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CONSTRAINED IDENTIFICATION OF CONSTANTS IN POLYNOMIAL MODELS OF HYPERELASTIC MATERIALS

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Summary: *Constant identification is an important step in the structural analysis of rubber components. Performing the constant identification only from stress strain data measured in one or two simple deformation modes often gives physically inadmissible results. In this paper a constrained constant identification of hyperelastic materials is presented and verified.*

1. Introduction

Mathematical models in form of polynomial functions of principal stretches or invariants of principal stretches are widely used in simulation of hyperelastic materials. Constants of polynomials are determined from measured stress-strain data in simple deformation modes by means of least squares method. Higher order polynomials better approximate measured data than lower order polynomials, but outside the measured strain interval, the higher order polynomials often lead to physically inadmissible models.

Strictly spoken, a mathematical model should not be used outside the interval of the independent variable for which the constant identification was carried out. But in the structural analysis of rubber components such an assumption is difficult to fulfill, because in an area the calculated strain often exceeds the highest value reached on the specimens used for the material model constant identification.

In this paper, an approach diminishing the space of material model constants is presented. The presented approach uses a system of constraints. The system of constraints assures, that in an interval of strain larger than the interval of measured strain, a calculated mathematical model gives physically admissible results. The presented approach of the material model constant identification is verified in MATLAB. In this paper a constrained approach is described and the results of constrained and unconstrained constant identifications are graphically depicted and compared.

2. Constitutive models of rubber

Two different approaches are used in characterization of rubber elasticity. The first one called statistical theory derives rubber elastic behavior from idealized molecular structures of

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vulcanized rubber; the other one called phenomenological theory, treats rubber elasticity from the point of view of continuum mechanics. In this paper, we will focus on the characterization of rubber elasticity by means of the phenomenological theory.

According to Rivlin rubber elastic behavior may be described by strain energy function in form of infinite power series of strain invariants I_1 and I_2. For incompressible materials the strain energy function is expressed by the following formula

$$W(I_1, I_2) = \sum_{p,q=0}^{\infty} c_{pq} \cdot (I_1 - 3)^p \cdot (I_2 - 3)^q, \quad (1)$$

where c_{pq} are constants, I_1 and I_2 are invariants of the principal stretch λ, $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$, $I_2 = \lambda_1^2 \cdot \lambda_2^2 + \lambda_3^2 \cdot \lambda_3^2 + \lambda_1^2 \cdot \lambda_3^2$. The principal stretches $\lambda_1, \lambda_2, \lambda_3$ are the principal invariants of the right Cauchy-Green deformation tensor.

Ogden proposed another well-known strain energy function. Ogden's function for incompressible material is given by infinite series

$$W(\lambda_1, \lambda_2, \lambda_3) = \sum_{p,q=0}^{\infty} a_{pq} \cdot \{[\lambda_1{}^p \cdot (\lambda_2{}^q + \lambda_3{}^q) + \lambda_2{}^p \cdot (\lambda_3{}^q + \lambda_1{}^q) + \lambda_3{}^p \cdot (\lambda_1{}^q + \lambda_2{}^q)] - 6\}, \quad (2)$$

where a_{pq} are constants and $\lambda_1, \lambda_2, \lambda_3$ are principal invariants of the right Cauchy-Green deformation tensor.

In structural analysis of rubber components, models with finite number of terms are widely used. From the equation (1) we can derive the following strain energy functions

$$W = c_{10} \cdot (I_1 - 3), \quad (3)$$

$$W = c_{10} \cdot (I_1 - 3) + c_{01} \cdot (I_2 - 3), \quad (4)$$

$$W = c_{10} \cdot (I_1 - 3) + c_{01} \cdot (I_2 - 3) + c_{11} \cdot (I_1 - 3) \cdot (I_2 - 3), \quad (5)$$

$$W = c_{10} \cdot (I_1 - 3) + c_{01} \cdot (I_2 - 3) + c_{20} \cdot (I_1 - 3)^2, \quad (6)$$

$$W = c_{10} \cdot (I_1 - 3) + c_{01} \cdot (I_2 - 3) + c_{11} \cdot (I_1 - 3) \cdot (I_2 - 3) + c_{20} \cdot (I_1 - 3)^2, \quad (7)$$

$$W = c_{10} \cdot (I_1 - 3) + c_{01} \cdot (I_2 - 3) + c_{11} \cdot (I_1 - 3) \cdot (I_2 - 3) + c_{20} \cdot (I_1 - 3)^2 + c_{30} \cdot (I_1 - 3)^3, \quad (8)$$

$$W = c_{10} \cdot (I_1 - 3) + c_{20} \cdot (I_1 - 3)^2 + c_{30} \cdot (I_1 - 3)^3. \quad (9)$$

The strain energy function given by formula (3) is called the Neo-Hookesn model, the expression (4) is known as the Mooney-Rivlin model and formula (9) is denoted as the Yeoh model.

3. Constant identification

Determination of constants of a material model is carried out from the measured stress-strain data in simple deformation modes such as uniaxial tension, uniaxial compression, simple shear, pure shear, biaxial tension by means of least squares method. Constants of a material model are obtained by minimization function

$$res(c_{ij}) = \sum \left(\sigma_1(c_{ij}) - \hat{\sigma}_1\right)^2, \tag{10}$$

where c_{ij} are sought-after material model constants, σ_1 is the engineering stress determined from a strain energy function, $\hat{\sigma}_1$ is the engineering stress measured on testing specimens.

The lower order strain energy functions are capable of reasonable accuracy in the fitting of the measured data in the range of engineering strain $\varepsilon \in \langle -0.2; +0.2 \rangle$, see in Fig. 1. On the other hand, the higher order polynomials can fit the measured data with high accuracy as it is shown in Fig. 2, but their prediction of strain energy function outside the range of the measured data sometimes leads to the decrease of stress upon straining. The fitting of data of one deformation mode does not necessarily predict rubber behavior in another deformation mode (James, 1975). In Fig. 3 and 4 we can see the examples of inadequate curve fits performed on data from uniaxial compression and tension. At least two physical reasons necessarily lead to different stress-strain data obtained from the measurement of different deformation modes. The first cause of dissimilar rubber behavior in different deformation modes is rubber anisotropy under straining. The second reason of dissimilarity between the measured stress-strain data in different deformation modes comes from different temperature history in the course of vulcanization of specimens.

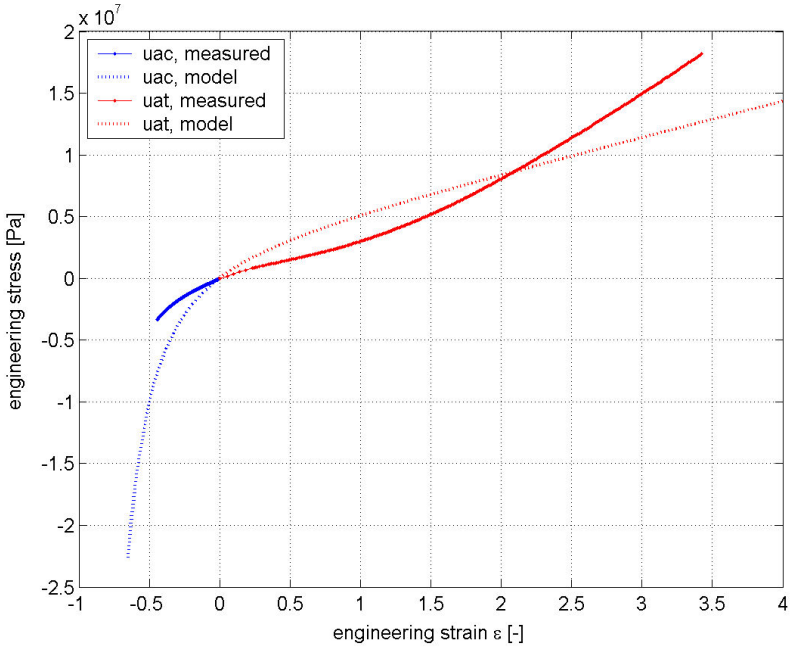

Fig. 1 Unconstrained fit of the measured stress-strain data in uniaxial tension and compression by the lower order polynomial function given by equation (3)

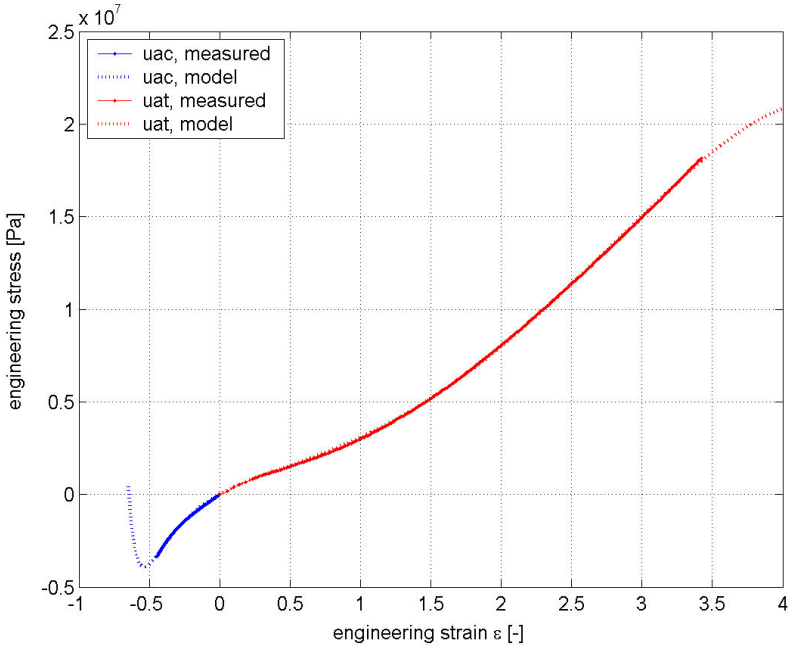

Fig. 2 Unconstrained fit of the measured stress-strain data in uniaxial tension and compression by the higher order polynomial function given by equation (8)

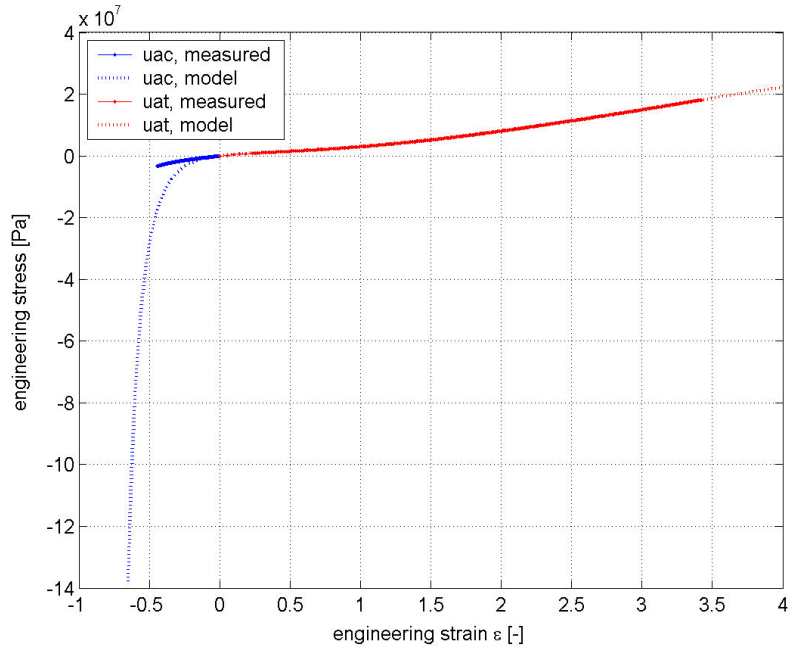

Fig. 3 Unconstrained fit of the measured stress-strain data in uniaxial tension by the higher order polynomial function given by equation (8)

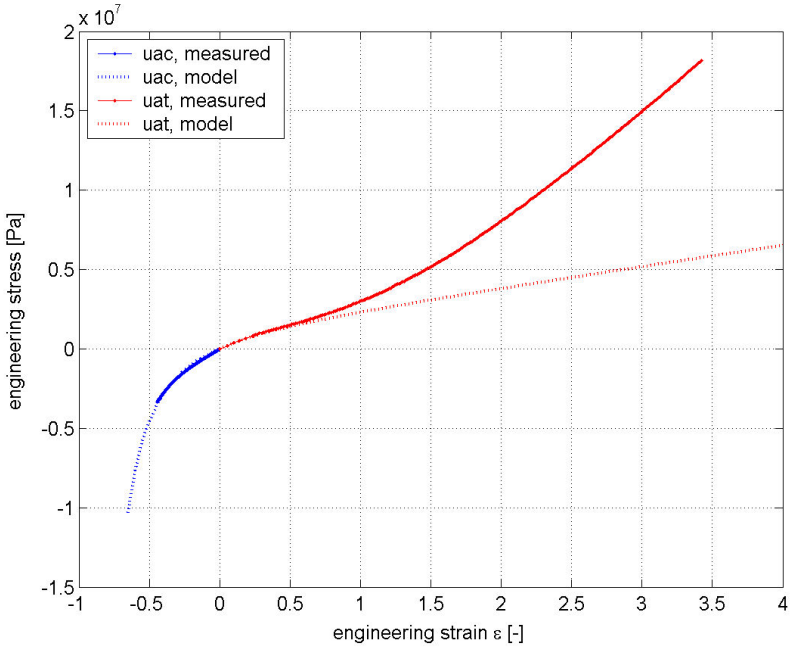

Fig. 4 Unconstrained fit of the measured stress-strain data in uniaxial compression by the lower order polynomial function given by equation (3)

To overcome some of the above mentioned issues, James (1975), ANSYS (2003) and other authors recommend performing of the constitutive model constant identification from more than one deformation mode. MSC Software recommends keeping the constants of a rubber model positive. Yeoh (1997) found that some of the Ogden model constants could be held constant for a known rubber material. Twizell (1983) used the Levenberg-Marqurdt non-linear least squares optimization algorithm for the improvement of fitting accuracy. Yeoh (1990) proposed a strain energy function to predict filled rubber behavior with a reasonable accuracy.

Trelour pointed out, that the phenomenological theory of rubber elasticity does not have any connection to rubber molecular structure. The phenomenological theory is merely a mathematical framework thus there are no direct restrictions on the values of constitutive model constants following from theory (Treloar, 1949, 1974).

4. A constrained constant identification procedure

In industrial applications of structural analysis of rubber components we require high accuracy of the material model and proper behavior in the range of strain we achieve during computation. We can increase the reliability of a material model by including the measured data from more than one deformation mode and loading testing specimens to high deformation. Nevertheless, an increasing number of the measured deformation modes increases the manufacturing costs and for some deformation modes measurement requires an expensive equipment and a time consuming and costly procedure. Moreover for all homogeneous deformation modes, the actual design of testing specimens and the boundary conditions following from the testing fixtures limit the uniformity of deformation modes. For a trouble-free structural analysis we need a procedure that is capable of determination of filled rubber material model from the measured data in uniaxial tension or uniaxial tension and

compression, which is stable in the prescribed range of strain that is larger than the range measured.

Keeping in mind that constants of a phenomenological model do not have a relation to the molecular structure of rubber we could think of the constant identification as a purely mathematical procedure.

From the point of view of mathematical procedure we can formulate the constant identification as a constrained minimization of the objective function

$$\underset{c_{ij} \in < clb,\, cup >}{\text{minimize}} \quad res\left(c_{ij}\right), \tag{11}$$

subject to the Drucker criterion.

$$d\sigma_{ij} : d\varepsilon_{ij} > 0\,, \qquad \varepsilon \in < \varepsilon lb, \varepsilon ub > \tag{12}$$

where $res(c_{ij})$ is the sum calculated according to equation (10) for all measured deformation modes, c_{ij} are sought-after material model constants, clb and cub are the lower and upper bounds of the constants c_{ij}, $\sigma_{ij}, \varepsilon_{ij}$ is the conjugate stress and strain couple and εlb and εub are the lower and upper bounds of strain. The Drucker criterion is imposed on specified simple deformation modes.

5. Verification of the constrained constant identification

The constrained constant identification formulated in (11) and (12) for hyperelastic material functions (3) to (9) was verified in MATLAB, utilizing the function *fmincon* from MATLAB Optimization toolbox.

The results of the curve fitting to the measured stress-strain data on the filled natural rubber of hardness $Hd = 70\ Shore\ A$ in uniaxial tension and compression are presented in Fig. 5 and 6. The calculated constants of the material model given by (7) have values $c_{10} = 845\,260$, $c_{01} = \text{-}19\,7780$, $c_{11} = 48\,288$ and $c_{20} = 25\,469$. The curves in Fig. 5 show a good agreement between the stresses calculated by the mathematical model and the measured stress-strain data in uniaxial tension and compression. The curves depicted in Fig. 6 show that calculated material model satisfies the Drucker criterion (12) in the prescribed range of engineering strain $\varepsilon =< -0.65, 4 >$.

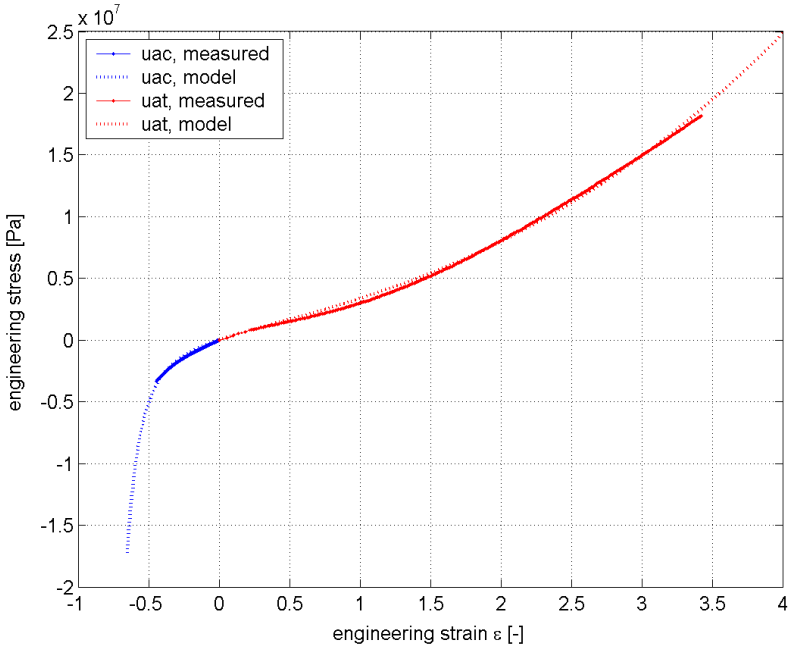

Fig. 5 Constrained fit of the measured stress-strain data in uniaxial tension and compression by the higher order polynomial function given by equation (7)

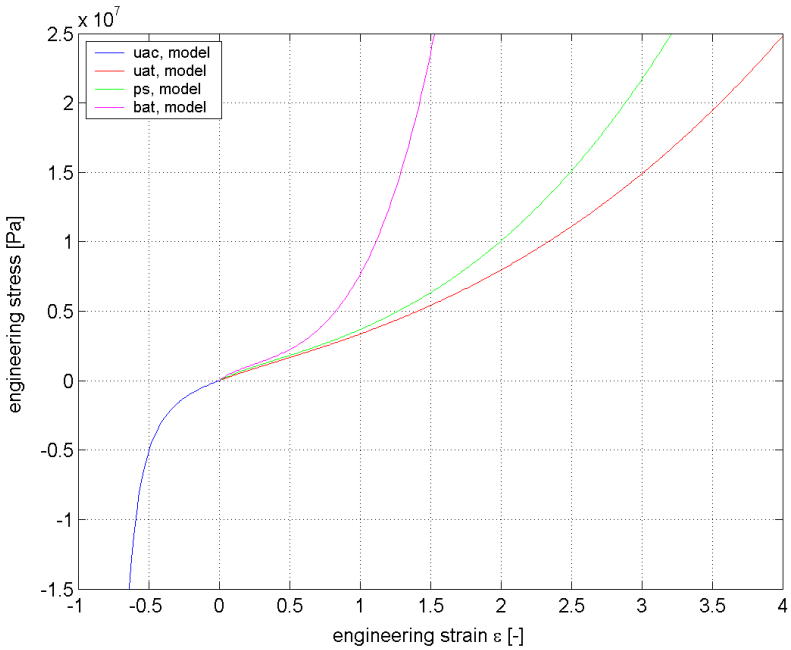

Fig. 6 Stress-strain curves calculated by the mathematical model from Fig. 5 corresponding to uniaxial tension (uat), uniaxial compression (uac), pure shear (ps) and biaxial tension (bat).

6. Conclusion

In this paper, the same set of stress-strain data measured on filled rubber of hardness $Hd = 70\ Shore\ A$ in uniaxial tension and compression is used for the comparison of the constrained and unconstrained constant identification.

The curves depicted in Fig. 1 and 2 show either the inaccurate fit of the lower order polynomials to the measured data, see Fig. 1 or the faulty behavior of the higher order strain energy functions outside the range of the measured strain, see Fig. 2.

The curves drawn in Fig. 5 and 6 shows that the properly constrained constant identification applied to a higher order polynomial model leads to a very good fit to the measured data and the admissible strain energy function behavior outside the range of the measured strain.

7. Literature

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