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STRESS SINGULARITY ANALYSIS OF BODIES WITH A BI-MATERIAL NOTCH

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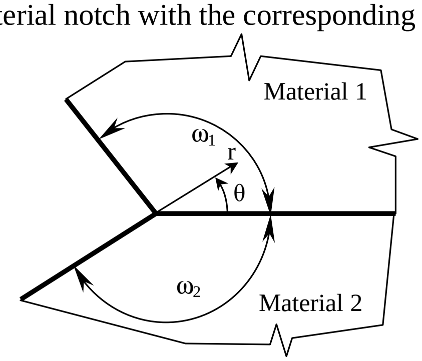
Summary: *The article deals with analysis of a stress state in the case of constructions with a bi-material notch. The eigenvalues* l *as well as the stress singularity exponents are determined and discussed for three specific geometries of a bi-material notch and for varying material combinations. The values* l *are presented in dependence on the composite parameters* a, b *in so called Dundurs' parallelograms. A numerical example of critical stress estimation is given at the end of the article.*

1. Introduction

In practical engineering structures, geometrical and material discontinuities are frequently responsible for their final failure. Most of such discontinuities with step change of material characteristics can be mathematically modelled as bi-material notches (fig. 1).

fig. 1 A bi-material notch with the corresponding polar coordinate system



In the following article a bi-material notch is analysed from the perspective of linear elastic fracture mechanics, i.e. the validity of small scale yielding conditions is assumed. It is further assumed that the bi-material interface is of a welded type and the notch radius R → 0 (a sharp bi-material notch).

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2. The stress distribution in the vicinity of a bi-material notch

The expressions for the singular stress distribution referring to plane problems in the vicinity of a bi-material notch are introduced in this chapter. The results are based on the solution of the Airy stress function. The singular stress components may be written (in polar coordinates *r*, *q*, see fig. 1) in the following form:

$$s_{irr} = -\frac{H_k}{\sqrt{2p}} r^{l_k-1} l_k (a_{ik}\sin((l_k+1)q) + b_{ik}\cos((l_k+1)q) - 3c_{ik}\sin((l_k-1)q) - 3d_{ik}\cos((l_k-1)q) +$$

$$+ l_k a_{ik}\sin((l_k+1)q) + l_k b_{ik}\cos((l_k+1)q) + l_k c_{ik}\sin((l_k-1)q) + l_k d_{ik}\cos((l_k-1)q))$$

$$s_{iqq} = \frac{H_k}{\sqrt{2p}} r^{l_k-1} (l_k+1)(a_{ik}\sin((l_k+1)q) + b_{ik}\cos((l_k+1)q) + c_{ik}\sin((l_k-1)q) + d_{ik}\cos((l_k-1)q)) l_k$$

$$s_{irq} = -\frac{H_k}{\sqrt{2p}} r^{l_k-1} l_k (a_{ik}\cos((l_k+1)q) - b_{ik}\sin((l_k+1)q) - c_{ik}\cos((l_k-1)q) + d_{ik}\sin((l_k-1)q) +$$

$$+ l_k a_{ik}\cos((l_k+1)q) - l_k b_{ik}\sin((l_k+1)q) + l_k c_{ik}\cos((l_k-1)q) - l_k d_{ik}\sin((l_k-1)q)) \qquad (1)$$

where the real part of values l_k is in the interval Re(l_k) ∈ (0; 1). The subscript *i* refers to material 1 or 2. The value H_k is the so called generalized stress intensity factor (GSIF) and its value results from a numerical solution for a certain construction with a notch and given boundary conditions. The coefficients a_{ik}, b_{ik}, c_{ik}, d_{ik} for *i* = 1, 2 are known parameters corresponding to l_k and depending on the material combination and notch geometry. Generally, there exist one or two singularities of type (1) corresponding to one or two different values of l_k (*k* = 1 or *k* = 1, 2).

Regardless of the number of eigenvalues λ, the combined mode of loading occurs inherently in the vicinity of a bi-material notch. The presence of both sine and cosine terms in each equation of (1) leads to the fact that the existence of even one eigenvalue λ leads to a combination of modes I and II.

3. Study of the stress singularity of a bi-material notch

As mentioned above, the stress components have a singular character for eigenvalues in range 0 < λ < 1. The values λ_k have been determined for some basic geometrical configurations as shown in fig. 2 and in dependence on Dunders` composite parameters α, β (2), see [1] for details.

$$a = \frac{-m_1(k_2+1) + m_2(k_1+1)}{m_1(k_2+1) + m_2(k_1+1)} \qquad b = \frac{-m_1(k_2-1) + m_2(k_1-1)}{m_1(k_2+1) + m_2(k_1+1)}, \qquad (2)$$

where the shear modulus $m_i = \frac{E_i}{2(1+n_i)}$ (E_i is Young's modulus), the parameters $k_i = \frac{3-n_i}{1+n_i}$ for the case of plane stress or $k_i = 3-4n_i$ for plain strain (*i* = 1, 2). It is –1 ≤ α ≤ 1, –0.5 ≤ β ≤ 0.5 and the special case α = β = 0 corresponds to a homogeneous body.

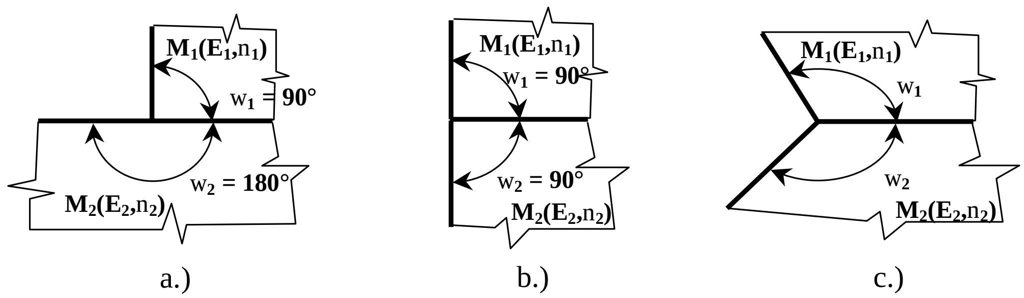

fig. 2 Ordinary geometrical configurations of bi-material notches (materials M_1 and M_2)

It is well known that the eigenvalues depend on the elastic constants of the two media and on the wedge angles, but do not depend on the body dimensions or on the external or residual stresses:

$$l = l(w_1, w_2, m_1, n_1, m_2, n_2) = l(w_1, w_2, a, b) \quad (3)$$

This enables us, for a fixed geometry, i.e. for fixed angles ω_1, ω_2, to discuss the dependence of the eigenvalues λ on the Dundurs` parameters $\alpha$, $\beta$ inside the so-called Dundurs` parallelogram. For this article, the eigenvalues λ have been numerically determined for the following geometries (see fig. 2):
Rectangular bi-material notch: $\theta_1 = 90°$, $\theta_2 = 180°$
interface with a free surface: $\theta_1 = 90°$, $\theta_2 = 90°$
symmetrical notch: $\theta_1 = 150°$, $\theta_2 = 150°$.

The corresponding Dundurs` parallelograms are shown in fig. 3, fig. 4, fig. 5, and fig. 6

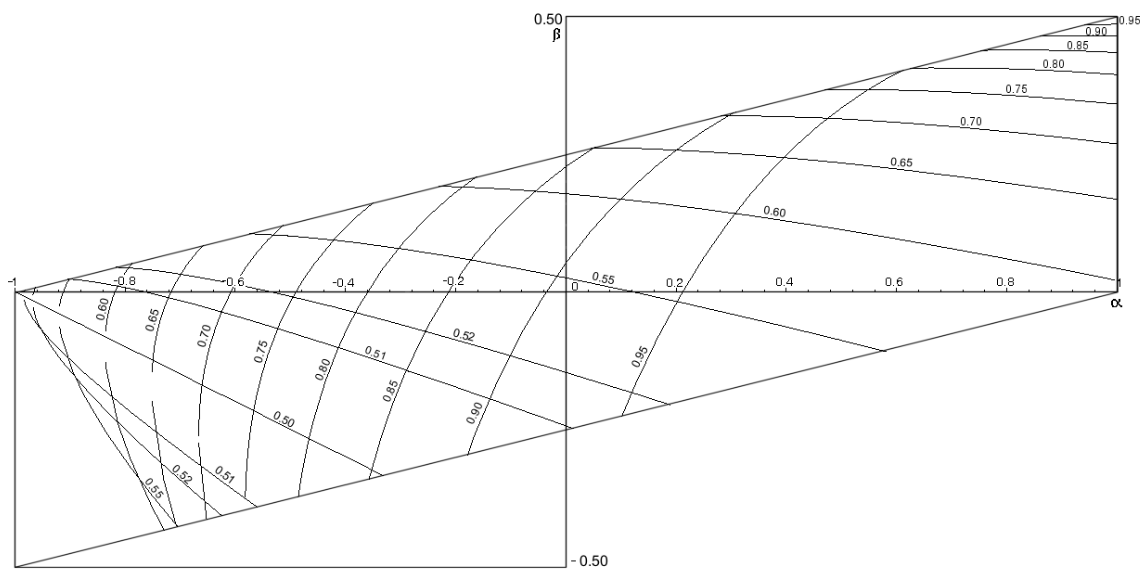

fig. 3 Eigenvalues λ_1, and λ_2 for a rectangular bi-material notch ($\theta_1 = 90°$, $\theta_2 = 180°$)

In the case of a rectangular bi-material notch - fig. 2a.), two different eigenvalues occur in the majority of material combinations, see fig. 3. In technical practice it is possible to neglect one of the two eigenvalues if the value of λ is close to 1 (weak singularity).

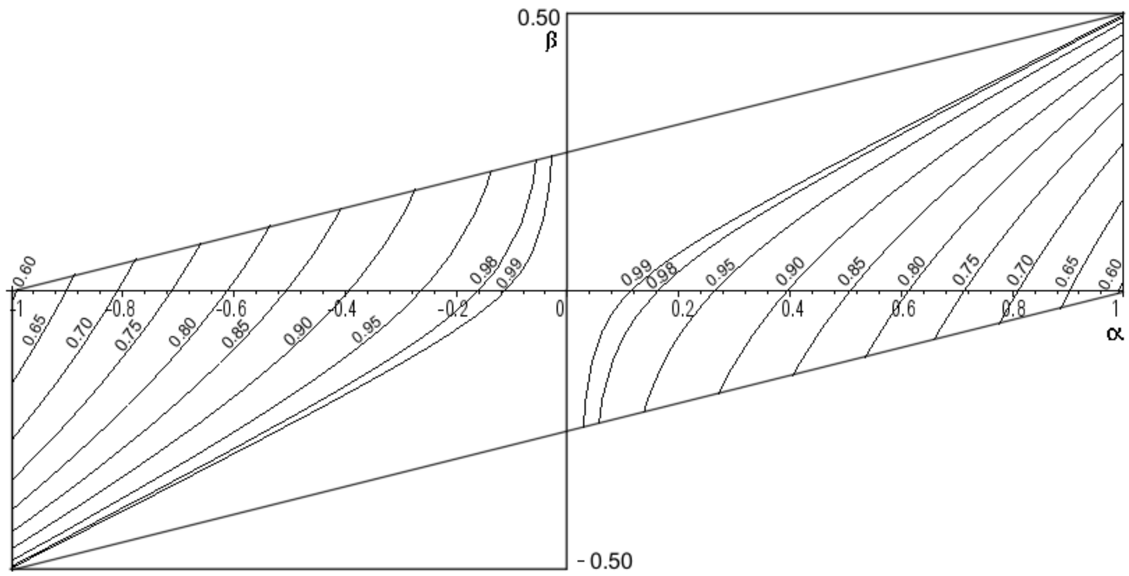

fig. 4 Eigenvalues λ_1 for an interface with a free surface (θ_1 = 90°, θ_2 = 90°)

The geometry of a bi-material notch shown in fig. 2 b.) – an interface with a free surface – leads only to one eigenvalue λ_1 (see fig. 4). It is expected that the singularity is weak for most of the combinations of two materials. The values of λ are close to $1 > \lambda > 0.9$ in quite a large region of values of α, β. The stronger singularity λ about 0.65 is reached only for a combination of two materials with markedly different material properties.

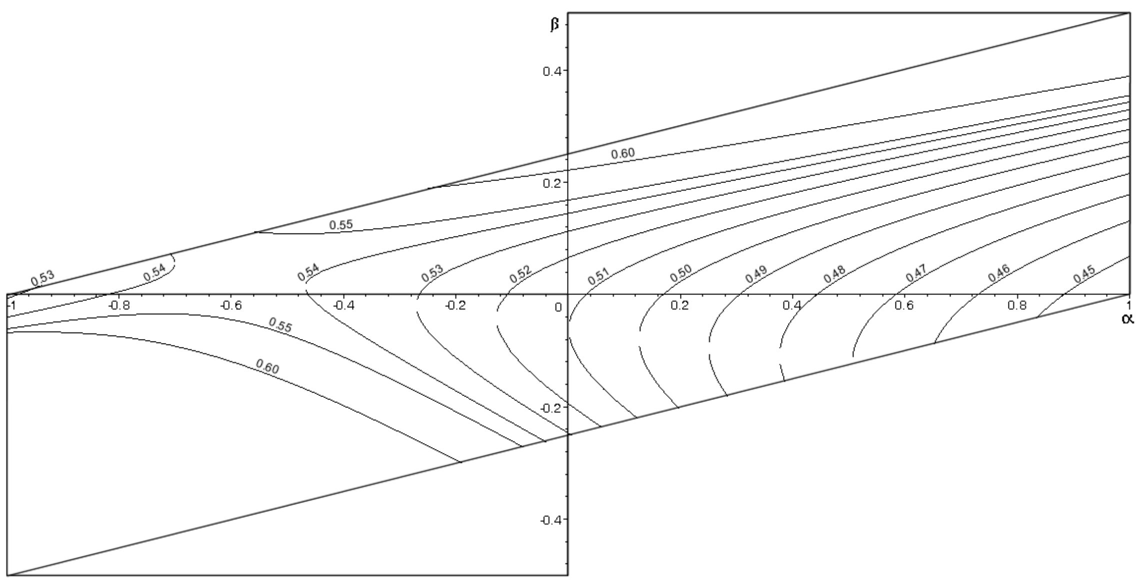

fig. 5 Eigenvalues λ_1 for a symmetrical notch: θ_1 = 150°, θ_2 = 150°

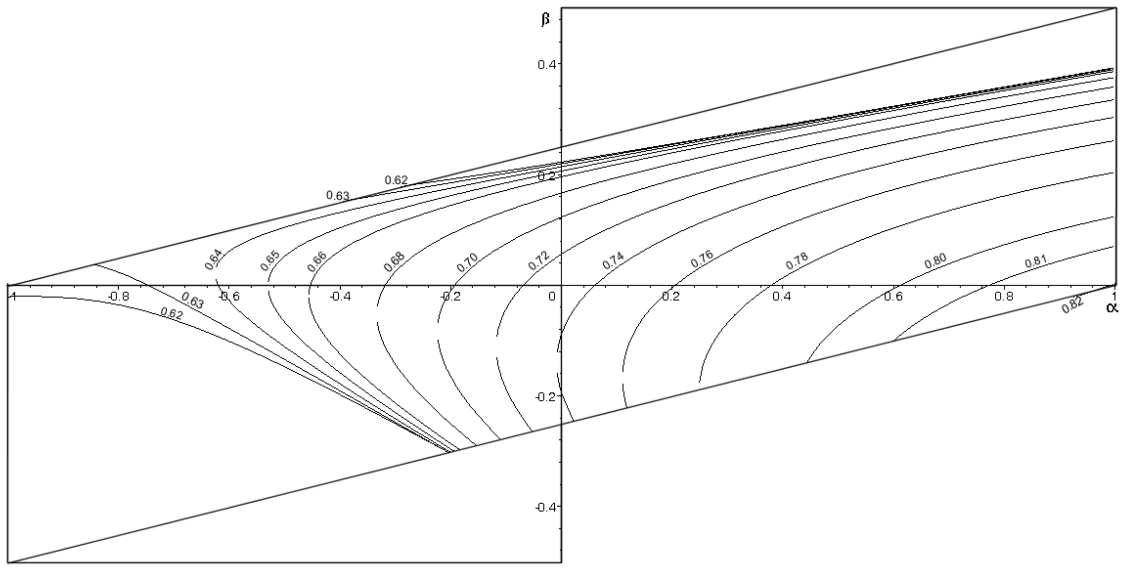

fig. 6 Eigenvalues λ_2 for a symmetrical notch: θ_1 = 150°, θ_2 = 150°

For one of the more general geometries (a geometrically symmetrical notch with θ_1 = 150°, θ_2 = 150°) two eigenvalues λ_1 and λ_2 are found in almost the whole region of possible combinations of the values α and β. The values λ_1 and λ_2 are separated into two parallelograms (fig. 5, and fig. 6) because of their better lucidity but both eigenvalues have to be considered generally. Note that – as in the first case – one singularity should be neglected in technical practice if it is significantly weaker then the second one.

4. Discussion on the stress singularity

Generally two eigenvalues occur in the case of a bi-material notch. But the stress singularity differs from a case of a sharp notch in homogeneous material. Without respect to the number of eigenvalues λ, the combined mode of loading occurs inherently in the vicinity of a bi-material notch. Even one eigenvalue λ leads to a combination of modes I and II, owing to the presence of both sine and cosine terms in each equation of (1). Unlike the homogeneous case, as far as a bi-material notch is concerned, the two singularities do not correspond to the two loading modes, nor is it possible to separate them as easily as in the homogeneous case. If one of the two eigenvalues λ leads to a strong singularity and the other leads to a significantly weaker singularity, it is possible to neglect the eigenvalue closer to value 1. But generally the fact that there are two singularities – in most cases of bi-material notches – makes it unpleasant or even impossible to use the standard methods of life time evaluation.

For the bi-material notch and also for other general singular stress concentrators a special methodology of quantification of crack initiation conditions was suggested in e.g. [2]. The procedure is based on a comparison of a quantity describing the behaviour of a crack in a homogeneous body with a magnitude with the same physical meaning, but corresponding to a bi-material notch. This magnitude has to have a clear physical interpretation and it is chosen based on a mechanism of rupture. Because of the fact that the combined mode of loading occurs inherently in this case, the strain energy density seems to be a suitable quantity for life time estimation [3], [4], [5], [6].

5. Numerical example

Let us demonstrate the critical stress value estimation of a bi-material notch on a simple numerical example. The notch geometry is shown in fig. 7, where a = 0.001m, b = 0.002m, properties of material 1: $E_1 = 3.8\times10^5$ MPa, $\nu_1 = 0.26$, $K_{IC,1} = 5$ MPa.m$^{1/2}$, material 2: $E_2 = 2.1\times10^5$ MPa, $\nu_2 = 0.31$, $K_{IC,2} = 9.5$ MPa.m$^{1/2}$. It results $\alpha = -0.28814$ and $\beta = -0.09051$, and consequently $\lambda = 0.9755$ from the specified geometry and material properties.

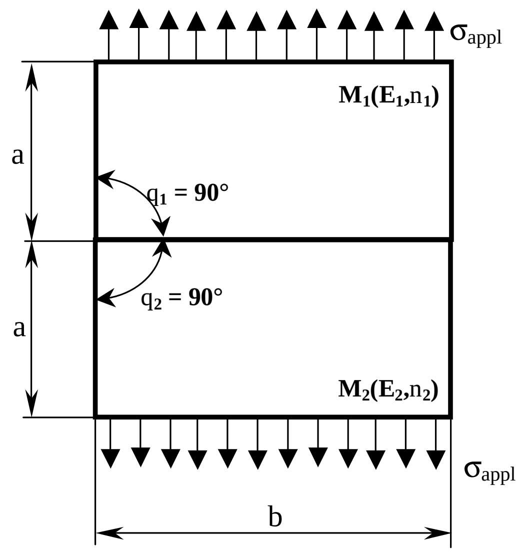

fig. 7 Model of an evaluated notched body

The strain energy density (SED) was used as a controlling quantity for the estimation of the critical stress of the studied configuration. If the applied stress σ_{appl} reaches the critical stress σ_{crit}, a crack is initiated at the notch tip and the growth of the crack can lead to a final rupture of the construction. The general methodology of quantification of the critical stress estimation was suggested in e.g. [2]. Applying the strain energy density concept to an evaluation of the general singular stress concentrator, it is necessary to opt for a dimensional parameter r where the value of the strain energy is calculated. It can be taken in dependence on rupture mechanism as a plastic zone size, a grain size or another suitable parameter. In this case $r = 1.2\times10^{-5}$m. The case of plane stress is considered. Than the trend of strain energy density was obtained from numerical solution by ANSYS. The local minimum of SED was found in material 1 in angle $q_m = 30.7°$, where the value of strain energy density w_{min1} = 14812 J/m^3, see fig. 8.

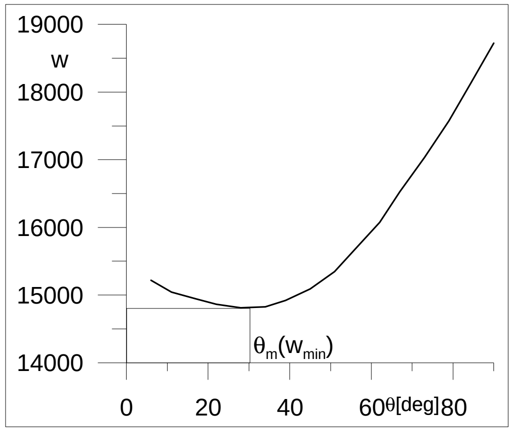

fig. 8 The strain energy density distribution in material 1, q_m in minimum of w is the assumed angle of crack initiation

Then the value of the generalized stress intensity factor H can be obtained from the relation (4) for SED distribution. It results H = 56.97 MPa.m$^{1-\lambda}$.

$$w_i = \frac{1}{4} H^2 r^{(2l-2)} l^2 [2a_i \sin((l+1)q)\left(c_i \sin((l-1)q) + d_i \cos((l-1)q)\right)\left(l^2-1\right) +$$
$$+2b_i \cos((l+1)q)\left(c_i \sin((l-1)q) + d_i \cos((l-1)q)\right)\left(l^2-1\right) + 2a_i \cos((l+1)q)$$
$$\left(c_i \cos((l-1)q) - d_i \sin((l-1)q)\right)\left(l^2-1\right) + 2b_i \sin((l+1)q)\left(d_i \sin((l-1)q) - c_i \cos((l-1)q)\right)$$
$$\left(l^2-1\right) + a_i^2(1+2l+l^2) + b_i^2(1+2l+l^2) + c_i^2(1-2l+l^2+4\overline{k}_i - 4\cos((l-1)q)^2 \overline{k}_i) +$$
$$+d_i^2(1-2l+l^2+4\cos((l-1)q)^2\overline{k}_i) + 8c_i \sin((l-1)q) d_i \cos((l-1)q)\overline{k}_i] / (pm_i) \tag{4}$$

Then comparing the relation (4) (under critical condition and for the material where crack initiation is assumed) with those for a crack we get:

$$H_C = K_{iIC} r^{\left(\frac{1}{2}-l\right)} \{\overline{k}_i / [l^2 (2a_i \sin((l+1)q_m)\left(c_i \sin((l-1)q_m) + d_i \cos((l-1)q_m)\right)\left(l^2-1\right) +$$
$$+2b_i \cos((l+1)q_m)\left(c_i \sin((l-1)q_m) + d_i \cos((l-1)q_m)\right)\left(l^2-1\right) + 2a_i \cos((l+1)q_m)$$
$$\left(c_i \cos((l-1)q_m) - d_i \sin((l-1)q_m)\right)\left(l^2-1\right) + 2b_i \sin((l+1)q_m)\left(d_i \sin((l-1)q_m) - c_i \cos((l-1)q_m)\right) \quad (5)$$
$$\left(l^2-1\right) + a_i^2(1+2l+l^2) + b_i^2(1+2l+l^2) + c_i^2(1-2l+l^2+4\overline{k}_i - 4\cos((l-1)q_m)^2\overline{k}_i) +$$
$$+d_i^2(1-2l+l^2+4\cos((l-1)q_m)^2\overline{k}_i) + 8c_i \sin((l-1)q_m) d_i \cos((l-1)q_m)\overline{k}_i)]\}^{\frac{1}{2}}$$

for the critical value of the generalized stress intensity factor H_C = 644.37 MPa.m$^{1-\lambda}$.

Finally the critical stress is obtained as :

$$s_{crit} = s_{appl} \frac{H_C}{H(s_{appl})} \tag{6}$$

For the values of H and H_C determined previously we get σ_{crit} = 1131 MPa, which is the stress that should be applied in order to initiate a crack from the stress concentrator.

6. Conclusion

The relations for the singular stress distribution in the vicinity of a bi-material notch have been discussed. The eigenvalues λ describing the stress singularity were evaluated for three particular basic geometries of a bi-material notch. Generally there are two eigenvalues λ as well as two corresponding singularities. In contrast with the case of a sharp notch in a homogeneous body, neither singularity – in the case of a bi-material notch – belongs to one loading mode, but each of the singularities leads to a combination of loading modes I and II. This fact makes it unpleasant or even impossible to use the standard methods of linear elastic fracture mechanics for evaluation of crack initiation. The special method of evaluation of rupture initiation was suggested and tested. This method leads to estimation of the critical stress necessary for crack initiation in a bi-material notch.

The numerical example of estimation of the critical stress was presented. Note that if the applied stress is less than critical, no crack is initiated in the bi-material notch of specified geometry and boundary conditions.

7. Acknowledgement

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8. References

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