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CONTACT OF CYLINDRICAL SURFACES WITH ELASTIC INCOMPRESSIBLE COATINGS

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Summary: *The paper deals with a frictionless contact problem of two parallel congruent rigid cylinders, one encased in the other. The cylindrical surfaces are coated with thin elastic, transversely isotropic and incompressible layers. A simplifying approximation for the displacement in the coating enables to formulate the problem using the stress and strain averaged through the coating thickness (for the method see Matthewson M. J., J. Mech. Phys. Solids 29(1981), 89-113). Analytical results are obtained for the contact width and contact pressure distribution. The results are applied to the human ankle joint and generalized for articular cartilage with depth-dependent properties.*

1. Introduction

Coated bodies are widely used in building materials, tribosystems, microelectronic industry, etc. Rubber linings have important applications in the design of polygraphic, textile and paper machinery. Compliant coatings are frequently applied to suppress noise, quell vibrations, delay laminar-to-turbulent transition and reduce skin-friction drag on naval vessels and airplanes. Elastomeric materials have found a wide use for protecting components from impact damage. Incompressible elastomers are successful partly because of their large strain to fracture, resulting in ability to absorb large amounts of the impact energy in elastic deformation. These materials are inherently weak and are reinforced by adhesion to the substrate. This reduces the strains in the coating, which would otherwise be large. Good adhesion is essential for their optimal performance.

The mathematical problem of the contact of a rigid punch pressed against a compliant layer bonded to a rigid half-space is of great practical interest. Cylindrical, spherical and flat-ended indenters were discussed by the Russian writers Lebedev, Ufliand, Aleksandrov and Vorovich in the early sixties. Meijers (1967) obtained an asymptotic solution to the problem of a rigid cylinder pressed on an isotropic elastic layer of any thickness and any Poisson's ratio.

Matthewson (1981) published an interesting solution for the indentation of a thin elastic coating, bonded to a rigid half-space, by a rigid axially symmetric punch. The essence of the

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method lies in the fact that a simple polynomial approximation across the layer thickness for the displacement vector is assumed and an averaging technique through the layer thickness is applied. Due to the approximation of the displacement vector, most equations and conditions can be satisfied only averaged through the coating thickness. Matthewson obtained simple analytic formulas for the contact pressure distribution and contact width. Especially, in the case of a volumetrically incompressible compliant coating, these analytic results agreed surprisingly well, when compared to the numerical results of other authors. The coating was considered plain (i. e. not curved), homogeneous and isotropic. The indenter had no coating.

In order to simplify problems, the curved coatings are often considered plane and only their relative (effective) curvature is taken into account. However, it is apparent that both the contact width and maximum contact stress can be considerably higher for curved coatings and the same total load, compared with the plane case. This is the case if the contact width and the curvature radius are of the same order of magnitude. In fact, the surface traction in a frictionless contact is normal to the coating surface. In an axially symmetric curved case, only one component of the traction (that parallel with the symmetry axis) contributes to the total load. The other part (that perpendicular to this axis) cancels out.

Using the Matthewson averaging method, the contact of two parallel cylindrical rigid and highly congruent bodies, one encased in the other and pressed against each other, both coated with soft incompressible elastic transversely isotropic coatings of a constant thickness, is analysed in the paper. The effect of coating curvature is taken into account using cylindrical co-ordinates. With a high surface congruency, the difference of the curvature radii of the mating surfaces is small compared with their curvature radii. Moreover, the total compressive load is assumed as high that the contact width is of the same order of magnitude as the curvature radii, while the coating thickness should be much smaller than the contact width.

2. Contact of encased coated cylinders

Two parallel circular rigid cylinders, a solid cylinder of radius R_{S} encased in a hollow one of radius $R_{\mathrm{H}} > R_{\mathrm{S}}$, with $R_{\mathrm{H}} - R_{\mathrm{S}} = \Delta R \ll R_{\mathrm{S}}$ (high congruency), are coated with thin elastic layers of a constant thickness $h \ll R_{\mathrm{S}}$ (Fig. 1). In what follows, subscript C (C for coating) substitutes subscript S (S for solid) and H (H for hollow), used with the quantities referred to the coated solid and hollow cylinder, respectively. Two parallel cylindrical coordinate systems $r_{\mathrm{C}}, \phi_{\mathrm{C}}, z$, with the z-direction along the cylinder axes and with the plane of symmetry $\phi_{\mathrm{S}} = \phi_{\mathrm{H}} = 0$, are introduced. For the coatings, $R_{\mathrm{H}} - h \le r_{\mathrm{H}} \le R_{\mathrm{H}}$ and $R_{\mathrm{S}} \le r_{\mathrm{S}} \le R_{\mathrm{S}} + h$. Due to symmetry, it is sufficient to consider only one half of the cylinders and choose ϕ_{C} in the interval $0 \le \phi_{\mathrm{C}} < \pi$. The cylinders are compressed by a vertical force W (per unit cylinder length), acting in the plane of symmetry.

Materials of the coatings are incompressible and transversely isotropic with the isotropy axis normal to the coating surface. The physical components of the stress and strain tensors $\sigma_{\mathrm{C}ik}$ and $\varepsilon_{\mathrm{C}ik}\,(i,k = r,\phi,z)$, referred to unit base vectors, are used. The form of the constitutive equations (the coating is incompressible, linear elastic, transversely isotropic and homogeneous) in the cylindrical co-ordinates $r_\mathrm{C}, \phi_\mathrm{C}, z$ is given by the equations (Pipkin, 1976)

$$\begin{bmatrix}\sigma_{\mathrm{C}rr}\\ \sigma_{\mathrm{C}\phi\phi}\\ \sigma_{\mathrm{C}zz}\end{bmatrix} = -\begin{bmatrix}p_\mathrm{C}\\ p_\mathrm{C}\\ p_\mathrm{C}\end{bmatrix} + \begin{bmatrix}C_r & 0 & 0\\ 0 & C_\phi & 0\\ 0 & 0 & C_\phi\end{bmatrix}\cdot\begin{bmatrix}\varepsilon_{\mathrm{C}rr}\\ \varepsilon_{\mathrm{C}\phi\phi}\\ \varepsilon_{\mathrm{C}zz}\end{bmatrix}, \qquad \begin{bmatrix}\sigma_{\mathrm{C}r\phi}\\ \sigma_{\mathrm{C}rz}\\ \sigma_{\mathrm{C}z\phi}\end{bmatrix} = 2\begin{bmatrix}\mu_{r\phi}\\ \mu_{r\phi}\\ \mu_{z\phi}\end{bmatrix}\cdot\begin{bmatrix}\varepsilon_{\mathrm{C}r\phi}\\ \varepsilon_{\mathrm{C}rz}\\ \varepsilon_{\mathrm{C}z\phi}\end{bmatrix}. \qquad (1)$$

p_C is the hydrostatic pressure. There are three independent material parameters: the two Young moduli E_r, E_ϕ and the shear modulus $\mu_{r\phi} = \mu$. Due to incompressibility, the Poisson ratios $\nu_{r\phi}, \nu_{z\phi}$ take the values $\nu_{r\phi} = 1/2, \nu_{z\phi} = 1 - E_\phi / 2E_r$. Transverse isotropy yields $\mu_{z\phi} = E_\phi / 2(1 + \nu_{z\phi})$. The compliances C_r, C_ϕ in (1) take the forms

$$C_\phi = \frac{2E_r E_\phi}{4E_r - E_\phi}, \qquad C_r = \frac{2E_r(2E_r - E_\phi)}{4E_r - E_\phi}. \qquad (2)$$

The elastic coating is assumed to undergo small plane deformation in the $r_\mathrm{C}\phi_\mathrm{C}$-plane and touch along an unknown cylindrical surface *F*. ϕ_S and ϕ_H of a fixed point at *F* differ little for thin coatings and need not be distinguished. The same goes for R_S and R_H (except for the difference $R_\mathrm{H} - R_\mathrm{C}$). To simplify, if it is not the coating but the magnitude that matters, subscript C is often left with $\phi_\mathrm{C}, R_\mathrm{C}$, writing ϕ, R instead. Let a bar over a quantity refers to that at the edge of *F*. For example, $\bar{\phi}$ denotes ϕ of the point at the contact edge. The contact half-width is $\bar{s} = R\bar{\phi}$. In the paper, *W* is assumed high enough that $h/\bar{s} \ll 1$, $\bar{s}$ and R being of the same order of magnitude, i. e. $\bar{s} \sim R$. Also, it should hold that $\bar{\phi} < \pi$. C_r, C_ϕ may be one order higher than μ. The above assumptions (i. e. $h \ll R, h \ll \bar{s}, \bar{s} \sim R, \Delta R \ll R$) are necessary for the use of the applied averaging method. They also simplify the results as some terms in the equations are small and can be neglected. As $\bar{s} \sim R$, the coating curvature effect is taken into account by using the cylindrical co-ordinates. For $h \ll \bar{s} \ll R$ it would be sufficient to consider the coatings plane.

The non-zero physical components of the displacement vector in the C-coating, referred to the location of the rigid cylinder after deformation, are approximated by finite power series in r_C in the form

$$u_{j\mathrm{C}\phi}(r'_\mathrm{C}, \phi) = \mp\alpha_{j\mathrm{C}} r'_\mathrm{C} + \beta_{j\mathrm{C}} r'^2_\mathrm{C}, \qquad u_{j\mathrm{C}r}(r'_\mathrm{C}, \phi) = \gamma_{j\mathrm{C}} r'_\mathrm{C}, \qquad (3)$$

$$r'_\mathrm{C} = r_\mathrm{C} - R_\mathrm{C}, \quad r_\mathrm{S} \in \langle R_\mathrm{S}, R_\mathrm{S} + h\rangle, \qquad r_\mathrm{H} \in \langle R_\mathrm{H} - h, R_\mathrm{H}\rangle.$$

Here and in what follows, the lower sign is taken with $\mathrm{C} = \mathrm{S}$ and the upper with $\mathrm{C} = \mathrm{H}$. $\alpha_{j\mathrm{C}}, \beta_{j\mathrm{C}}$ and $\gamma_{j\mathrm{C}}$ are unknown smooth functions of ϕ. For $j = 1$, they are defined inside (with $0 \le \phi \le \bar{\phi}$) and for $j = 2$ outside the contact region (with $\pi \ge \phi \ge \bar{\phi}$). Due to the symmetry it holds that $\alpha_{1\mathrm{C}}(0) = \beta_{1\mathrm{C}}(0) = \gamma_{1\mathrm{C},\phi}(0) = 0$ and $\alpha_{2\mathrm{C}}(\pi) = \beta_{2\mathrm{C}}(\pi) = \gamma_{2\mathrm{C},\phi}(\pi) = 0$. Note that $u_{j\mathrm{C}\phi}, u_{j\mathrm{C}r}$ in (3) meet the bonding conditions $u_{j\mathrm{C}r} = u_{j\mathrm{C}\phi} = 0$ at $r_\mathrm{C} = R_\mathrm{C}$. The non-zero physical components of the infinitesimal strain tensor for plane strain are

$$\varepsilon_{jCrr} = u_{jCr,r}, \quad \varepsilon_{jC\phi\phi} = u_{jCr}/r_C + u_{jC\phi,\phi}/r_C, \tag{4}$$

$$\varepsilon_{jC\phi r} = (u_{jCr,\phi}/r_C - u_{jC\phi}/r_C + u_{jC\phi,r})/2.$$

A comma followed by r or ϕ denotes a partial derivative with respect to r_C or ϕ_C. The equilibrium equation in the ϕ_C-direction in plane strain expressed in the physical components of the stress tensor is

$$\sigma_{jC\phi r,r} + \sigma_{jC\phi\phi,\phi}/r_C + 2\sigma_{jC\phi r}/r_C = 0. \tag{5}$$

Due to the simple approximation (3), most equations and boundary conditions can be satisfied only averaged through the coating thickness h, while some boundary conditions are met point-wise. The average $\underline{f}_C(\phi_C)$ of a function $f_C(r_C, \phi_C)$ in the element $h \times R_C \mathrm{d}\phi_C$ extending through h is denoted by $\underline{f}_C = \int f_C r_C \mathrm{d}r_C / hR_C$. The integral is taken from R_S to $R_S + h$ and from $R_H - h$ to R_H for the S- and H-coating, respectively. Note that the averaging procedure through h makes the equilibrium equation in the r_C-direction irrelevant. Introduce (3) into (4) and integrate over h to obtain (after leaving small terms) the averaged strains for any ϕ_C as

$$\underline{\varepsilon}_{jCrr} = \gamma_{jC}, \qquad \underline{\varepsilon}_{jC\phi\phi} = \frac{h}{R}\left(\frac{1}{2}\alpha_{jC,\phi} + \frac{h}{3}\beta_{jC,\phi}\right), \tag{6}$$

$$\underline{\varepsilon}_{jCr\phi} = \mp\frac{1}{2}\left(\alpha_{jC} + h\beta_{jC} + \frac{h}{2R}\gamma_{jC,\phi}\right).$$

The following continuity conditions are introduced at point $\bar{\phi}$:

$$\bar{\underline{\varepsilon}}_{1Crr} = \bar{\underline{\varepsilon}}_{2Crr}, \quad \bar{\underline{\varepsilon}}_{1C\phi\phi} = \bar{\underline{\varepsilon}}_{2C\phi\phi}, \quad \bar{\underline{\varepsilon}}_{1Cr\phi} = \bar{\underline{\varepsilon}}_{2Cr\phi}, \quad \bar{\underline{p}}_{1C} = \bar{\underline{p}}_{2C}. \tag{7}$$

These conditions also guarantee the continuity of all averaged components of the stress tensor at $\bar{\phi}$. (6) yield the average incompressibility condition, $\underline{\varepsilon}_{jCrr} + \underline{\varepsilon}_{jC\phi\phi} = 0$, in the form

$$h\alpha_{jC,\phi}/2 + h^2\beta_{jC,\phi}/3 + R\gamma_{jC} = 0. \tag{8}$$

The contact is assumed frictionless and the condition $\sigma_{jC\phi r} = 0$ at $r_S = R_S + h$ with $C = S$ and at $r_H = R_H - h$ with $C = H$ is valid for any ϕ, or,

$$\alpha_{jC} + 2h\beta_{jC} + h\gamma_{jC,\phi}/R = 0. \tag{9}$$

Use (8-9) to get α_{jC}, β_{jC} as functions of γ_{jC} in the form

$$\alpha_{jC} = -\frac{3R}{h}\int_0^{\phi}\gamma_{jC}\mathrm{d}\phi + \frac{h}{2R}\gamma_{jC,\phi}, \qquad 2h\beta_{jC} = \frac{3R}{h}\int_0^{\phi}\gamma_{jC}\mathrm{d}\phi - \frac{3h}{2R}\gamma_{jC,\phi}. \tag{10}$$

Insert the equations $\sigma_{jC\phi\phi} = -p_{jC} + C_\phi\varepsilon_{jC\phi\phi}$, $\sigma_{jC\phi r} = 2\mu\,\varepsilon_{jC\phi r}$ with the use of (4) into (5) and apply the incompressibility condition to obtain the equilibrium equation (5) as

$$r_C\,p_{jC,\phi} = -C_\phi r_C u_{jCr,r\phi} + \mu(r_C^2 u_{jC\phi,rr} + r_C u_{jC\phi,r} - u_{jC\phi} - u_{jC\phi,\phi\phi}).$$

Use the expansion (3) for $u_{jC\phi}$, u_{jCr}, integrate over h, neglect small terms and obtain

$$\underline{p}_{jC,\phi} = (\mu - C_\phi)\gamma_{jC,\phi} + 2\mu R \beta_{jC} . \tag{11}$$

(3-6), (8-11) are valid for any ϕ. Now, deduce the equations valid either inside or outside the contact.

Outside the contact (with $j = 2$), the condition $\underline{\sigma}_{2Crr} = -\underline{p}_{2C} + C_r \underline{\varepsilon}_{2Crr} = 0$ is introduced, yielding with the help of $(6)_1$

$$\underline{p}_{2C} = C_r \gamma_{2C} . \tag{12}$$

Inside the contact, cut the deformed coatings off the rigid substrate. Roll out the cylinder with radius R_S (it could be also R_H) in the way that it becomes plane. The coatings remain joined and their deformed thickness unchanged. Radius R_H (alternatively R_S) changes into radius R_{eq}, with $1/R_{eq} = 1/R_S - 1/R_H$. Approximate quadratically in $s = R\phi$ the total deformed thickness $\rho(\phi)$ (the sum of both deformed thicknesses) for $\phi \le \bar{\phi}$ in the rolled-out state as

$$\rho(\phi) = 2h + u_{1Sr}(h,\phi) - u_{1Hr}(-h,\phi) \approx \tilde{\rho}_0 + \frac{s^2}{2R_{eq}} = \tilde{\rho}_0 + \frac{\Delta R}{2}\phi^2 . \tag{13}$$

$(3)_2$ yields $u_{1Sr}(h,\phi) = h\gamma_{1S}$, $u_{1Hr}(-h,\phi) = -h\gamma_{1H}$ and (13) can be written as

$$h\left(\gamma_{1S} + \gamma_{1H}\right) = \rho_0 + \frac{\Delta R}{2}\phi^2 , \tag{14}$$

where $\rho_0 = \tilde{\rho}_0 - h_S - h_H$ is an unknown constant. Having obtained (14) in the rolled-out state, come back to the curved state. As the extension in the normal direction is not changed, (14) remains also valid for the curved coatings. Inside the contact, beside the geometric condition (14), introduce the condition for the averaged normal stress at any ϕ ($|\phi| \le \bar{\phi}$):

$$\underline{\sigma}_{1Hrr} = \underline{\sigma}_{1Srr} . \tag{15}$$

Differentiating (15) and using (1), (6), (10-11) and (14) give $\alpha_{1C}, \beta_{1C}, \gamma_{1C}$ the same for both coatings in the end. A detailed deduction is omitted here.

Outside the contact, eliminate $\underline{p}_{2C,\phi}$ from (11-12) to have

$$(C - \mu)\gamma_{2C,\phi} = 2\mu R \beta_{2C} , \qquad C = C_r + C_\phi . \tag{16}$$

At first, insert here for $\gamma_{2C,\phi}$ from (9) to have

$$\beta_{2C} = -\frac{C - \mu}{2hC}\alpha_{2C} . \tag{17}$$

Then, insert into (16) for $\gamma_{2C,\phi}$ from (8) differentiated with respect to ϕ and use (17) to obtain a differential equation for α_{2C} as

$$\alpha_{2\mathrm{C},\phi\phi} - m^2\,\alpha_{2\mathrm{C}} = 0\,, \qquad m = \frac{R}{h}\left(\frac{6\mu}{2C+\mu}\right)^{1/2}$$

whose solution (meeting the condition $\alpha_{2\mathrm{C}}(\pi) = 0$) is

$$\alpha_{2\mathrm{C}} = \alpha_{2\mathrm{C}0}\left\{\exp\left(-m\phi\right) - \exp\left[-m\left(2\pi-\phi\right)\right]\right\}. \tag{18}$$

$\alpha_{2\mathrm{C}0}$ are unknown constants. (8), (17) and (18) give

$$\gamma_{2\mathrm{C}} = \frac{1}{e}\alpha_{2\mathrm{C}}\,, \quad e(\phi) = n(\phi)\left(\frac{6C^2}{\mu(2C+\mu)}\right)^{1/2}, \tag{19}$$

$$n(\phi) = \frac{\exp\left[2m(\pi-\phi)\right]-1}{\exp\left[2m(\pi-\phi)\right]+1}.$$

$(7)_{1,3}$ yield with the help of $(6)_{1,3}$ and (9)

$$\bar{\gamma}_{1\mathrm{C}} = \bar{\gamma}_{2\mathrm{C}}\,, \quad \bar{\alpha}_{1\mathrm{C}} = \bar{\alpha}_{2\mathrm{C}}\,. \tag{20}$$

It is clear now that also $\alpha_{2\mathrm{C}}, \beta_{2\mathrm{C}}, \gamma_{2\mathrm{C}}$ are the same for both coatings. Thus, in what follows, for the same coatings, subscript C is left with $\alpha_{j\mathrm{C}}, \beta_{j\mathrm{C}}, \gamma_{j\mathrm{C}}$ and the other quantities. (20) and $(10)_1$ for $j = 1$ at $\bar{\phi}$, with the aid of (19), yield

$$\frac{h}{3R}\bar{e}\bar{\gamma}_1 - \frac{h^2}{6R^2}\bar{\gamma}_{1,\phi} + \int_{\phi}^{\bar{\phi}}\gamma_1 \mathrm{d}\phi = 0\,. \tag{21}$$

Constants α_{20}, ρ_0 now can be found using (20-21). ρ_0 takes the form

$$\rho_0 = -\frac{\bar{s}^2 L(\bar{s})}{6R_{\mathrm{eq}}}\,, \qquad L\left(\bar{s}\right) = 1 + \frac{4C}{\left[6\mu\left(2C+\mu\right)\right]^{1/2}}\frac{h}{\bar{s}}n\left(\bar{s}\right). \tag{22}$$

$\underline{p}_1(\phi)$ for any ϕ can be found by integrating (11) with respect to ϕ, except for the constant $\underline{p}_{10}$. Substitute $\underline{p}_1$ from the integrated equation (11) into the equation $\underline{\sigma}_{1rr} = -\underline{p}_1 + C_r\underline{\varepsilon}_{1rr}$ and for $\underline{p}_{10}$ use condition $(7)_4$ together with (12):

$$\underline{\sigma}_{1rr} = (C-\mu)(\gamma_1 - \bar{\gamma}_1) + 2R\mu\int_{\phi}^{\bar{\phi}}\beta_1 \mathrm{d}\phi\,. \tag{23}$$

(23), with the help of $(10)_2$ and (14), takes the form

$$\underline{\sigma}_{1rr} = \frac{\mu}{8hR_{\mathrm{eq}}}\left\{\left[\frac{\bar{s}^2}{h^2}L(\bar{s}) + \frac{2C+\mu}{\mu}\right]\left(s^2 - \bar{s}^2\right) - \frac{1}{2h^2}\left(s^4 - \bar{s}^4\right)\right\}. \tag{24}$$

It remains to find the contact half-width $\bar{s}$ that is obtained from the total vertical load condition

$$W = -2\int_0^{\bar{s}} \sigma_{1rr} \cos\frac{s}{R}\,\mathrm{d}s\,,$$

which together with (24) yields after integration

$$\frac{2R_{\mathrm{eq}}W}{\mu R^2}\left(\frac{h}{\bar{s}}\right)^3 - \left(\frac{R}{\bar{s}}\sin\frac{\bar{s}}{R} - \cos\frac{\bar{s}}{R}\right)\left[L(\bar{s}) + \frac{2C+\mu}{\mu}\frac{h^2}{\bar{s}^2}\right]$$

$$-\left(1 - 6\frac{R^2}{\bar{s}^2}\right)\cos\frac{\bar{s}}{R} - 3\frac{R}{\bar{s}}\left(2\frac{R^2}{\bar{s}^2} - 1\right)\sin\frac{\bar{s}}{R} = 0\,. \qquad (25)$$

The total normal load W_{n} becomes

$$W_{\mathrm{n}} = -2\int_0^{\bar{s}} \sigma_{1rr}\,\mathrm{d}s = \frac{\mu\bar{s}^5}{2h^3 R_{\mathrm{eq}}}\left\{-\frac{1}{5} + \frac{1}{3}\left[L(\bar{s}) + \frac{2C+\mu}{\mu}\frac{h^2}{\bar{s}^2}\right]\right\}. \qquad (26)$$

The term with $h^2/\bar{s}^2$ is kept in (25-26) as for anisotropic coatings it may be of order $h/\bar{s}$ (if $C \gg \mu$). Note that W_{n} in (26) depends on $R_{\mathrm{H}}, R_{\mathrm{S}}$ only through R_{eq}, and on $\bar{s}$. Take the limit $R_{\mathrm{C}} \to \infty$ with R_{eq} kept constant in (25). $\bar{\phi}$ becomes small for large R_{C}, but in the limit, $\bar{s} = R\bar{\phi}$ tends to a positive constant. For low $\bar{\phi}$, expand sines, cosines and the exponential function in $(19)_3$, (25) into the Taylor series. In the limit $R_{\mathrm{C}} \to \infty$ (R_{eq} fixed), $W \to W_{\mathrm{n}}$ and (25) tends to (26).

3. A model of the human ankle joint

As an illustration, consider a human ankle joint loaded by the body weight. This joint can be simply taken as cylindrical (Medley et al., 1983) as it enables rotation in the sagittal plane only. The ankle joint, where the tibia, fibula and talus join, can be modelled as parts of parallel infinite circular rigid cylinders in the inner contact, coated with a thin deformable layer – articular cartilage of constant thickness *h*. Due to a high content of water and very low permeability of the cartilage matrix, articular cartilage may be considered incompressible and under small deformation for physiological short-term loading. However, under long-term loading, such as in long standing, fluid flow through the matrix pores should be taken into account and a biphasic model for cartilage would be adequate (Mow et al., 1980). After tens of minutes of creep, cartilage becomes consolidated. The equilibrium elastic moduli of the cartilage matrix in tension and compression differ considerably (Donzelli et al., 1999). As the matrix compression is predominantly perpendicular and the extension parallel to the articular surface, cartilage can be considered transversely isotropic. Garcia et al. (1998) have shown that if the fluid transport in the mixture does not occur or is not yet apparent (for example, at the moment of a step-load application or some tens of seconds after), the biphasic material is equivalent to a single-phase incompressible material. For zero equilibrium Poisson ratios (Donzelli et al., 1999), by using the formulas deduced by Garcia et al. and (1-2), it can be shown that the moduli C_r, C_ϕ in (2) of the single-phase incompressible material equal, interestingly, the equilibrium Young moduli of the matrix.

In agreement with the matrix microstructure, the equilibrium moduli of the cartilage matrix are also depth-dependent. Due to the measurements by Schinagl et al. (1997) and Akizuki et al. (1986), $C_r(r'_C), C_\phi(r'_C)$ are taken in the form

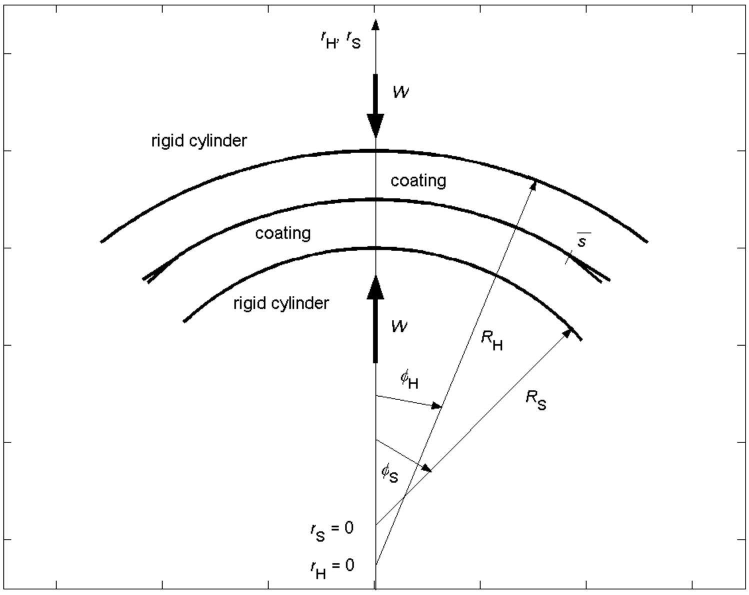

Fig. 1 Contact problem

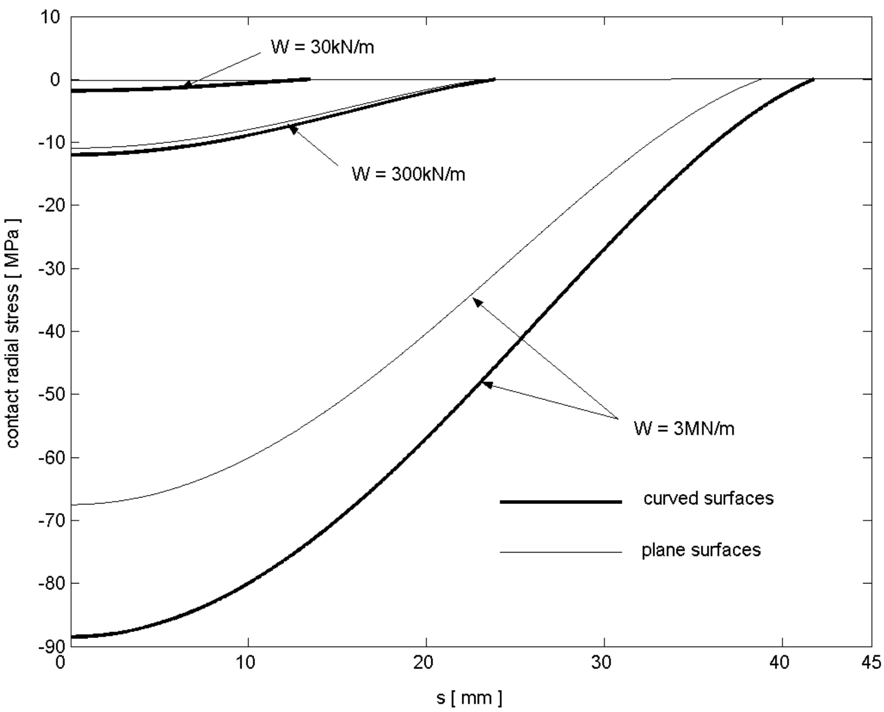

Fig. 2 Normal contact stress distribution vs. circumferential co-ordinate s

$$C_r(r'_{\mathrm{C}}) = C_{\mathrm{a}r} + (C_{\mathrm{b}r} - C_{\mathrm{a}r})\left(\frac{r'_{\mathrm{C}}}{h} \pm 1\right)^2, \quad C_\phi(r'_{\mathrm{C}}) = C_{\mathrm{b}\phi} + (C_{\mathrm{a}\phi} - C_{\mathrm{b}\phi})\frac{r'^2_{\mathrm{C}}}{h^2}. \tag{27}$$

The constants $C_{\mathrm{a}r}$, $C_{\mathrm{a}\phi}$ refer to the values at the articular surface, and $C_{\mathrm{b}r}$, $C_{\mathrm{b}\phi}$ to those at the cartilage-bone interface. As no measurement of variations of μ with the depth seems available, μ is assumed constant through the layer thickness.

Now, the whole procedure can be carried on also for inhomogeneous moduli (27). Caution must be taken whenever integration over the layer thickness is taken. As a result, all equations remain the same if only $\left(2C_{\mathrm{a}r} + C_{\mathrm{b}r}\right)/3$ and $\left(C_{\mathrm{a}\phi} + 2C_{\mathrm{b}\phi}\right)/3$ are written instead of C_r and C_ϕ, respectively. Thus, the constants C_r, C_ϕ in the equations must be replaced by the averages of $C_r(r'_{\mathrm{C}}),\ C_\phi(r'_{\mathrm{C}})$ through the layer thickness.

Take the following values for the parameters (Medley et al., 1983; Schinagl et al., 1997; Akizuki et al., 1986): $h = 1.5\mathrm{mm}$, $R_{\mathrm{H}} = 22.1\,\mathrm{mm}$, $R_{\mathrm{S}} = 20.8\,\mathrm{mm}$, $C_{\mathrm{a}r} = 0.2\mathrm{MPa}$, $C_{\mathrm{b}r} = 1.1\mathrm{MPa}$, $C_{\mathrm{a}\phi} = 9\mathrm{MPa}$, $C_{\mathrm{b}\phi} = 4.5\mathrm{MPa}$, $\mu = 0.4\,\mathrm{MPa}$, $W = 3\times10^4\,\mathrm{N/m}$. Here W has been estimated for the body weight of 800N, multiplied twice due to the muscle forces and calculated for one joint of the length of 28mm. R_{eq} becomes 0.35m, averaged moduli $C_r = 0.5\mathrm{MPa}$, $C_\phi = 6\mathrm{MPa}$. Fig. 2 shows the variation of the contact radial stress, denoted by $\underline{\sigma}_{1rr}$ in this paper, calculated for the above parameters, as well as for W by one and two orders higher (thick lines). The last two high values are not encountered with the ankle joint. They are used to see the effect of the surface curvature. The thin lines represent the stresses for plane layers and the same R_{eq} and axial load. The curves are calculated using (25-26), the last equation taken with $W_{\mathrm{n}} = W$. It is apparent that the effect of surface curvature is small in the case of the ankle joint in standing (the thick line practically merge with the thin one: the maximum value is 1.89MPa vs. 1.83MPa and the contact half-width 13.3mm vs. 13.2mm). One half of the contact angle, $\bar{\phi}$, becomes 36.6, 64.9 and 114.4 degrees for the above three values of W and curved surfaces. For the highest value of W (with $\bar{\phi} = 114.4$ degrees), the difference in the maximum contact stress for the curved and plane contacts is high (88.4MPa vs. 67.6MPa).

4. Conclusion

A contact of two coated parallel congruent rigid cylindrical surfaces has been tackled in the current paper. The coating is linearly elastic (transversely isotropic) and incompressible. Using an averaging procedure due to Matthewson (1981), analytical results have been obtained for the contact width and contact pressure distribution.

The coating curvature effect becomes apparent as the angle of the contact increases. For the half contact angle of $\pi/2$, the maximum contact pressure and contact width are considerably higher in the curved contact than in the plane contact and the same axial load. The results are also generalized for an inhomogeneous coating, i. e. if the moduli vary quadratically through the coating thickness.

An illustrative example of the human ankle joint loaded by the body weight indicates that the curvature effect is small in this case and the surface curvature need not be taken into account. The physiological load in standing is relatively low. In walking, running or jumping the maximum load is much higher. The maximum contact pressure would be also high, but not due to the curvature effect. In fact, the contact area is given by the tibial width (about 30mm) and the contact angle remains relatively small.

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6. References

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