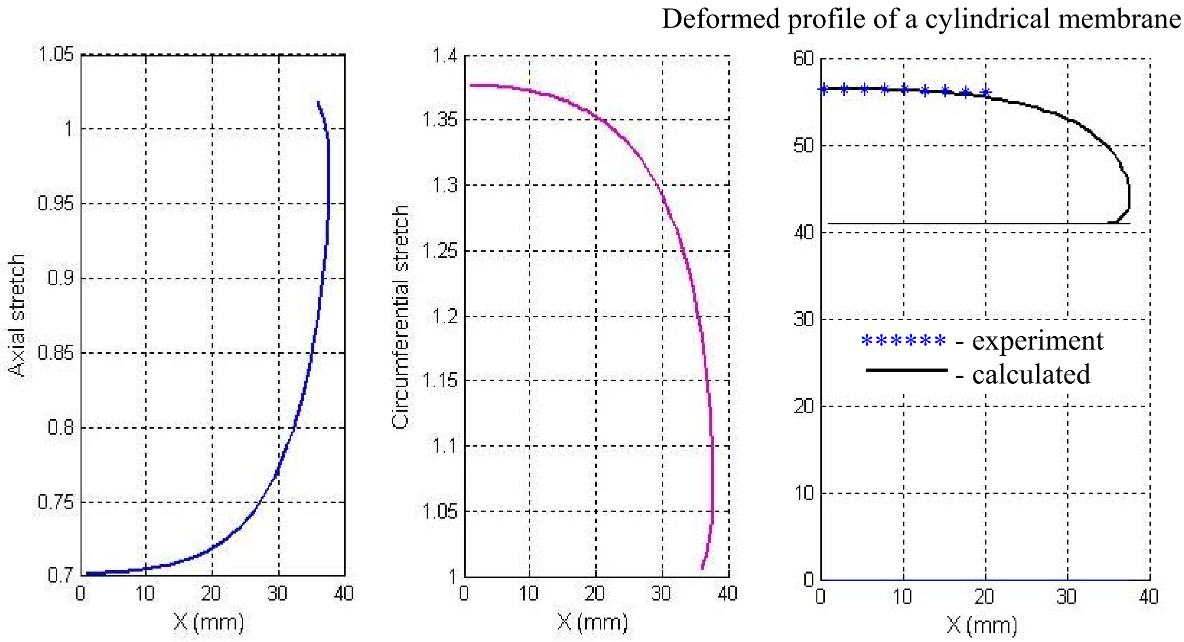

Fig. 3 Results of deformations of cylindrical membrane of air-spring

6. Conclusions

The deformations of the nonlinear composite membrane were determined experimentally. The problem of the identification of the material parameters was solved. The proposed strain energy function was implemented into the calculus of deformations of the cylindrical membrane of air-spring. The deformations were determined by numerical solution the system of ordinary differential equations based on the membrane theory. The method will be used for the inverse identification of material parameters of the inflatable structures namely air-springs.

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