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RELIABILITY ANALYSIS AND OPTIMISATION OF COMPONENTS WITH RANDOM INPUT VARIABLES

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Summary: *The behaviour of components with input quantities showing random variations can be studied using the Monte Carlo method together with the Response Surface method. Both methods can also be used in searching for the optimum design parameters, preferably with low sensitivity to the input scatter. The paper explains effective ways for the construction of the response surface, among others the use of dimensional analysis and similarity theory. The use of both methods (MC + RSM) is illustrated on an example of dimensioning of an elastic-plastic beam subjected to impact by a moving body.*

1. Introduction

A reliable structure is such which keeps the output parameters (deformations, dynamic or kinematic characteristics, etc.) in the allowable limits. If the input quantities (load, geometry, material properties, etc.) vary or can deviate from nominal values, it is also necessary to analyse the corresponding variability of the response. If these deviations have random character, this analysis can be done using the simulation Monte Carlo method: the response is calculated many times for randomly generated input values, and the results form a histogram, which informs about the probability distribution. If the calculation of response is time consuming (made, e.g., by the finite element method), it is reasonable to use the Response Surface Method: the detailed analysis is done only for selected combinations of input parameters, and the results are fitted by a suitable regression function (response surface). The Monte Carlo simulations are then carried out only with this simple analytical function. The response surface also enables one to find such design point (i.e. the nominal values of input quantities), for which the response is only little sensitive to their variability (so-called robust design, Fig. 1). Having known the allowable limit values of the output parameters, one can also set the tolerances of input quantities. The paper gives instructions for the construction of response surface. It also shows how to reduce the extent of the necessary computer modeling by using the theory of similarity and dimensional analysis. The use of the described methods is explained on an example of dimensioning of a beam subjected to impact.

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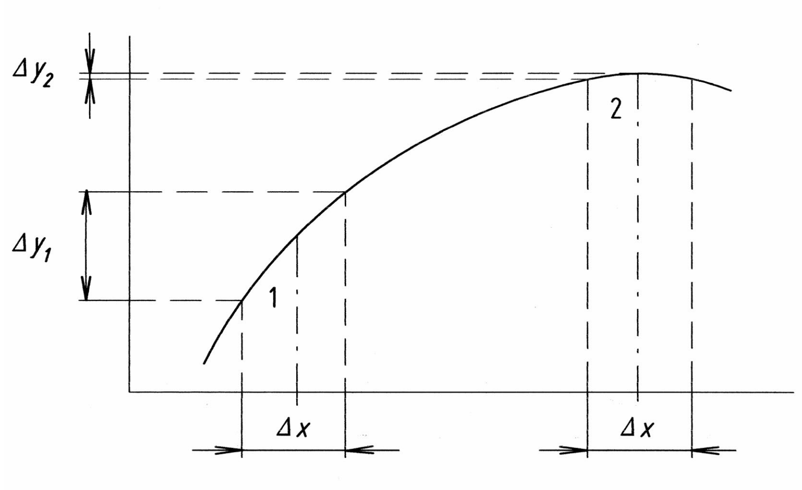

Fig. 1 Principle of robust design.

2. Response surface method

The relationship between the output quantity y and input variables can be expressed generally as $y = f(x_1, x_2, \ldots x_i, \ldots x_n)$. The response is calculated for selected combinations of input quantities, and the results are fitted by a regression function, usually a polynomial

$$y = a_0 + \sum a_i x_i + \sum b_i x_i^2 + \ldots + \sum d_{ij} x_i x_j + \ldots \qquad (1)$$

This approximation is suitable if the actual relationship between input and output has such character, or if the output quantity changes in the considered interval only little. If it differs significantly from a polynomial (e.g. $y \sim 1/x^3$ or $y \sim x^{1/2}$), Equation (1) cannot give a good approximation in a wider interval. There are several ways for improvement, the starting point being a visual judgement of the character of response. With several input quantities, this information can be obtained by creating several regression functions, each with only one or two input quantities as independent variable (= „cuts“ through the response surface), Fig. 2.

Linear or polynomial function can also be used for the approximation of other relationships if suitable transformations are made. For example, the relationship $y = a/x^3$ can

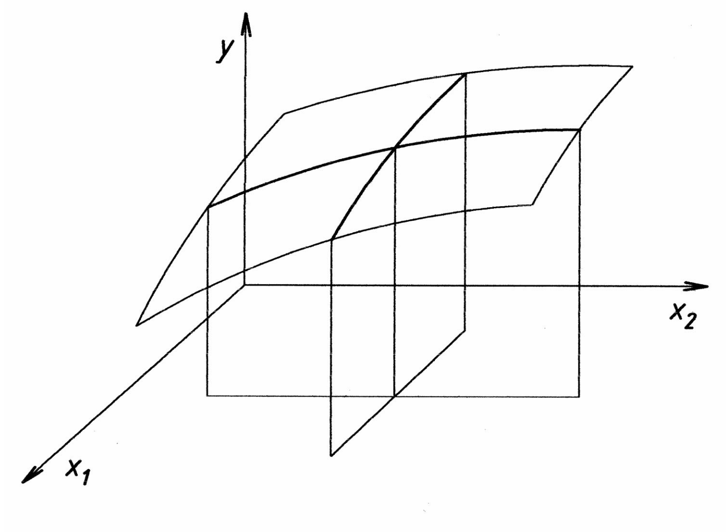

Fig. 2 Response surface (a schematic)

be expressed as $y = az$ by introducing new variable $z = 1/x^3$; the relationship $y = ax_1/x_2^2$ can be converted to multiple linear regression $Y = A_0 + A_1X_1 + A_2X_2$ by logarithmic transformation, etc. Tools like Solver enable the determination of regression coefficients in complicated functions by direct application of the least squares method, without transformations.

The fit of a response function to the real data can sometimes be improved by dividing the definition interval of some quantities into sub-intervals and using different regression functions for each of them. This is sometimes substantiated by the physical character of the problem; for example, the elastoplastic deflection obeys another law than elastic one.

3. Use of dimensional analysis and theory of similarity

The form of response function should correspond to the physical character of the problem. When looking for such function, it is reasonable to utilise the known analytical solutions of similar problems. Useful also is dimensional analysis (Kožešník, 1983), which can help one to find a proper form of the response function and to reduce the amount of computer work.

For example, let us look for the expression for critical load F_{cr} for buckling of a compressed column. Assume that we do not know how this load depends on its length L. In buckling, the beam deflects. Thus, we can deduct that the critical force will be directly proportional to its bending stiffness EJ (E is elastic modulus and J the moment of inertia of the cross section), so that the general form of the formula will be

$$F_{cr} = k\,E\,J\,L^s \quad ; \qquad (2)$$

k is a constant to be found (for the given fixing of ends) by experiment or computer modeling. After replacing individual quantities by their units, one obtains $[N] = [1]\times[Nm^{-2}]\times[m]^4\times[m]^s$, which gives $s = -2$ (corresponding to the known solution $F_{cr} = kEJ/L^2$).

The use of dimensional analysis in computer modeling is useful in another respect also. Every physical equation can be expressed by means of nondimensional parameters, whose number is usually lower than the number of original variables. If n is the number of variables, and r is the number of basic dimensions necessary to express them, the pertinent equation can be written as a relationship of $(n - r)$ nondimensional parameters. The reduction of variables also means the reduction of computer work necessary for obtaining the data for the response surface. The number of points needed is generally

$$n = \prod_{i=1}^{m} u_i \quad , \qquad (3)$$

where m is the number of factors and u_i is the number of levels for i-th factor. For example, the response „surface“ for four factors, each on three levels, needs $3^4 = 81$ points. Reduction of the variables from 4 to 3 would mean the reduction of points to $3^3 = 27$, i.e. to one third! For example, Eq. (2) can be rewritten as

$$F_{cr}\,L^2 / (EJ) = k \quad , \qquad (4)$$

so that even only one „experiment“ is necessary for obtaining the unknown constant k !

There is one more important advantage of using the theory of similarity and formulating the problem in terms of dimensionless quantities: the solution is valid not only for the particular case, but for the whole class of similar objects !

Some dimensionless arguments can be created as the ratios of quantities with the same dimension; e.g. the ratio of deflection of a beam to its length, the height of the cross section to the length, stress to elastic modulus, etc. For example, the dimensionless expression for maximum stress σ in a plate with characteristic lengths *a, b, c* and thickness *t,* loaded by pressure *p* (Nm^{-2}) can be written as

$$\sigma / p \ = \ f\,[\ (a/t\}^2, (b/a), (c/a), \mu\ ] \quad , \tag{5}$$

where μ is Poisson's ratio (also dimensionless), and *f*[...] is some function to be determined. If plastic deformations are expected, one more argument, e.g. the ratio of yield stress and elastic modulus must enter the equation. If the compressed column from the above example had two parts with different cross sections, the expression (4) should be extended by two dimensionless arguments J_1/J a L_1/L, where J_1 a L_1 are the moment of inertia and length of the part with different cross section. Nondimensional quantities can also be formed as the ratio

$$X \ = \ x \,/\, x_{\mathrm{char}} \quad , \qquad \text{or} \quad X \ = \ (x - x_{\mathrm{min}}) \,/\, (x_{\mathrm{max}} - x_{\mathrm{min}}) \tag{6}$$

where x_{char} is a characteristic value of *x*, and x_{min} and x_{max} are the minimum and maximum (or starting and final) values.

After having found the response surface, we return (in the Monte Carlo simulations) back to the original (dimensional) expressions like Eq. (2), and assign random values to all individual variables (for example *a* and *b* instead of *b/a*). An example is is given below.

4. Sensitivity analysis and determination of tolerances of input quantities

The design point with low sensitivity to the variations of input quantities can be found using the analytical form of the response surface, or various optimisation methods. Effective is the so-called simplex method, which approaches to the optimum continuously according to a very simple algorithm. A fictitious body with $n + 1$ vortices (a simplex) is created, the number of vortices being by one higher than the number of input variables (e.g. a triangle for two independent variables, and a tetrahedron for three variables). The coordinates of the vortices correspond to the values of input parameters. For all these points, the output quantity of interest is calculated.The individual values are compared. In the next step, the new simplex is created by replacing the vortex with the worst value of the dependent quantity by a new one, whose coordinates are mirror-symmetrical. For this new point, the dependent variable is calculated. Now, in the new simplex (consisting of the new vortex and all vortices from the previous simplex except the worst one), the values of the dependent variable for all vortices are compared, and again the worst vortex is omitted and the new one is created in the same manner. In this manner we proceed until the quantity of interest attains the extreme or acceptable value. More details can be found, e.g., in (Tichomirov, 1974).

After having found the position of the design point, the sensitivity of the output to the variations of input quantities should be analysed, in order to find the quantities which contribute most to the output variation, and to set them appropriate tolerances. Various methods and formulae for sensitivity analysis using analytical and simulation methods were described in detail in (Menčík, 2000), together with the procedures for setting the tolerances of input quantities.

5. Practical example

The use of the Monte Carlo method in combination with the response function will be illustrated on an example of a simply supported beam hit in the middle by a moving rigid body (Fig. 3). The beam should absorb the energy of impact. Plastic deformations are allowed, but only such that no plastic collapse occurs, and the permanent deflection does not exceed a permitted value. The mass and velocity of the moving body, and the dimensions and yield strength of the beam are random quantities. Thus, also the resultant load and bending moment and residual deformation are random variables. The task is to find the cross-section dimensions of the beam such that the probability of exceeding the allowed limit values is low.

Theory

The moving body deflects the beam, and its kinetic energy $U = mv^2/2$ changes into the energy of elastic strains and the work of plastic deformations (m is the mass of the body and v is its velocity). The maximum force F acting on the beam is determined from the energy via the $F(y)$, $U(F)$ and $F(U)$ relationships; y is the deflection. These calculations depend on whether the deformations are elastic or elastoplastic. The permanent deflection after unloading is calculated as the difference between the total elastoplastic deflection and the elastic deflection caused by the same force. The beam material is assumed ideally elastic-plastic without strain hardening (Fig. 3), the cross section is rectangular, of width w and height h.

Remark. The weight of the body can play a role if potential energy is released by its movement in the gravitational field during the beam deflection. For simplicity, these effects are not considered here (movement in horizontal direction).

The elastic deflection is $y = FL^3/(48EJ)$. E is elastic modulus, L is the beam span, and $J = wh^3/12$ is the moment of inertia of the cross section. The maximum stress $\sigma = M/Z$, where $M = FL/4$ is the maximum bending moment and $Z = wh^2/6$ is the section modulus. The beam deforms elastically until the maximum stress reaches the yield strength σ_y of the material. The corresponding force and deflection are

$$F_y = \frac{2}{3}\,\sigma_y \frac{wh^2}{L} \quad ; \qquad y_y = \frac{\sigma_y L^2}{6Eh} \quad . \qquad (7)$$

For higher loads, the maximum stress remains constant (no strain hardening), while the plastically deformed region grows. At the ultimate load ($F_u = 1.5\,F_y$), the whole cross-section in the middle of the beam is plastified (a plastic hinge). Higher force would cause a collapse.

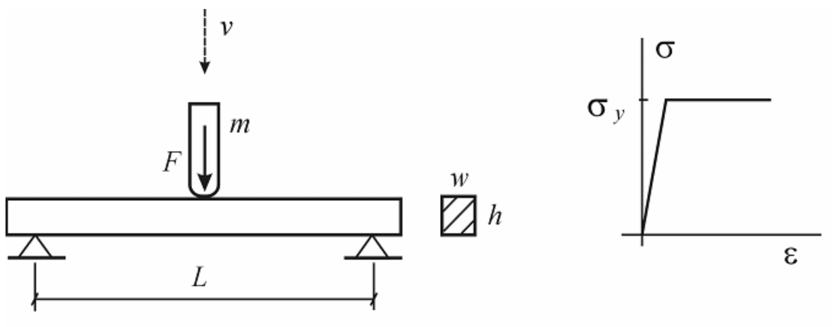

Fig. 3 Elastoplastic beam under impact

The pertinent theory for the elastoplastic deflections can be found in (Šmiřák et al., 1966), so that only the main results will be presented here. In order to get general formulae, we introduce relative (nondimensional) quantities

$$F^* = F / F_y \qquad , \qquad y^* = y / y_y \quad . \tag{8}$$

The relative elastic deflection (for $0 \le F^* \le 1$) can be expressed as

$$y^* = F^* \ , \tag{9}$$

while the elastoplastic deflection (for $1 \le F^* \le 1.5$) depends on F^* as (Fig. 4)

$$y^* = [5 - 3^{1/2}(3 + F^*)(1 - 2F^*/3)^{1/2}] / F^{*2} \quad . \tag{10}$$

The accumulated (and dissipated) energy U can be obtained by integration of the $F(y)$ relationship. For elastoplastic deformations, the integration must be performed numerically. For universality, nondimensional energy U^* is introduced, defined as the ratio of the deformation work corresponding to the force F and that corresponding to F_y :

$$U^* = U / U_y = \int F\mathrm{d}y / U_y = \int F_y F^* y_y \mathrm{d}y^* / (F_y y_y/2) = 2 \int F^* \mathrm{d}y^* \quad . \tag{11}$$

(The relative energy could also be defined as $U^{**} = \int F^* \mathrm{d}y^*$. In this case, the relative energy at the elastic-plastic transition would be $U^{**} = 1/2$, while $U^*(F_y y_y) = 1$. The difference between both definitions is only formal.)

Having found the $U^*(F^*)$ values, the force can be determined from energy. For elastic deformations ($0 \le F^* \le 1$, or $0 \le U^* \le 1$), the relationship

$$F^* = \sqrt{U^*} \tag{12}$$

is exact. For the elastoplastic range ($1 \le F^* \le 1.5$, or ($1 \le U^* \le 4.333$), a suitable expression can be obtained by fitting the F^* values calculated for several values of U^* by a regression function (= Response Surface). A relatively good approximation (coefficient of determination $R^2 = 0.9997$) is

$$F^* = 0{,}3308 + 0{,}8806U^* - 0{,}2278U^{*\,2} + 0{,}0201U^{*\,3} \quad . \tag{13}$$

The relative residual deflection for $1 \le U^* \le 4.333$, can be expressed ($R^2 = 0{,}99996$) as

$$y_{\mathrm{res}}^* = 0.1467 - 0.2847U^* + 0.1508U^{*2} - 0.0126U^{*3} \quad . \tag{14}$$

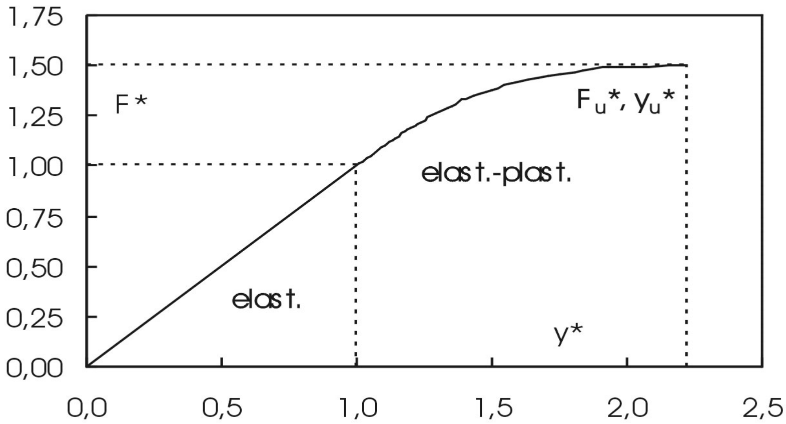

Fig. 4 Deflection y^* of the elastoplastic beam as a function of (normalised) load F^*.

In the Monte Carlo procedure, random numbers are assigned to the dimensions, yield strength of the beam, and the mass and velocity of the moving body. Then, the corresponding values F_y, y_y are calculated, as well as the energy U of impact and the characteristic value $U_y = F_y\, y_y /2$. Then, the relative energy U^* is determined. If $U^* \leq 1$, the deformations are only elastic. If $U^* > 1$, the corresponding force F^* and relative residual deflection y_{res}^* are calculated from Eqs. (13) and (14).

Two limit states are considered: unacceptable permanent deflection after the impact, and creation of plastic hinge. The permanent deflection should not exceed some value δ. Thus, each relative value y_{res}^* obtained in M.C. trials from (8) must be converted into absolute value $y_{res} = y_y \times y_{res}^*$, which is then compared with δ. The probability of failure is determined as the ratio of the number of cases where y_{res} has exceeded δ, and the number of all MC trials. The plastic hinge arises when bending moment attains the ultimate value M_u. For higher safety, however, the limit state is defined here as some fraction of M_u. As the bending moment is directly proportional to the forces F and F^*, it is sufficient to compare the calculated value F^* with some number k (< 1.5). As F^* is related to the relative energy U^*, it is also possible to check this limit state by comparing the calculated U^* value with the value $K = U^*(F^* = k)$.

<u>Numerical input values and results</u>

The parameters of input variables (distribution, mean, standard deviation) are: m (normal, 500 kg, 50 kg), v (normal, 1.0 m.s^{-1}, 0.1 m.s^{-1}), w (normal, 50 mm, 0.3 mm), L (constant, 2000 mm), E (constant, 210 000 Mpa), f_y (steel S 235, the histogram in Guštar & Marek, 2001). The quantity to be found is the beam height h in the force direction; it can be chosen from the series of nominal values 100 - 110 - 120 ... mm (normal dist. with standard deviation 0.5 mm). The allowable permanent deflection after the impact is $\delta = 1$ mm, with permitted probability of exceeding $P_{f1,a} = 0.001$. The allowable maximum force is $0.9F_u$ (corresponding to $k = 0.9\times1.5 = 1.35$ and $K = 2.042$) with the permitted probability of exceeding $P_{f2,a} = 0.001$.

The simulations were performed using the Ant-Hill program (Guštar & Marek, 2001), Fig. 5. The starting beam thickness was 100 mm, and was gradually increased with respect to the results of simulations. For each height, 100,000 trials were made. The standard normal variable u, used in generating the quantities m, v, b, h, was bounded by the interval $u = \pm$ 3,5.

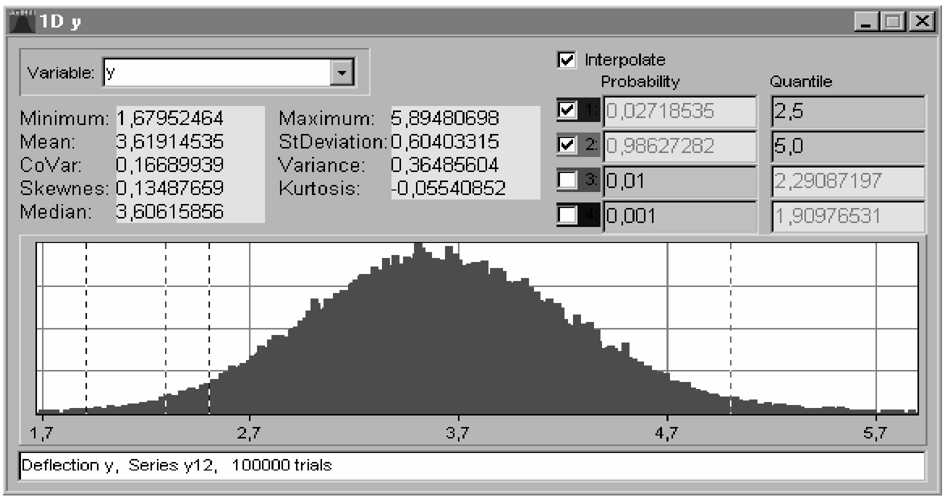

Fig. 5 Screen of the Ant-Hill program for Monte Carlo simulations

Table 1. Results of the Monte Carlo analysis of elastoplastic beam under impact

h (mm)	$P_{f1}(y_{res}>1\text{mm})$	$y_{res,\,max}$ (mm)	$P_{f2}(U^*>2.042)$	$U^*_{\ max}$
100	0.02557	5.155	0.0591	4.501
110	0.00813	2.792	0.0301	3.404
120	0.00224	2.291	0.0152	3.289
130	0.00061	2.158	0.0073	3.382
140	0.00006	1.265	0.0033	2.797
150	0.00001	0.926	0.0022	2.631
160	0.00001	1.140	0.0006	2.865

The main results are shown in Table 1. U_{max}^* and $y_{res,\,max}$ are the maximum values obtained ($U^* = 4.333$ corresponds to the plastic hinge). The condition of acceptable permanent deformation $P_{f1}(y_{res}>1\text{mm}) < 0.001$ was fulfilled for $h = 130$ mm, but with the maximum value in the 100,000 trials $y_{res,\,max} = 2.16$ mm and unacceptably high probability of exceeding $0.9M_u$ (or $U^* = 2.042$). The condition $P(U^*>2.042) \le 0.001$ is fulfilled only for $h \ge 160$ mm. Thus, the beam height satisfying all conditions is $h = 160$ mm.

The table shows how the beam height is related to the probability of exceeding individual limit states, and also shows the expectable maximum values, which can be used for assessing the consequences. As the size of the cross section is related to the costs, it is possible to find in this way the optimum size (sometimes also the reasonable limit values and the allowable probabilities of their exceeding) which guarantee the minimum costs for the structure including those following from exceeding the limit states.

6. Conclusion

The behaviour of components with input quantities of random character can be studied using the Monte Carlo method together with the Response Surface method. Both methods can also be used in searching for the design parameters with low sensitivity to the input scatter. The paper has explained methods for the construction of response surface, among others the use of dimensional analysis and similarity theory. The use of the Monte Carlo and Response Surface methods was illustrated on an example of dimensioning of an elastic-plastic beam subjected to impact by a moving body.

7. Acknowledgement

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