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HEAT TRANSFER WITH FUZZY COEFFICIENTS

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Summary: *The fuzzy set theory found its application in numerous industrial fields. In this article, we present a simulation of heat transfer in a material whose thermal coefficients are expressed in terms of fuzzy numbers. The proposed method of solution is based on the engineers' presumptions and utilizes the basic fuzzy arithmetic operations, which is shown in an illustrative example of the nonstationary case of heat transfer in a concrete wall.*

1. Introduction

The uncertainty in engineering problems can be tackled in several ways, among which the probabilistic approach is the most common. However, it is quite a usual case that gathering satisfactorily large statistical data is rather expensive, or merely impossible, which leaves the data uncertain. In such cases, it is difficult to justify the use of statistical methods. On the contrary, the fuzzy set theory a priori takes into account the uncertain, or vague, nature of data.

Once the theory of fuzzy sets is considered, the imprecise quantities can be treated in terms of fuzzy numbers. The traditional definition of a fuzzy number is splitting the fuzzy number into so-called α-cuts, which correspond to a membership degree denoted by α. α ranges in the interval $< 0; 1 >$. For $\alpha = 0$ the widest possible range of the quantity is obtained. For $\alpha = 1$ we assume the quantity to acquire the most common value. The bounds of a α-cut of a fuzzy number, which represents the value of a quantity, is acquired either by a feasible experiment or by experience, which means there is no need to carry out experiments in order to generate large sets of probable events.

In this paper, we propose a method for simulation of heat transfer, where the material and load coefficients are set as fuzzy numbers. The objective of the work is to provide the heat distribution with its lower and upper bounds throughout a structure and time, which takes into account the imprecision of input data. The method and the interpretation of results are explained and shown in an illustrative numerical example of heat transef in a concrete wall.

2. Fuzzy arithmetics

In the following, these notations will be used; the membership function of a fuzzy set A is μ_A, where $\mu_A = \sup_{x \in A_\alpha} \alpha$. A_α , for any fuzzy set A, denotes its α-level set for $\alpha \in (0; 1]$, i.e., $A_\alpha = \{x \in \mathbf{R} : \mu_\mathbf{A}(\mathbf{x}) \geq \alpha\}$.

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2.1 Fuzzy number

A fuzzy set A over the real line is called a fuzzy number if the following conditions are fulfilled;

- A is normal, i.e., there is a member with membership 1;
- A is compactly supported, i.e., the support of A is bounded;
- The membership function μ_A is upper semi-continuous and quasi-concave, which ensures that the α-cuts A_α are closed intervals.

A fuzzy number A may also be defined as a set of all its α-cuts, denoted by $[\underline{a}_\alpha, \overline{a}_\alpha]$.

2.2 Fuzzy arithmetics

The usual arithmetic operations, such as addition (+), subtraction (-), multiplication (×), and division (/) are extended to fuzzy numbers with the use of the extension principle. Knowing that a fuzzy number may be equally represented by its α-cuts, which are closed intervals of real numbers, and knowing how to use arithmetic operations on closed intervals, the above mentioned four operations applied to two fuzzy numbers A and B may be easily defined, e.g. (Wagenknecht et al., 1999). These basic fuzzy operations are used in the proposed method.

3. Heat transfer

The governing equation of heat transfer is derived from Fourier's law of heat conduction $g_{x_i} = -k_{x_i}\frac{\partial T}{\partial x_i}$, where q_{x_i} is the heat flow conducted per unit area in the direction of the axis x_i. Assuming the heat flow equilibrium in the analyzed area, we obtain

$$\frac{\partial}{\partial x}\Big(k_x\frac{\partial T}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(k_y\frac{\partial T}{\partial y}\Big) + \frac{\partial}{\partial z}\Big(k_z\frac{\partial T}{\partial z}\Big) + f = \rho c\frac{\partial T}{\partial t}, \tag{1}$$

where f is the rate of heat generated per unit volume, ρ is the mass density and c is the heat capacity. The boundary condition may be defined in the terms of either Dirichlet type $T = g$, where g is a function prescribing temperature on the boundary, or Neumann type $\frac{dT}{dn}k = j_q$, which defines the amount of heat transfered through the boundary. To obtain the governing equation for FEM we use Galerkin method, which yields the governing equation in the form

$$\boldsymbol{C}\dot{\boldsymbol{r}} + \boldsymbol{K}\boldsymbol{r} = \boldsymbol{f} + \boldsymbol{b}, \tag{2}$$

where $\boldsymbol{C}$ is the capacity matrix, $\boldsymbol{K}$ is the conductivity matrix, $\boldsymbol{r}$ is the vector of nodal values, $\boldsymbol{f}$ is the vector of internal heat supplies and b is the vector describing the boundary conditions. The equation (2) describes a general nonstationary distribution of temperature.

The equation (2) is used for solving the nonstationary case. The functions f and b are now functions of time. Therefore, the initial condition (**??**) is considered. Assume that the solution r_{i-1} at time t_{i-1} is known. Then, consider a time step $\Delta t = t_i - t_{i-1}$ at which the vector of unknowns is approximated linearly, which leads to

$$\Big[\boldsymbol{K}\psi + \frac{\boldsymbol{C}}{\Delta t}\Big]\boldsymbol{r}_i = \boldsymbol{q}_{i-1}(1-\psi) + \boldsymbol{q}_i\psi + \Big[\frac{\boldsymbol{C}}{\Delta t} - \boldsymbol{K}(1-\psi)\Big]\boldsymbol{r}_{i-1}. \tag{3}$$

It is desirable to choose ψ from the interval $1/2 \geq \psi \geq 1$. If ψ is a fixed value, then the relation (3) is a linear algebraic equation system for the unknowns r_i.

4. Proposed solution

The solution should consist of two stages. First, linguistic terms and imprecise information are translated to numeric values. Such procedures are proposed, for example, in (Valliappan & Pham, 1993; Valliappan & Pham, 1995). Second, the algorithm which models heat transfer is carried out. In this paper, only the second stage will be dealt with.

According to the fuzzy set theory, the material characteristics (heat conductivity k, heat capacity c and mass density ρ) are taken for fuzzy because of the nature of concrete, which is its inhomogeniety. These material characteristics are modelled in the terms of fuzzy numbers and as such are used in the finite element method analysis of heat transfer. The definition of a fuzzy number was proposed in section . The fuzzy material characteristics, then, are denoted by $[\underline{k}_\alpha, \overline{k}_\alpha]$, $[\underline{c}_\alpha, \overline{c}_\alpha]$ and $[\underline{\rho}_\alpha, \overline{\rho}_\alpha]$. Also temperature at the boundary is treated as fuzzy, stemming from the imprecise information, and bears the same denotation $[\underline{T}_\alpha, \overline{T}_\alpha]$.

Although the relation (2) is originaly a differential equation, it is, with the use of numerical integration, converted to a linear algebraic equation (3). Moreover, time is regarded as crisp value, not fuzzy. Thus we are not forced to use special techniques treating differential equations, e.g. (Ma et al., 1999; Kaleva, 1987; Buckley, 1992), and we can last with the basic fuzzy arithmetic operations mentioned in section .

The fuzzy matrices of capacity and conductivity in the finite element method are given by (4) and (5).

$$\boldsymbol{C} = \int_\Omega \rho c [\boldsymbol{N}]^T [\boldsymbol{N}] d\Omega, \tag{4}$$

$$\boldsymbol{K} = \int_\Omega \left(\left[\frac{\partial \boldsymbol{N}^T}{\partial x_i} \right] k_{x_i} \left[\frac{\partial \boldsymbol{N}}{\partial x_i} \right] \right) d\Omega, \qquad i = 1, 2, 3, \tag{5}$$

where N is the vector of shape functions. N is a crisp function, since we do not treat geometry and dimensions as fuzzy. For 1D and the linear approximation, the matrices of capacity C and conductivity K are derived as follows

$$\boldsymbol{C} = \frac{c\rho A}{l} \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \tag{6}$$

$$\boldsymbol{K} = \frac{kA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{7}$$

To determine the lower and upper bounds of the solution for (3), we choose the lower and upper bounds of c, ρ and k for (6) and (7), respectively. Which means that the upper bound of the capacity matrix $\boldsymbol{C}_{up}$ is acquired when the upper bounds of c_{up} and ρ_{up} are used

$$\boldsymbol{C}_{up} = \frac{c_{up}\rho_{up} A}{l} \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}. \tag{8}$$

The upper bound of the conductivity matrix $\boldsymbol{K}_{up}$ is acquired when the upper bound of k_{up} is used

$$\boldsymbol{K}_{up} = \frac{k_{up}A}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{9}$$

Similarly, to get the lower bounds of the conductivity $\boldsymbol{K}_{low}$ and capacity $\boldsymbol{C}_{low}$ matrices, the lower bound values of c, ρ and k are used

$$\boldsymbol{C}_{low} = \frac{c_{low}\rho_{low}A}{l}\begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \tag{10}$$

$$\boldsymbol{K}_{low} = \frac{k_{low}A}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{11}$$

In matrices (6) to (11), A and l denotes the cross-section area and the length of a finite element, respectively. Both these geometric characteristics are set as crisp, not fuzzy numbers.

To propose the algorithm for solving the problem of fuzzy heat transfer, we can proceed the following reasoning. If a material is being warmed, a combination of low heat conductivity, great heat capacity and high mass density is needed to obtain slowest warming rate of the material. When fastest warming rate is being sought, then, a combination of high heat conductivity, low heat capacity and mass density is to be used. If a material is being cooled, a combination of low heat conductivity, high heat capacity and high mass density is needed to get slowest cooling rate. And finally, when fastest cooling rate is sought, a combination of high heat conductivity, low heat capacity is desired. From this reasoning it is obvious that only two combinations of fuzzy matrices $\boldsymbol{K}$ and $\boldsymbol{C}$, defined by (8) to (11), in (3) are enough to secure the lower and upper bounds of the solution of fuzzy heat transfer. It also depends on the ration between the heat conductivity k and the heat capacity c and mass density ρ, but this fact does not need any special treatment.

In the 1D case, the derivation of the fuzzy governing equation is quite straigtforward. In the 2D and 3D cases, where heterogeniety usually occurs, it should be considered in advance which of the directions of heat transfer is of interest. The condition of thermal equilibrium must hold, and thus it is clear that a higher amount of heat transferred in one direction leads to a lower amount of heat which is transferred in another direction. Therefore, it is impossible to transfer maximum amounts of heat in all directions at one time, because this would violate the thermal equilibrium. Speaking in the terms of fuzzy numbers, upper bound of heat flux in one direction means lower bound of heat flux in the other direction, provided that we seek the maximum. This problem may also be understood the other way round and treated as an optimization task.

5. Numerical example

A concrete wall is considered. The thickness of the wall is 40 centimeters. The modal values of the material characteristics are quoted in Table 1 and are changed by ± 10 %. The geometry of the task is shown in Fig 1. The inner and outer surfaces of the wall are both at the temperature 20 °C changed by ± 10 % at the starting point of the analysis. During the first minute the inner surface is heated to 800 °C changed by ± 10 % and

conductivity	1.67 $J/(msK)$
heat capacity	840 $J/(kgK)$
mass density	2400 kg/m^3

Table 1: Material characteristics

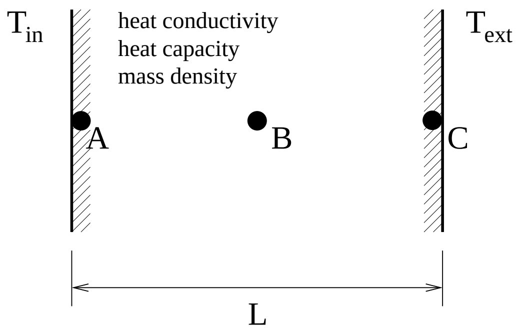

Figure 1: Example 1

maintains this temperature for one day. After the 24 hours from the initial heating the temperature of the inner surface decreases down to the initial value 20 °C changed by ± 10 % during next 24 hours and at that temperature it rests. The wall is investigated in the points A to C, where A and C are right below the surfaces and B is in the middle of the wall. Because the problem is axisymmetric, it is modeled with 1D elements along the axis.

The results are in the form of the nodal temperatures over the time of analysis in the A, B and C points in Figs 2 and 3 and the heat distributions in the wall after 1 hour and after 1, 2, 3 and 4 days in Figs 3 to 5, respectively. The membership function is scaled in grey shades from light grey for $\alpha \doteq 0$ to black for $\alpha = 1$.

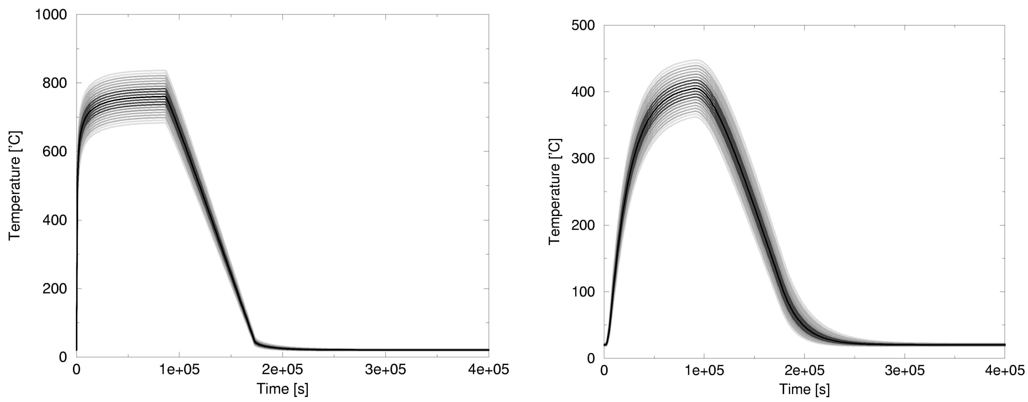

Figure 2: Temperature in A and B

6. Conclusions and discussion

In this paper, we propose a method for solving the problem of heat transfer with fuzzy material and load coefficients as another engineering application to prove the good applicability of the theory of fuzzy sets to engineering problems. The objective of this

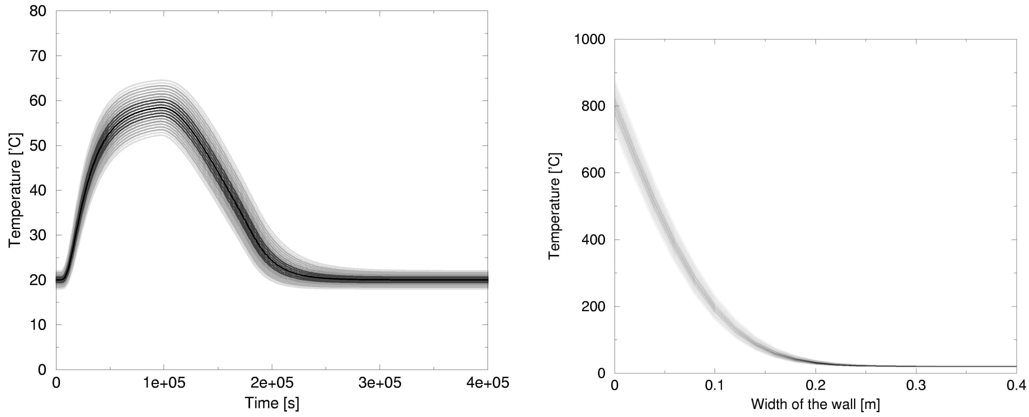

Figure 3: Temperature in C and temperature in wall after 1 hour

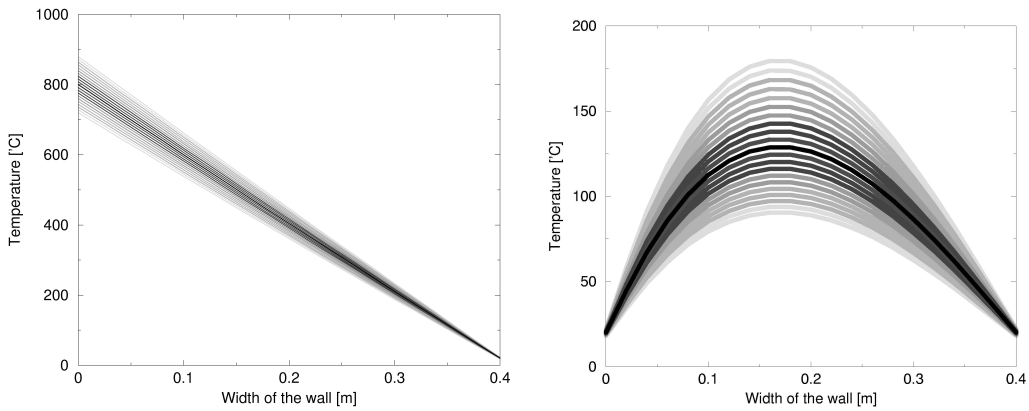

Figure 4: Temperature in wall after 1 day and 2 days

work is to assess the most and the least possible temperatures in investigated parts of structures over time due to imprecise information about the material and the load. The results are expressed in the terms of fuzzy numbers, where the membership function has the meaning of the degree of possibility. This analysis covers inhomogeniety, which is typical for most building materials, and the imprecise information of input data and their effect on the temperature distribution in structures. The computation is carried out with the use of fuzzy arithmetics on fuzzy level cuts. The number of α-cuts depends on how much it is to be learnt about the shape of the membership function of a nodal temperature. One should carefully choose the number of α-cuts. A large number of α-cuts may slow down the computation considerably. On the other hand, the knowledge of the modal value and the least lower and most upper bounds is not satisfactory. To remedy this situation we investigate the possibility of using the fast computation formulation of fuzzy numbers, also known as (L, R)-numbers, to this model. The arithmetics on these numbers has already been developed, see (Štemberk & Wagenknecht, 1999; Wagenknecht et al., 1999).

7. Acknowledgement

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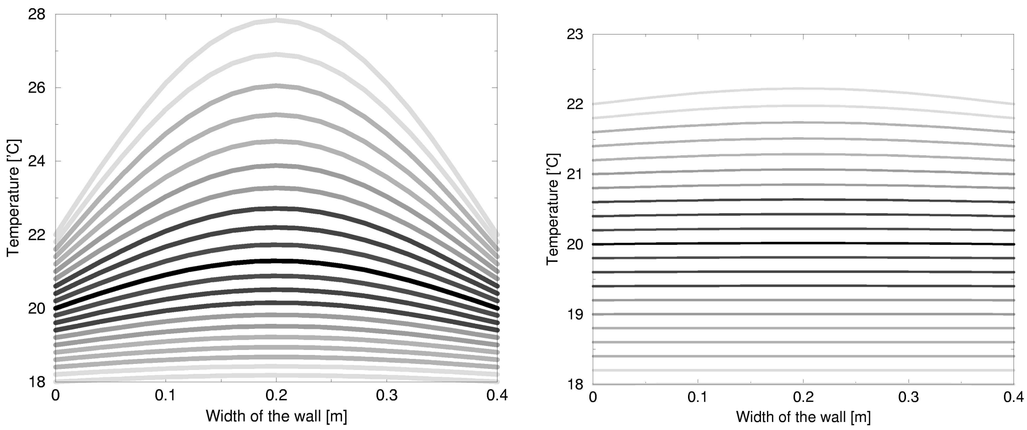

Figure 5: Temperature in wall after 3 days and 4 days

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