National Conference with International Participation
ENGINEERING MECHANICS 2003
Svratka, Czech Republic, May 12 – 15, 2003
paper no. 277

GEOMETRIC SYNTHESIS OF FACE HELICOIDS

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Summary: *The basic problem of analytical and computer design of the spatial gears is the synthesis of their tooth surfaces. The technology of many gear pairs manufacture is based on the second Olivier's principle. It means that one of the gear of the gear pair is generated by a tool, which is a counterpart of the other gear. For this reason we are exclusively interested in the synthesis of the active flanks of the first gear (the tool).*
This study is dedicated to the synthesis of face helicoids of convolute, involute and Archimedean type. These helical surfaces could be used as active flanks of the threads of flat worm and flat hob, respectively.

1. Introduction

Helical surfaces are widely applied as active tooth gear surfaces. They are also used as basic elements in the construction of various cutting and milling tools.

The precise and understanding theory of the helicoids performance makes easier the creation of the mathematical models of the gear drive and tool synthesis.

The cylindrical helicoids are studied in a number of publications. They treat the mathematical modelling for synthesis of gear drives and instrumental sets in case when the flanks of some gears are cylindrical helicoids (Litvin, 1968; Ljukshin, 1968; Lashnev, & Julikov, 1975; Golovanov, Ginzburg & Firun1967; Semenchenko, Matjushin, & Saharov, 1962).

There are scientific publications (Abadjiev & Minkov, 1981; Abadjiev, 1984; Georgiev, 1965; Ganshin, 1970; Abadjiev & Abadjieva, 2002) dedicated to the synthesis of the conic linear helicoids (convolute, involute and Archimedean). The equations and analytical relations of these helical surfaces are applicable for the synthesis of cylindrical helical surfaces when the angle defining the conic form of the helicoids is equal to zero.

Analogically to the cylindrical helicoid, when the conic form angle of the conic linear helicoid is assumed to be equal to 90 degrees it is transformed in the face linear helicoid. The study is dedicated to the face linear helicoids research connected with creation of the mathematical model for synthesis. The derived in the study face helicoid equations and the analytical descriptions of their cross and axial sections are substantial geometrical characteristics of the active tooth surfaces of the flat worms, and of the flat hobs, respectively. They are related to the technological synthesis organization and to a new type gear design. They are also related to the manufacture of instrumental equipment and to the control measuring equipment of the active tooth surfaces.

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2. Geometrical synthesis of face helical surfaces

2.1. Face convolute helicoid

In Fig. 1 is illustrated the process of generating of a face right-hand face helicoids $\Sigma_f^{(j)}$ $(j=1\ ,\ 2)$ with a constant pitch in the coordinate system $S_f(O_f, x_f, y_f, z_f)$. The generatrix $L^{(j)}$ does not intersect the axis $O_f z_f$. The angle between generatrix and the axis is $\pi/2 < \xi^{(j)} < \pi$.

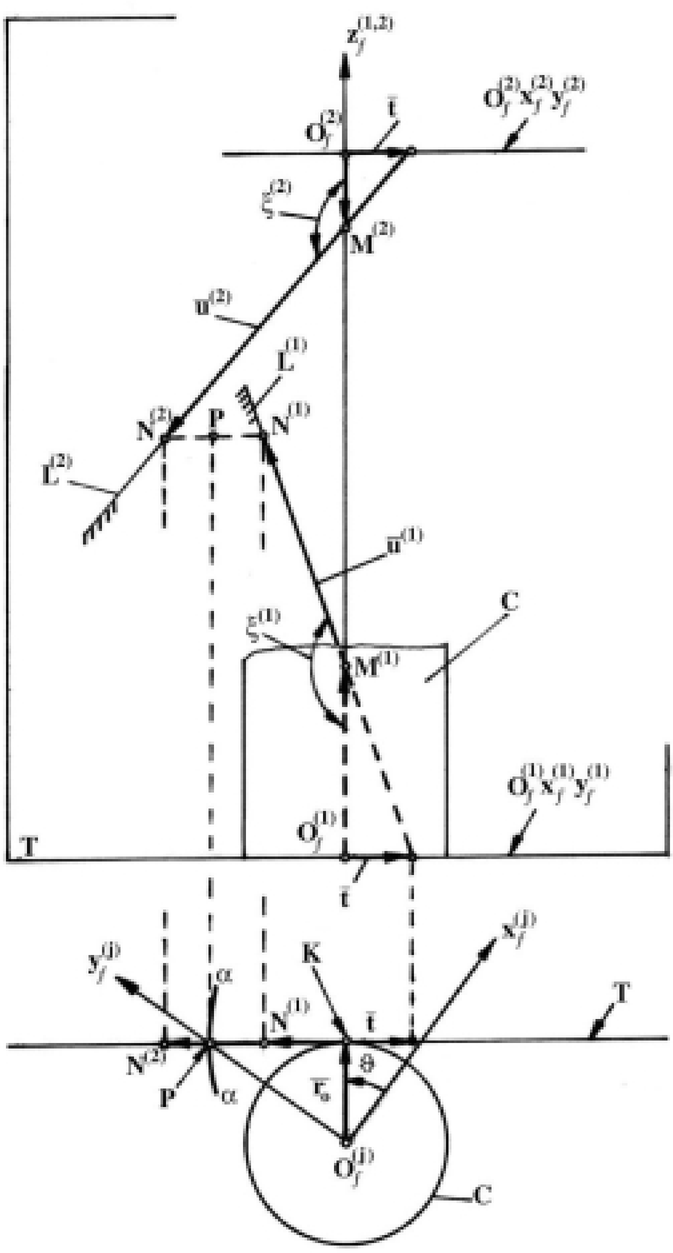

Fig. 1: Generation of face helicoid: P -characteristically (calculatingly) point from face worm thread; α - α is the helix line of the face worm thread in the point P;

The smallest distance between $O_f z_f$ and $L^{(j)}$ is $r_0^{(j)}$. It is the basic cylinder radius for which is fulfilled the condition $r_0^{(j)} = constant$ for the entire process of face convolute helicoid $\Sigma_f^{(j)}$ generation. The generation of $\Sigma_f^{(j)}$ is a result of a crossed (it is tangential with respect to the basic cylinder C) helical motion of the straight line $L^{(j)}$ with helical parameter $p_t^{(j)} = constant$. In this case it is accepted that the generatrix $L^{(j)}$ lies in the plane T, which is tangential to the directed cylinder C. To the used index j here and onwards are assigned numbers $j = 1,\ 2$ and j=1 is referred to the parameters, which are connected with face helicoid $\Sigma_f^{(1)}$ generation, $j = 2$ is referred –to the parameters of face helicoid $\Sigma_f^{(2)}$. The parts of the surfaces $\Sigma_f^{(1)}$ and $\Sigma_f^{(2)}$ are used as active tooth surfaces of a face worm and a face hob, respectively. The active tooth surfaces $\Sigma_f^{(1)}$ are displaced closer to the $O_f z_f$ then to the active tooth surfaces $\Sigma_f^{(2)}$. In this study in the generation process of the $\Sigma_f^{(j)}$ $(j = 1\ ,\ 2)$, from technological reasons connected with real tooth cutting is assumed $r_0^{(1)} = r_0^{(2)} = r_0$, $\vartheta^{(1)} = \vartheta^{(2)} = \vartheta$ and $p_t^{(1)} = p_t^{(2)} = p_t = constant$.

Let point $N^{(j)}$ belongs to the straight line $L^{(j)}$,which generates the helicoid $\Sigma_f^{(j)}$, when $L^{(j)}$ performs cross helical motion. Then for the vector equation of $\Sigma_f^{(j)}$ can be written:

$$\overline{\rho}_f^{(j)} = \overline{r}_0 + \overline{t} + \overline{u}^{(j)}, \tag{1}$$

where

$t = p_t\vartheta$ is the value of the cross helical motion of the straight line $L^{(j)}$;

$u^{(j)}$, ϑ - curvilinear coordinates of the $\Sigma_f^{(j)}$.

In the coordinate system S_f for (1) it is obtained:

$$\begin{aligned} x_f^{(j)} &= r_0 \cos\vartheta - (u^{(j)} \sin\xi^{(j)} - p_t\vartheta) \sin\vartheta, \\ y_f^{(j)} &= r_0 \sin\vartheta + (u^{(j)} \sin\xi^{(j)} - p_t\vartheta) \cos\vartheta, \\ z_f^{(j)} &= \mp u^{(j)} \cos\xi^{(j)}. \end{aligned} \tag{2}$$

Substituting in (2)

$$R_0^{(j)} = u^{(j)} - \frac{p_t\vartheta}{\sin\xi^{(j)}}, \quad p^{(j)} = -p_t \cot\xi^{(j)} > 0 \tag{3}$$

the system equations (2) can be written in the following form:

$$\begin{aligned} x_f^{(j)} &= r_0 \cos\vartheta - R_0^{(j)} \sin\xi^{(j)} \sin\vartheta, \\ y_f^{(j)} &= r_0 \sin\vartheta + R_0^{(j)} \sin\xi^{(j)} \cos\vartheta, \\ z_f^{(j)} &= \pm p^{(j)}\vartheta \mp R_0^{(j)} \cos\xi^{(j)}. \end{aligned} \tag{4}$$

The system of equations (4) could be treated as a system of equations, which describe the cylindrical helicoid with an axial helical parameter $p^{(j)} = constant$. It could be considered that the system (4) is obtained from the vector equation:

$$\overline{\rho}^{(j)} = \overline{O_f K} + \overline{KM^{(j)}} + \overline{M^{(j)}N^{(j)}} \tag{5}$$

Where $M^{(j)}N^{(j)} = R_0^{(j)}$ is a curvilinear coordinate of the helicoid with an accounting origin in the point $M^{(j)}$.

2.2. Face involute helicoid

It is known (Ljukshin, 1968; Rashevsky, 1956), that basic characteristic of the surfaces, (the surfaces generatrix is a straight line) is their distribution parameter. For these surfaces the generatrix passes from one position to the closest nearest position, simultaneously rotating to some angle and displacing to some distance. These two quantities are infinity small, but their relation has a limit. The limit is the distribution parameter of these type surfaces.

Let (4) is presented in the type:

$$\overline{\rho}_f^{(j)} = \overline{\rho}_0^{(j)} + R_0^{(j)}\bar{l}^{(j)} \tag{6}$$

Here $\overline{\rho}_0^{(j)} = \overline{\rho}_0^{(j)}(\vartheta)[\chi_x^{(j)},\chi_y^{(j)},\chi_z^{(j)}]^T$ could be considered as the directed helical line equation, which is placed on the basic cylinder C. The $\bar{l}^{(j)}[l_x^{(j)},l_y^{(j)},l_z^{(j)}]^T$ is the directed vector of the generatrix $L^{(j)}$. It is evident that:

$$\begin{aligned} &\chi_x^{(j)} = r_0 \cos\vartheta, \quad \chi_y^{(j)} = r_0 \sin\vartheta, \quad \chi_z^{(j)} = \pm p^{(j)}\vartheta, \\ &l_x^{(j)} = -\sin\xi^{(j)}\cos\vartheta, \quad l_y^{(j)} = \sin\xi^{(j)}\sin\vartheta, \quad l_z^{(j)} = \mp\cos\xi^{(j)}. \end{aligned} \tag{7}$$

The distribution parameter of the face convolute helicoid is defined, by using the relation:

$$h^{(j)} = \frac{[d\overline{\rho}_0^{(j)},\bar{l}^{(j)},d\bar{l}^{(j)}]}{d\bar{l}^2} = \frac{\begin{vmatrix} d\chi_x^{(j)} & l_x^{(j)} & dl_x^{(j)} \\ d\chi_y^{(j)} & l_y^{(j)} & dl_y^{(j)} \\ d\chi_z^{(j)} & l_z^{(j)} & dl_z^{(j)} \end{vmatrix}}{(dl_x^{(j)})^2 + (dl_y^{(j)})^2 + (dl_z^{(j)})^2}. \tag{8}$$

After rudimentary transformations for the case, from (7) and (8) is obtained:

$$h^{(j)} = \pm\cot\xi^{(j)}(r_0 - p_t). \tag{9}$$

The systems of equations (2) define a developable face helicoids, if the following condition is fulfilled:

$$h^{(j)} = \pm\cot\xi^{(j)}(r_0 - p_t) = 0. \tag{10}$$

It is obvious, that relation (10) is fulfilled if $r_0 = p_t$ and if $\xi^{(j)} = \pi$ when $r_0 \neq p_t$.

Hence for the face involute helicoid can be written:

$$\begin{aligned} x_f^{(j)} &= -u^{(j)}\sin\xi^{(j)} + r_0(\cos\vartheta + \vartheta\sin\vartheta), \\ y_f^{(j)} &= u^{(j)}\sin\xi^{(j)} + r_0(\sin\vartheta - \vartheta\cos\vartheta), \\ z_f^{(j)} &= \mp u^{(j)}\cos\xi^{(j)}. \end{aligned} \tag{11}$$

2.3. Face Archimedean helicoid

The face Archimedean helicoid is obtained, when the straight line $L^{(j)}$ crosses the screw axis $O_f z_f$, i.e. $r_0 = 0$. In this case from (2) for the system of equations defining this helicoid it can be written:

$$\begin{aligned} x_f^{(j)} &= -(u^{(j)}\sin\xi^{(j)} - p_t\vartheta)\sin\vartheta, \\ y_f^{(j)} &= +(u^{(j)}\sin\xi^{(j)} - p_t\vartheta)\cos\vartheta, \\ z_f^{(j)} &= \mp u^{(j)}\cos\xi^{(j)}. \end{aligned} \tag{12}$$

Here the distribution parameter is $h^{(j)} = \mp p_t \cot\xi^{(j)} \neq 0$. In the system (12) ϑ is the angle, between the normal $\bar{r}_0$ to the axial plane $A \equiv T$ and the axis $O_f x_f$.

If it is substituted with $\theta = \pi/2 - \vartheta$ (this angle is the angle between the plane A and the plane $O_f z_f x_f$), then the equations (12) obtain the following type:

$$x_f^{(j)} = -[u^{(j)} \boldsymbol{sin}\,\xi^{(j)} - p_t(\pi/2-\theta)]\boldsymbol{cos}\,\theta,$$
$$y_f^{(j)} = +[u^{(j)} \boldsymbol{sin}\,\xi^{(j)} - p_t(\pi/2-\theta)]\boldsymbol{sin}\,\theta, \qquad (13)$$
$$z_f^{(j)} = \mp u^{(j)} \boldsymbol{cos}\,\xi^{(j)}.$$

3. Basic geometrical characteristics of face helicoid

3.1. Cross section

The equations of the face convolute helicoid cross section when $z_f^{(j)} = C_z^{(j)} = \mp u^{(j)} \boldsymbol{cos}\,\xi^{(j)}$ are obtained from the system of equations (2), i.e.:

$$x_f^{(j)} = r_0 \boldsymbol{cos}\,\vartheta + b^{(j)}\vartheta\, \boldsymbol{cos}\,\vartheta,$$
$$y_f^{(j)} = r_0 \boldsymbol{sin}\,\vartheta - b^{(j)}\vartheta\, \boldsymbol{sin}\,\vartheta, \qquad (14)$$
$$b^{(j)} = \frac{C_z^{(j)}}{\vartheta\, \boldsymbol{cos}\,\vartheta} + p_t.$$

If the face helicoid is an involute, i.e. $r_0 = p_t$ and if $C_z^{(j)} = 0$ then from (14) are obtained:

$$x_f^{(j)} = r_0(\boldsymbol{cos}\,\vartheta + \vartheta\, \boldsymbol{sin}\,\vartheta),$$
$$y_f^{(j)} = r_0(\boldsymbol{sin}\,\vartheta + \vartheta\, \boldsymbol{cos}\,\vartheta). \qquad (15)$$

The system (15) describes the involute curve in the plane $O_f z_f x_f$. When $r_0 = 0$, the equations (14) circumscribe the cross section of the face Archimedean helicoid, i.e.:

$$x_f^{(j)} = b^{(j)}\vartheta\, \boldsymbol{cos}\,\vartheta,$$
$$y_f^{(j)} = -b^{(j)}\vartheta\, \boldsymbol{sin}\,\vartheta. \qquad (16)$$

The system of equations (16) illustrates in the plane $z_f = C_z$ an Archimedean spiral.

3.2. Axial section

After substitution $y_f^{(j)} = 0$ in (2) for the axial section equations of the face convolute helicoid are obtained:

$$x_f^{(j)} = \frac{r_0}{\boldsymbol{cos}\,\vartheta} = r_0\, \boldsymbol{sec}\,\vartheta,$$
$$z_f^{(j)} = \mp p^{(j)}(k^{(j)}\, \boldsymbol{tan}\,\vartheta - \vartheta) = \mp p^{(j)} conv(\vartheta, k^{(j)}), \qquad (17)$$
$$k^{(j)} = -\frac{r_0\, \boldsymbol{cot}\,\xi^{(j)}}{p^{(j)}} > 0.$$

If $k^{(j)} = 1$, i.e. $r_0 = p_t$, then (17) obtaines the type:

$$x_f^{(j)} = r_0\, \boldsymbol{sec}\,\vartheta,$$
$$z_f^{(j)} = \mp p^{(j)}(\boldsymbol{tan}\,\vartheta - \vartheta) = \mp p^{(j)} inv\vartheta. \qquad (18)$$

and describe face involute helicoid .

It is derived from (17) using the substitution $r_0 = 0$:

$$x_f^{(j)} = y_f^{(j)} = 0,$$
$$z_f^{(j)} = \mp p^{(j)} \boldsymbol{cot}\, \xi^{(j)} \vartheta = \pm p^{(j)} \vartheta\,. \tag{19}$$

It is easy to establish that (19) is the straight line equation.

3.3. Elimination of the face helicoid surfaces cutting points

If the system of equations (2), which describe the face convolute helicoid $\Sigma_f^{(j)}$ is presented in the form:

$$\overline{\rho}_f^{(j)} = \overline{\rho}_f^{(j)}(u^{(j)}, \vartheta) \tag{20}$$

Then for the normal vector of this surface in arbitrary point $N^{(j)}$ can be written:

$$\overline{n}_f^{(j)} = \frac{\partial \overline{\rho}_f^{(j)}}{\partial u^{(j)}} \times \frac{\partial \overline{\rho}_f^{(j)}}{\partial \vartheta}\,. \tag{21}$$

Here $\partial \overline{\rho}_f^{(j)} \big/ \partial u^{(j)}$ and $\partial \overline{\rho}_f^{(j)} \big/ \partial \vartheta$ are vectors, which are tangential to $\Sigma_f^{(j)}$ in the point $N^{(j)}$. If it is used the projection of the $\overline{n}_f^{(j)}$ in the coordinate system S_f, then it can be written:

$$n_{f,x_f}^{(j)} = \begin{vmatrix} \dfrac{\partial y_f^{(j)}}{\partial u^{(j)}} & \dfrac{\partial y_f^{(j)}}{\partial \vartheta} \\ \dfrac{\partial z_f^{(j)}}{\partial u^{(j)}} & \dfrac{\partial z_f^{(j)}}{\partial \vartheta} \end{vmatrix} = \pm \boldsymbol{cos}\, \xi^{(j)} [\boldsymbol{cos}\, \vartheta\, (r_0 - p_t) - \boldsymbol{sin}\, \vartheta\, (u^{(j)} \boldsymbol{sin}\, \xi^{(j)} - p_t \vartheta)],$$

$$n_{f,y_f}^{(j)} = \begin{vmatrix} \dfrac{\partial z_f^{(j)}}{\partial u^{(j)}} & \dfrac{\partial z_f^{(j)}}{\partial \vartheta} \\ \dfrac{\partial x_f^{(j)}}{\partial u^{(j)}} & \dfrac{\partial x_f^{(j)}}{\partial \vartheta} \end{vmatrix} = \pm \boldsymbol{cos}\, \xi^{(j)} [\boldsymbol{sin}\, \vartheta\, (r_0 - p_t) + \boldsymbol{cos}\, \vartheta\, (u^{(j)} \boldsymbol{sin}\, \xi^{(j)} - p_t \vartheta)], \tag{22}$$

$$n_{f,z_f}^{(j)} = \begin{vmatrix} \dfrac{\partial x_f^{(j)}}{\partial u^{(j)}} & \dfrac{\partial x_f^{(j)}}{\partial \vartheta} \\ \dfrac{\partial y_f^{(j)}}{\partial u^{(j)}} & \dfrac{\partial y_f^{(j)}}{\partial \vartheta} \end{vmatrix} = \boldsymbol{sin}\, \xi^{(j)} (u^{(j)} \boldsymbol{sin}\, \xi^{(j)} - p_t \vartheta).$$

The modulus of the vector $\overline{n}_f^{(j)}$ is:

$$n_f^{(j)} = \sqrt{\boldsymbol{cos}^2\, \xi^{(j)} (r_0 - p_t)^2 + (u^{(j)} \boldsymbol{sin}\, \xi^{(j)} - p_t \vartheta)^2}\,. \tag{23}$$

If in (22) and (23) $r_0 = p_t$ and $r_0 = 0$ are substituted, then the analytical presentation of the normal vector to the $\Sigma_f^{(j)}$ in the point $N^{(j)}$ is derived, when $\Sigma_f^{(j)}$ is a face involute helicoid and face Archimedean helicoid, respectively. It is obvious from (22) and (23) that the face convolute and the face involute helicoid are constructed only from ordinary points $N^{(j)}$, thus for them the following condition $\overline{n}_f^{(j)} \neq \overline{0}$ is always fulfilled. If the face helicoid is a face involute surface and for some its points the following conditions are satisfied:

$$u^{(j)} \boldsymbol{sin}\, \xi^{(j)} - p_t \vartheta = 0 \tag{24}$$

then $\overline{n}_f^{(j)} = \overline{0}$ in these points from $\Sigma_f^{(j)}$.

It is easy to specify that this condition is feasible for all the common points of the generatrix $L^{(j)}$ and for the basic cylinder C.

4. Conclusion

The present study shows that the face linear helicoids are synthesized, when the common methodology for geometrical synthesis of conic linear helicoids is observed. This is the reason that the flat gear with face helical surfaces is called flat worm or face worm, and the analogical tool – flat (face) hob.

The illustrated and the described method for the surfaces generation shows the possibility for their feasible manufacturing using the lathe tool when the thread-milling machine is applied.

The derived equations for the face linear helicoids as well as the analytical descriptions of their crossed and axial sections are very important geometrical characteristics of the active tooth surfaces of the flat worms and the flat hobs, respectively. They treat the technological synthesis of the gears with application of this helical surfaces type (helical surfaces, which have working or instrumental meshing). Also the obtained analytical dependencies could be used for the organization of the measuring and control procedures of the surfaces parameters in the process of their manufacturing.

The conclusions for the appearance of the surfaces cutting points are also important for their technological synthesis.

4. Acknowledgment

This is a paper under the project No TN 820/98 supported by Ministry of Education and Science, Bulgaria

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