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RELATIONS BETWEEN FREQUENCY EQUATIONS OF BEAMS

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Summary: *The analysis of the structure of frequency expressions corresponding to an arbitrarily restrained uniform single-span Euler-Bernoulli beam results in derivation of simple formulas, which allow to express the frequency equation of a beam with more complex boundary conditions in terms of fundamental frequency expressions corresponding to beams with classical boundary conditions, such as clamped, pinned, free and guided.*

1. Introduction

The purpose of the present paper is to demonstrate a simple method that allows one to find the frequency equations (characteristic equations) for a single-span uniform Euler-Bernoulli beam with rotationally and/or translationally flexible supports directly from the most simple frequency equations $\varphi_i(\lambda) = 0$, $i = 1,2,...,6$; corresponding to classical boundary conditions. By classical boundary conditions we mean end configurations such as pinned, clamped, guided or free, in which all the four end compliances are zero or infinite.

2. Frequency equations revisited

We consider an arbitrarily supported uniform Euler-Bernoulli beam. Modal and spectral properties are given by the relevant homogeneous eigenvalue boundary problem. For harmonic vibration the governing equation is

$$w^{IV}(x) - \lambda^4 w(x) = 0, \tag{1}$$

and in the case of generally restrained beam the boundary conditions are

$$\begin{aligned} &w(0) + T_1 w^{III}(0) = 0, w^I(0) - R_1 w^{II}(0) = 0, \\ &w(1) - T_2 w^{III}(1) = 0, w^I(1) + R_2 w^{II}(1) = 0. \end{aligned} \tag{2}$$

Here x is the nondimensional spatial variable normalized by the beam's length L and

$$\lambda^4 = \frac{L^4 \omega^2 \rho A}{EJ}, \quad T_i = \frac{EJ}{K_{ti} L^3}, \quad R_i = \frac{EJ}{K_{ri} L},$$

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where T_i and R_i are the nondimensional translational and rotational compliances, respectively, λ is the nondimensional frequency parameter, K_{ti} and K_{ri} are the translational and the rotational spring constants, respectively. Subscript 1 corresponds to the left end $x = 0$, and the subscript 2 to the right end $x = 1$.
Substitution of the general solution of the differential equation (1)

$$w(x) = A\cos\lambda x + B\sin\lambda x + C\cosh\lambda x + D\sinh\lambda x \tag{3}$$

into the boundary conditions yields a homogeneous system of linear equations for unknown costants A, B, C and D, viz.,

$$\mathbf{G}(\lambda)\mathbf{a} = \mathbf{0}, \tag{4}$$

where $\mathbf{a}=[A,B,C,D]^T$ and $\mathbf{G}(\lambda)$ is a matrix of order 4 x 4, the elements of which are transcendental functions of the nondimensional frequency parameter λ. The form of $\mathbf{G}(\lambda)$ depends on the specified boundary conditions. In the case of a generally restrained beam one has

$$\mathbf{G}(\lambda) =$$
$$\begin{bmatrix} 1 & -\lambda^3 T_1 & 1 & \lambda^3 T_1 \\ \lambda^2 R_1 & \lambda & -\lambda^2 R_1 & \lambda \\ \cos\lambda - \lambda^3 T_2 \sin\lambda & \sin\lambda + \lambda^3 T_2 \cos\lambda & \cosh\lambda - \lambda^3 T_2 \sinh\lambda & \sinh\lambda - \lambda^3 T_2 \cosh\lambda \\ -\lambda(\sin\lambda + \lambda R_2 \cos\lambda) & \lambda(\cos\lambda - \lambda^3 T_2 \sin\lambda) & \lambda(\sinh\lambda + \lambda R_2 \cosh\lambda) & \lambda(\cosh\lambda + \lambda R_2 \sinh\lambda) \end{bmatrix} \tag{5}$$

For a non-trivial solution of equation (4), the coefficient matrix $\mathbf{G}(\lambda)$ must be singular, which leads to the following frequency equation for the unknown frequency parameter:

$$F(T_1, R_1, T_2, R_2) = \det[\mathbf{G}(\lambda)] = 0. \tag{6}$$

If the compliance Ti and/or Ri acquires the value of zero or infinity, we always substitute these limit values explicitlly in the frequency expression F as arguments. The symbols Ti and/or Ri are used only when the values of Ti and/or Ri are positive real. Thus, the frequency expression $F(0,0,0,0)$ corresponds to the clamped-clamped beam, $F(\infty,\infty,\infty,\infty)$ to the free-free beam, $F(\infty,R_1,T_2,\infty)$ to a beam with rotational spring on the left end and translational spring on the right end, etc.

Our aim is to find relations among frequency equations corresponding to closely related boundary conditions. For this purpose it is not appropriate to consider the equations $\varphi_i(\lambda)=0$ and $-\varphi_i(\lambda)=0$ to be equivalent. Neither can multiplicative factors be ignored, and therefore we shall distinguish between the equations $\varphi_i(\lambda)=0$, $2\varphi_i(\lambda)=0$ and $\lambda^n\varphi_i(\lambda)=0$. Further, let us accept the following rules that are to be observed when forming the frequency equations and frequency determinants:

Rule 1) The boundary conditions will be used in the order indicated by equations (2): that is, the first two rows of the matrix $\mathbf{G}(\lambda)$ result from the boundary conditions on the left side, while the last two rows of $\mathbf{G}(\lambda)$ result from boundary conditions on the right side of the beam.

Rule 2) When the value of the non-dimensional compliance is infinity, it is understood that the boundary condition $w(x) \pm T_i w^{III}(x) = 0$ or $w^I(x) \pm R_i w^{II}(x) = 0$ is reduced to $w^{III}(0) = 0$,

$-w^{III}(1)=0$, $-w^{II}(0)=0$ or $w^{II}(1)=0$. In other words, we keep the signs of the dynamical boundary conditions exactly as they appear in equations (2).

The above rules guarantee that the signs of the frequency expressions are under control and are not changed randomly: e.g., either due to possible exchanges of the order in which the boundary conditions are used to form the frequency equations or due to the change of the sign of the simplified equations (2). Under the above conventions, the elements of the matrix **G**(λ) for various boundary conditions are given by the row vectors summarized in Table 1.

Table 1: Entries of matrix **G**(λ) for various boundary conditions

Row of **G**	Compliance		
	Zero	Infinite	Finite $0<Ti<\infty$, $0<Ri<\infty$
First	$\mathbf{d}_0(0)$	$\mathbf{d}_3(0)$	$\mathbf{d}_0(0) + T_1\,\mathbf{d}_3(0)$
Second	$\mathbf{d}_1(0)$	$-\mathbf{d}_2(0)$	$\mathbf{d}_1(0) - R_1\,\mathbf{d}_2(0)$
Third	$\mathbf{d}_0(1)$	$-\mathbf{d}_3(1)$	$\mathbf{d}_0(1) - T_2\,\mathbf{d}_3(1)$
Fourth	$\mathbf{d}_1(1)$	$\mathbf{d}_2(1)$	$\mathbf{d}_1(1) + R_2\mathbf{d}_2(1)$

where

$$\mathbf{d}_i(x) = \lambda^i \frac{d^i}{dx^i}[\cos\lambda x, \sin\lambda x, \cosh\lambda x, \sinh\lambda x],$$

$$\mathbf{d}_0(0) = [1,0,1,0],\quad \mathbf{d}_1(0) = \lambda[0,1,0,1],\quad \mathbf{d}_2(0) = \lambda^2[-1,0,1,0],\quad \mathbf{d}_3(0) = \lambda^2[0,-1,0,1]$$

$$\mathbf{d}_0(1) = [\cos\lambda, \sin\lambda, \cosh\lambda, \sinh\lambda],\quad \mathbf{d}_1(1) = \lambda[-\sin\lambda, \cos\lambda, \sinh\lambda, \cosh\lambda],$$

$$\mathbf{d}_2(1) = \lambda^2[-\cos\lambda, -\sin\lambda, \cosh\lambda, \sinh\lambda],\quad \mathbf{d}_3(1) = \lambda^3[\sin\lambda, -\cos\lambda, \sinh\lambda, \cosh\lambda]$$

3. Relations among frequency equations

As is well known from the elementary theory of determinants, the determinant is a linear function of its rows (columns). For illustration, this means that

$$\begin{vmatrix} a_1+\alpha b_1 & a_2+\alpha b_2 \\ a_3 & a_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} + \alpha \begin{vmatrix} b_1 & b_2 \\ a_3 & a_4 \end{vmatrix} \tag{7}$$

From the structure of the matrix **G**, equation (5), and from the Table 1, it is obvious that for a restrained beam at least one row of the frequency determinant is a linear combination of two row vectors $\mathbf{d}_{i-1}(i-1)$ and $\mathbf{d}_{4-i}(i-1)$. By applying the above property (7) of determinants to the frequency expressions of the restrained beams, the following formulas are obtained:

$$F(T_1,.,.,.) = F(0,.,.,.) + T_1 F(\infty,.,.,.),\quad F(.,R_1,.,.) = F(.,0,.,.) + R_1 F(.,\infty,.,.),$$
$$F(.,.,T_2,.) = F(.,.,0,.) + T_2 F(.,.,\infty,.),\quad F(.,.,.,R_2) = F(.,.,.,0) + R_2 F(.,.,.,\infty) \tag{8}$$

Here the dots denote zero, positive real or infinite values of explicitly not specified compliances. According to formulas (8), the frequency expressions are linear functions of each of the finite compliance. In other interpretation, the frequency expression, seen as function of the finite compliance, is a linear combination of two other, simpler frequency expressions. One of them is the frequency expression for a beam, in which the corresponding compliance is set to zero, and the other one is the frequency expression for a beam, in which the same compliance is set to infinity. The more arguments of the function $F(T_1,R_1,T_2,R_2)$ are zero or infinity, the simpler the corresponding frequency equation is. As the finite compliance appears on the right side of equations (8) solely in the role of a multiplicative factor, each finite compliance can be eliminated from the frequency expression by subsequent expansion of frequency expressions in terms of each finite compliance. If this process of linear expansion is repeated, one ends up with the frequency expression where only the classical frequency expressions $\varphi_i(\lambda)$ appear. For details and examples see Nánási (1994).

4. Conclusion

Relations (8) are the desired formulas, describing the relations among frequency expressions corresponding to various boundary conditions. The usage of formulas (8) in conjunction with Table 2 allows one to build up the frequency equation of an arbitrarily restrained beam from only classical frequency expressions without the need to perform tedious evaluations of frequency determinants.

Table 2: Modified classical frequency expressions to be applied in formulas (8)

Right end → Left end ↓	-Clamped F(.,.,0,0)	-Free F(.,.,∞,∞)	-Guided F(.,.,∞,0)	-Pinned F(.,.,0, ∞)
Clamped- F(0,0,0,0)	$F(0,0,0,0) = -2\lambda^2\varphi_4$	$F(0,0,\infty,\infty) = 2\lambda^6\varphi_3$	$F(0,0,\infty,0) = 2\lambda^5\varphi_5$	$F(0,0,0,\infty) = 2\lambda^3\varphi_6$
Free- F(∞,∞,.,.)	$F(\infty,\infty,0,0) = 2\lambda^6\varphi_3$	$F(\infty,\infty,\infty,\infty) = -2\lambda^{10}\varphi_4$	$F(\infty,\infty,\infty,0) = -2\lambda^9\varphi_5$	$F(\infty,\infty,0,\infty) = -2\lambda^7\varphi_6$
Guided- F(∞,0,.,.)	$F(\infty,0,0,0) = 2\lambda^5\varphi_5$	$F(\infty,0,\infty,\infty) = -2\lambda^9\varphi_5$	$F(\infty,0,\infty,0) = -4\lambda^8\varphi_1$	$F(\infty,0,0,\infty) = 4\lambda^6\varphi_2$
Pinned- F(0, ∞,.,.)	$F(0,\infty,0,0) = 2\lambda^3\varphi_6$	$F(0,\infty,\infty,\infty) = -2\lambda^7\varphi_6$	$F(0,\infty,\infty,0) = 4\lambda^6\varphi_2$	$F(0,\infty,0,\infty) = 4\lambda^4\varphi_1$

Table 3: Classical frequency expressions in the traditional form

$\varphi_1(\lambda) = \sin\lambda \, \sinh\lambda$	= 0 for pinned-pinned and guided-guided beams
$\varphi_2(\lambda) = \cos\lambda \, \cos\lambda$	= 0 for pinned-guided beam
$\varphi_3(\lambda) = \cos\lambda \, \cos\lambda + 1$	= 0 for clamped-free beam
$\varphi_4(\lambda) = \cos\lambda \, \cos\lambda - 1$	= 0 for clamped-clamped and free-free beams
$\varphi_5(\lambda) = \sin\lambda \, \cosh\lambda + \cos\lambda \, \sinh\lambda$	= 0 for clamped-guided and free-giuded beams
$\varphi_5(\lambda) = \sin\lambda \, \cosh\lambda - \cos\lambda \, \sinh\lambda$	= 0 for clamped-pinned and free-pinned beams

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6. List of references

Nánási, T. (1994) Relations between frequency equations of single-span beams. *Journal of Sound and Vibration,* 171, 3, pp. 323-334.