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MODELING OF RESPONSE OF VISCOELASTIC MATERIALS TO HARMONIC LOADING

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Summary: *The response of polymeric materials to harmonic loading can often be described by means of linear viscoelastic models. The paper brings formulae for the resultant complex elastic modulus and phase angle for basic viscoelastic bodies (Maxwell, Kelvin-Voigt and Standard Linear Solid) and their generalised versions, shows how the constants in these models can be determined from the response measured at various frequencies, and gives recommendations for the choice of a suitable model.*

1. Introduction

The deformation of polymeric materials under harmonic loading is also harmonic, but lags behind the load (Fig. 1). The phase angle δ depends on the frequency, and so also does the complex modulus E^*, defined as the ratio of the stress amplitude σ_0 and strain amplitude ε_0. When designing a component, the properties corresponding to the frequency range of operation must be used; otherwise the actual response can differ from the expected. Thus, the dependence of stiffness and phase angle on frequency must by known for the particular material. This dependence can be obtained by fitting the data measured at various frequencies by a suitable analytical expression. This expression can be purely empirical (e.g. polynomial). A better way is to model the response by means of simple viscoelastic bodies consisting of springs and dashpots. The parameters (elastic moduli and viscosities) in these bodies are constant, independent of frequency, and the character of the model curves corresponds, in principle, to the behaviour of viscoelastic materials. This approach allows easy modelling of the response and is especially useful if also the relationship between the response and material composition or microstructure should be studied, or if one wants to describe the response in a wide frequency range, so that several groups of values, obtained by different methods of measurement (e.g. dynamic and quasistatic) must be combined.

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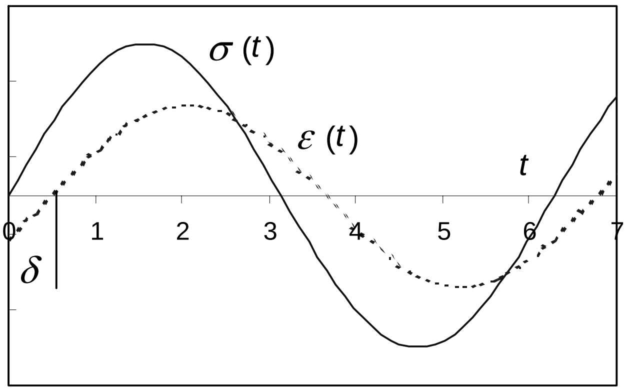

Fig. 1 Response of viscoelastic materials to harmonic loading.

The most often used viscoelastic models are the Maxwell body (Fig. 2a), the Kelvin (Voigt) body (Fig. 2b), and the Standard Linear Solid (2c, 3a). However, the frequency dependence of the response of real viscoelastic materials often differs from that predicted by these models. As it will be explained later, the simple bodies are able to describe the response only in a limited frequency range. Much better description can be obtained using more complex models. There are two principal general models: the general Kelvin-Voigt body, consisting of the Kelvin-Voigt elements in series, and the general Maxwell body, consisting of the Maxwell elements connected in parallel (cf. also Fig. 4). Often, better approximation is obtained by the general Standard Linear Solid, created from the above models by replacing the first body just by a spring (Fig. 4a, b).

The principal information about various viscoelastic models can be found in literature, e.g. (Haddad, 1995, or Ferry, 1980). In the following section, the formulae will be presented for the complex modulus and phase angle for the Kelvin-Voigt body, Maxwell body and Standard Linear Solid, and for their generalised versions – all under assupmtion of linear viscoelasticity. In the Practical part, we show characteristic features of these models, and give advice for the choice of a model.

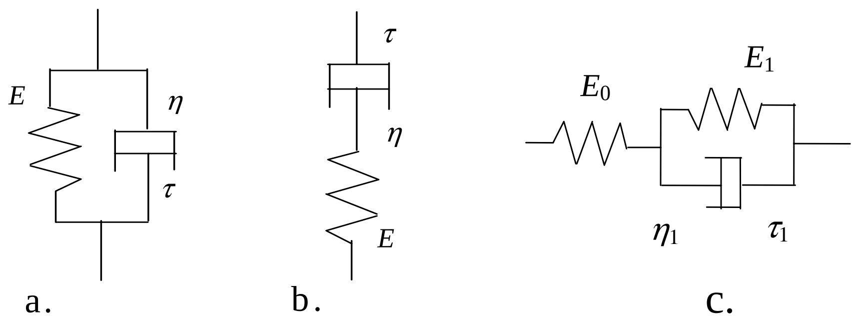

Fig. 2 Kelvin-Voigt body (a), Maxwell body (b), and Standard Linear Solid (c).

2. Theoretical part

In a spring, the stress is proportional to the strain, $\sigma = E\varepsilon$, while in a dashpot, the stress is proportional to the strain rate: $\sigma = \eta\, \mathrm{d}\varepsilon/\mathrm{d}t$; E is the elastic modulus, η viscosity, and t is time.

The Kelvin-Voigt body (Fig. 2a)

The strain ε is the same in the spring and the dashpot. The total stress σ is the sum of the stress in the spring (which is in phase with the strain) and the stress in the dashpot (which precedes the strain by 90^o), so that it precedes the strain by some phase angle δ. The stiffnes of the body can be characterised by complex modulus $E^* = \sigma_0/\varepsilon_0$; the subscript zero denotes the amplitude. For harmonic loading (Haddad, 1995),

$$\tan\delta = \omega\eta/E = \omega\tau \quad , \qquad (1)$$

$$E^* = E\sqrt{1+\tan^2\delta} = E\sqrt{1+(\omega\tau)^2} \quad , \qquad (2)$$

where ω is angular frequency and τ is the so-called relaxation (or retardation) time, calculated as $\tau = \eta/E$. The complex modulus can be decomposed into two components: the storage modulus $E' = E^* \cos\delta$, which characterises the stress component in phase with the strain, and the loss modulus $E'' = E^* \sin\delta$, corresponding to the out-of-phase component.

The Maxwell body (Fig. 2b)

The stress σ is the same in the spring and the dashpot. The total strain ε of this body is the sum of the strain in the spring (in phase with the stress) and in the dashpot (lagging behind the stress by 90^o). The resultant strain is delayed by the stress by the angle δ. The compliance of the element can be characterised by complex compliance C^* (= $1/E^*$), with the in-phase and out-of-phase components. For harmonic loading (Haddad, 1995),

$$\tan\delta = E/\omega\eta = 1/\omega\tau \quad , \qquad (3)$$

$$E^* = E\Big/\sqrt{1+\tan^2\delta} = E\Big/\sqrt{1+(\omega\tau)^2} \quad , \qquad (4)$$

The Standard Linear Solid (Fig. 3a)

This model consists of a spring in series with a Kelvin-Voigt body. The same stress $\sigma(t)$ acts in both the spring and the K-V body. The total strain $\varepsilon(t)$ equals the vector sum of the strain in the spring „0“ (in phase with the stress) and the strain of the K-V body, which lags behind the stress by the angle δ_1, given by Eq. (1). According to Fig. (3b), the phase angle can be obtained from the ratio of the out-of-phase component of the strain amplitude to the component which is in-phase with the stress:

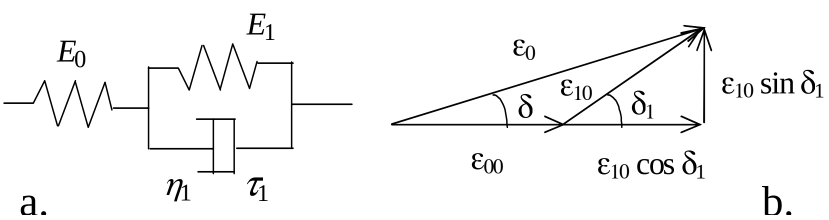

Fig. 3 Standard linear solid: arrangement (a), diagram of strain amplitudes (b)

$$\tan\delta \;=\; \frac{\varepsilon_{01}\sin\delta_1}{\varepsilon_{00} + \varepsilon_{10}\cos\delta_1} \quad , \qquad (5)$$

where ε_{00} and ε_{01} are the amplitudes of strain components in the spring and the K-V body. Using the following relationships between the stress and strain amplitudes,

$$\varepsilon_{00} = \sigma_0 / E_0 \quad ; \quad \varepsilon_{01} = \sigma_0 / E_1{}^* = \sigma_0 / E_1\sqrt{1+\tan\delta} \qquad , \qquad (6)$$

and known relationships between trigonometric functions sin, cos, tan (and $\omega\tau$), we obtain

$$\tan\delta \;=\; \frac{\dfrac{1}{E_1}\dfrac{\omega\tau_1}{1+(\omega\tau_1)^2}}{\dfrac{1}{E_0} + \dfrac{1}{E_1}\dfrac{1}{1+(\omega\tau_1)^2}} \quad . \qquad (7)$$

The resultant elastic modulus can be obtained from the ratio of the stress amplitude σ_0 to the total strain amplitude ε_0. According to Fig. 3b, ε_0 is calculateded as the hypotenuse of the vector triangle, generally as:

$$\varepsilon_0 \;=\; \sqrt{\left(\sum_i \varepsilon_{i0}\cos\delta_i\right)^2 + \left(\sum_i \varepsilon_{i0}\sin\delta_i\right)^2} \qquad , \qquad (8)$$

where $\Sigma\varepsilon_{i0}\cos\delta_i$ and $\Sigma\varepsilon_{i0}\sin\delta_i$ are the in-phase and out-of-phase components of the total strain amplitude. Using again Eq. (6) and the relationships sin…, cos…, tan…, $\omega\tau$…, we arrive at

$$\frac{1}{E^*} = \sqrt{\left(\frac{1}{E_0} + \frac{1}{E_1}\frac{1}{1+(\omega\tau_1)^2}\right)^2 + \left(\frac{1}{E_1}\frac{\omega\tau_1}{1+(\omega\tau_1)^2}\right)^2} \quad . \qquad (9)$$

Equation (9) also illustrates the fact that a single standard linear solid can describe the changes in response only in a limited range of frequencies. For relatively slow processes, with $(\omega\tau)^2 \ll 1$, the resistance of the dashpot is negligible compared to the spring E_1, and the whole body behaves as the springs E_0 and E_1 in series. For relatively high frequencies, $(\omega\tau)^2 \gg 1$, the resistance of the dashpot is very high compared to that of the spring, the Kelvin-Voigt part becomes stiff, and the whole body behaves as the spring E_0 alone. The sensitivity of the SLS modulus E^* to frequency changes is highest for such ω, where the expression $\omega\tau_1$ (for given retardation time τ_1) is comparable with 1, and becomes negligible for frequencies 10 – 100 times lower or higher. Thus, more model bodies must be usually combined in order to describe the response of viscoelastic materials in wider intervals of frequencies.

Remark: The described model is the Kelvin-Voigt variant of the Standard linear Solid. There is also the Maxwell variant, consisting of a Maxwell body in parallel with a spring.

<u>The general Kelvin-Voigt body</u>

This body is obtained by arranging several K-V bodies in series (Fig. 4a without the spring E_0). The same stress $\sigma(t)$ acts in all K-V bodies, each having its complex modulus and phase angle. The total strain is calculated as the vector sum of strains of individual elements (see also Fig. 3). The resultant phase angle δ and the total elastic modulus are obtained in a similar way as above. The formulae are not given here, but can be obtained easily from Eqs. (10), (11) for the general Standard Linear Solid, just by omitting the modulus E_0 in each.

The general Maxwell body

This body is obtained by arranging several Maxwell bodies in parallel (see Fig. 4b without spring E_0). Each Maxwell body has its complex modulus and phase angle. The strain $\varepsilon(t)$ in all M-bodies is the same. The total stress is calculated as the vector sum of stresses in individual bodies. The resultant phase angle is obtained from the ratio of the total out-of-phase stress component to the total in-phase component. The complex modulus E^* and tan δ are derived in a similar way as above. Again, the formulae are not given here, but can be obtained from Eqs. (12) and (13) by omitting the modulus E_0 in each expression.

The general Standard Linear Solid (GSLS)

The general Kelvin-Voigt body (with dashpots in all K-V elements) cannot properly describe the instantaneous deformation on loading, while the general Maxwell body cannot describe the final (asymptotic) deformation after long time of loading, and recovery after unloading. An improvement is obtained by replacing one body in the above general models by a spring. There is a Kelvin-Voigt variant and a Maxwell variant of this general standard linear solid.

The Kelvin-Voigt variant of GSLS (Fig. 4a)

The formulae for phase angle and complex modulus can be derived in a similar way as for the single standard linear solid above:

$$\tan\delta = \frac{\sum_i \frac{1}{E_i}\frac{\omega\tau_i}{1+(\omega\tau_i)^2}}{\frac{1}{E_0}+\sum_i \frac{1}{E_i}\frac{1}{1+(\omega\tau_i)^2}} \quad , \qquad (10)$$

$$\frac{1}{E^*} = \sqrt{\left(\frac{1}{E_0}+\sum_i \frac{1}{E_i}\frac{1}{1+(\omega\tau_i)^2}\right)^2+\left(\sum_i \frac{1}{E_i}\frac{\omega\tau_i}{1+(\omega\tau_i)^2}\right)^2} \quad , \qquad (11)$$

where E_0 is the elastic modulus of the „lonely" spring; the index i in the sums varies from 1 to the number n of Kelvin-Voigt bodies.

The Maxwell variant of the GSLS (Fig. 4b)

The resultant expresions are:

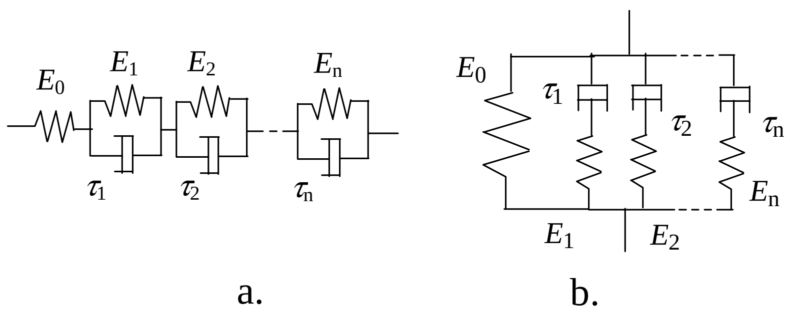

Fig. 4 General Standard Linear Solid:
a) Kelvin-Voigt variant, b) Maxwell variant.

$$\tan\delta \;=\; \frac{\displaystyle\sum_i E_i \frac{\omega\tau_i}{1+(\omega\tau_i)^2}}{\displaystyle E_0 + \sum_i E_i \frac{(\omega\tau_i)^2}{1+(\omega\tau_i)^2}} \quad , \qquad (12)$$

$$E^* \;=\; \sqrt{\left(E_0 + \sum_i E_i \frac{(\omega\tau_i)^2}{1+(\omega\tau_i)^2}\right)^2 + \left(\sum_i E_i \frac{\omega\tau_i}{1+(\omega\tau_i)^2}\right)^2} \quad . \qquad (13)$$

3. Practical part

There are several ways how to find the constants in a viscoelastic model from experimental data. A simple way is to minimize the squared differences between the E^* (or $\tan\delta$) values measured for several frequencies ω, and those calculated (for the same values ω) from the model, using, for example, Eqs. (10) and (11) written for the chosen number of elements. Excel's Solver or any other curve-fitting program can be used. As the response of viscoelastic materials is described by two quantities, always both curves, $E^*(\omega)$ and $\tan\delta(\omega)$ should be fitted. If only one of the functions is fitted (for example, $\tan\delta$), this fit can be very good, but – in some cases – the pertinent constants do not fit the other function (E^*) well. It can be recommended to fit $\tan\delta$ first, then E^*, then again $\tan\delta$, etc. Often 2 – 6 steps are sufficient.

In the choice of the model, two things should be kept in mind:

1. Despite of different arrangement of springs and dashpots, both variants of the general Standard Linear Solid (i.e. the Kelvin-Voigt and Maxwell variant) can describe the response of a viscoelastic material equally well.
2. The number of K-V (or M) bodies in the model can be chosen with respect to the shape of the empirical $\tan\delta(\omega)$ curve. For the curve with one „peak“, a simple Standard Linear Solid will be sufficient. A curve with two local maxima or steps can be approximated by a spring in series with two K-V bodies, etc.

The situation is illustrated in Fig. 5. The complex modulus and phase angle, plotted here as functions of angular frequency, correspond to the general Standard Linear Solid consisting of a spring and three Kelvin-Voigt elements in series (Fig. 4a), with the parameters chosen as E_0 = 1 GPa (spring alone), E_1 = 3 GPa, τ_1 = 0.1s, E_2 = 3 GPa, τ_2 = 1 s, and E_3 = 5 GPa, τ_3 = 10 s, using formulae (9) and (10). In the central part of the $\tan\delta(\omega)$ curve, we can see three steps. This is because the $\tan\delta(\omega)$ curve for a single standard linear solid resembles a Gaussian curve with one peak, while the body in our example can be interpreted as three standard linear solids in series. The distances between the peaks correspond to the steps between the relaxation times τ_i for individual K-V elements. For more flat curves, more elements are necessary. In the limit case of infinite number of elements, we could arrive at the continuous probability density of $E(\tau)$. However, good approximation for real materials is often obtained with a discrete model with a reasonably small number of elements, and the work with such models is much simpler than with probability densities (which must often be expressed in tabelar form). The discrete models are especially suitable for materials with more or less pronounced local maxima or steps in $\tan\delta(\omega)$.

Figure 5 also demonstrates the equivalence of the Kelvin-Voigt and Maxwell variant of the Standard Linear Solid. The curves in Fig. 5 are also plotted using formulae (12), (13) for

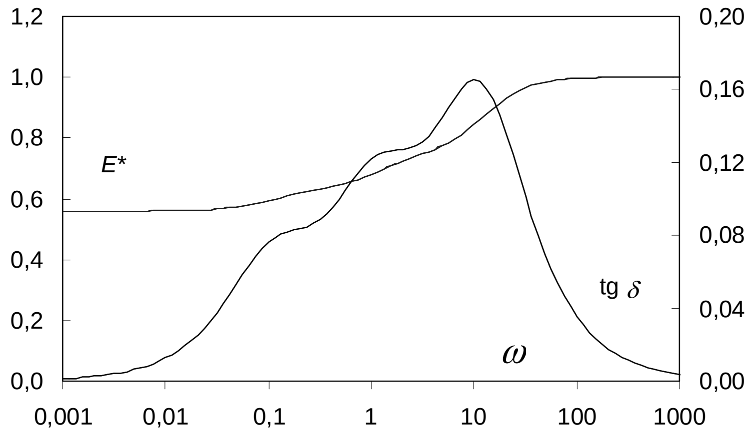

Fig. 5 Elastic modulus and phase angle as functions of frequency. Comparison of the Kelvin-Voigt and Maxwell variant of the standard linear solid.

the Maxwell variant (Fig. 4b) with constants: $E_0{}'$ = 0,560747 GPa, $E_1{}'$ = 0,258189 GPa, $\tau_1{}'$ = 0,0745872 s, $E_2{}'$ = 0,113314 GPa, $\tau_2{}'$ = 0,844688 s, $E_3{}'$ = 0,0677503 GPa, $\tau_3{}'$ = 8,900317 s. (These constants were obtained by minimizing the squared differences between the E* (and tan δ) values of the K-V and M variants. As we can see, both models overlap perfectly. The equivalence of K-V and M version of the general Standard Linear Solid is quite general. One can thus choose such model, which will better suit to the assumed use. (A remark. The characteristic relaxation times in the Maxwell variant were similar, but not identical with those of the K-V variant. In principle, it is possible to choose and fix the characteristic values τ_i in the model, and to fit only the moduli. However, the fit is often worse.

4. Conclusion

The response of polymeric materials to loading can be described by means of discrete linear viscoelastic models, such as the general Kelvin-Voigt body and the general Maxwell body, and the Kelvin-Voigt or Maxwell variant of the general Standard Linear Solid. In the paper, the formulae for the elastic modulus E* and phase angle δ of these bodies under harmonic loading were presented. These formulae enable easy determination of all constants in the model by fitting the $E^*(\omega)$, $\tan\delta(\omega)$ data measured for various frequencies ω. This was shown on an example, as well as the equivalency of K-V and M variants of the general standard linear solid. Also the recommendations for the choice of a model were given in the paper.

5. Acknowledgment

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