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# THE COMPARISON OF THE MODELS OF THE VIBRATION FOR CANTILEVER BEAM WITH A BACKSTOP WITH THE CONTINUOUSLY DISTRIBUTED AND LUMPED PARAMETERS

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**Summary:** *The contents of the paper is a comparison models of vibration of the cantilever beam with the backstop. The beam model is solved by analytical method and also by using finite element method. The backstop is on the end point of the free part of the beam and is solved as a spring with the meaning stiffness.*

## 1. Introduction

The main aim of the paper is to formulate a mathematical model and analyze of vibration of a cantilever beam with a free end, which is delimited by motion with a backstop. The requirement to solve this problem relates with an activity of a valve of a small piston compressor where in addition flow losses occurs. In generally so-called reed types of a valves are using in this type of compressors. These valves represents from design aspect cantilever beam with a small thickness of a various shapes.

In the solution of this problem is necessary to form the model as accurate as possible, which may be used in future to optimize valve parts of compressors. As noted previously, model is based on vibration of a cantilever beam whose free end is in contact with a solid part by his motion which determine a range of his deflection. This range present a backstop, which is also necessary to specify mathematically. In this paper, the backstop is perceive as a contact of a beam with a spring with a meaning stiffness in comparison with a stiffness of beam. This vibration may be solved by a model with continuously distributed parameters which arise from Bernoulli-Euler theory. A backstop whose mathematical description present a spring with a specific stiffness is mainly solved as a force applied to a beam in contact point. The value of this force varying in dependence on the deflection or on compression of the spring. This force acting only during contact of beam with

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backstop and is included in equation of vibration as a reversible force, what are addicted for example in [1], [2], [3]. The next possibility how to analyze this vibration is application of FEM, what represent a beam with a lumped parameters. This numerical method is in comparison with an analytical description of vibration of a beam more difficult because of time-consuming and proper choice of some parameters such as number of elements, formation of matrix of a damping, etc. However on the other side this method gives bigger possibility in areas where the analytical solution become considerably more complicated about what is dealing next part of this paper.

## 2. Analytical solution

Equation of motion of a forced bending vibration of a cantilever beam based on Bernoulli-Euler theory may be expressed in form

$$\rho.S(x)\frac{\partial^2 w}{\partial t^2}+E\frac{\partial^2}{\partial x^2}\left[J(x)\frac{\partial^2 w}{\partial x^2}\right]=q(x,t) \tag{1}$$

where $\rho$ [kg.m$^{-3}$] is the density, $S(x)$ [m$^2$] is the cross section area, $E$ [kg.s$^{-2}$.m$^{-1}$] is the Young's modulus and $J(x)$ [m$^4$] is the area moment of inertia, $q(x,t)$ [kg.s$^{-2}$] is the external load per unit length and $w$ [m] is beam deflection. In this equation are already applied some acceptable simplified assumption e.g. neglect of rotational inertia and also dismissed external and internal damping etc. Is evident that to motion of the beam will occur only in the direction of possible excitation and therefore the problem is solved as two-dimensional. And below we assume the uniform cross section over entire length of the beam. We assume the solution of equation (1) in form

$$w(x,t)=W(x)T(t) \tag{2}$$

where function $W(x)$ express the natural modes of vibration and function $T(t)$ determine the time change of vibration deflection. By the solution of the equation (1) we obtain function $W(x)$ in known form

$$W(x)=D_1\sinh(kx)+D_2\cosh(kx)+D_3\sin(kx)+D_4\cos(kx) \tag{3}$$

where

$$k^4=\Omega^2\frac{\rho S}{EJ} \tag{4}$$

where $\Omega$ is natural radian frequency. By declaring of boundary condition for the cantilever beam with free end

$$W_{(0)}=0,\quad \frac{\partial W_{(0)}}{\partial x}=0,\quad \frac{\partial^2 W_{(l)}}{\partial x^2}=0,\quad \frac{\partial^3 W_{(l)}}{\partial x^3}=0 \tag{5}$$

we would obtain the function $W(x)$ for every natural radian frequency. All external loading acting to the beam are on the right part of equation (1), what can be for example pressure or the force from spring dependent on beam deflection in the contact point. This force plays its role in particular solution of vibration. The homogenous solution is response to the initial condition of vibration. Their formulation are known in general and there is no point to deal with them now.

Other approach is possible to apply if we realize, that in the moment of the beam's free end contact with the backstop – spring, is coming up the case of cantilever beam vibration with a elastic bearing in the contact patch of beam with this backstop. This case of beam bedding is again solvable, if we assume the solution of function *W(x)* with specific boundary conditions, then it is possible to modify these condition for already mentioned flexible bedding. It's possible to express the frequency equation for this case of bedding, when boundary conditions in the place of flexible bedding are in form

$$\frac{\partial^2 W_{(x0)}}{\partial x^2} = 0\,, \quad EJ\frac{\partial^3 W_{(x0)}}{\partial x^3} = K.W_{(x0)} \tag{6}$$

where $K$ represents backstop stiffness and $x_0$ is the contact patch, that represents the force action with a stiffness $K$ in dependence on deflection in given place (Fig.1).

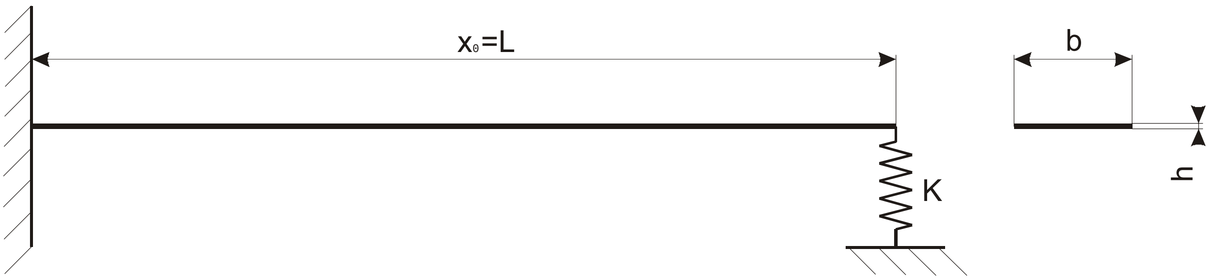


Fig. 1

From yet shown results, that in the case of positive beam deflection $w(x_0,t)$ in coordinate $x_0$ – contact patch, function *W(x)* represents the natural modes of vibration with a free end and at the moment of transition of deflection $w(x_0,t)$ to negative value, in the solution of equation (2) figures the function *W(x)* for beam with flexible bedding end. The moment of transition is then solved by function *T(t)*.

If we assume for simplicity, that none external force acts on the beam, the function *T(t)* will consist only of the homogenous part in the form

$$T(t) = A.\sin\Omega t + B.\cos\Omega t \tag{7}$$

where constants *A, B* are obtained from initial conditions of the solution. In this paper, as will be said later, is positive initial displacement *w(x,t)* from the static force $F_0$ acting on the free end considered. In the time of transition $t=\tau$ of the deflection to negative values in $x_0$, the initial condition for beam with flexible bedding will be exactly the deflection $w(x,\tau)$ and its corresponding derivation over time. This solution applies until the moment, when deflection in $x_0$ traverses again to positive values, where in the moment of transition, the deflection and her derivation are again initial conditions for the time function *T(t)* of the free end beam vibration. This process of "passing initial conditions" then continues analogical.

Based on initial conditions and by applying of Krylov's functions, whose considerably simplifies the work with the highest derivations, the frequency equation for the cantilever beam with the flexible bedding of the second end, where $x_0=l$, can be obtained

$$\sinh(k_n l).\cos(k_n l) - \cosh(k_n l).\sin(k_n l) = \frac{EJ}{K}.k_n{}^3.\left[1 + \cosh(k_n l).\cos(k_n l)\right] \qquad (8)$$

then function *W(x)* will be for every natural mode of vibration in the form

$$W_n(x) = E_{3n}\frac{1}{2}\left[\sinh(k_n x) - \sin(k_n x) - \frac{\sinh(k_n l) + \sin(k_n l)}{\cosh(k_n l) + \cos(k_n l)}\left(\cosh(k_n x) - \cos(k_n x)\right)\right] \qquad (9)$$

for *n=1,2,3,...* and constant $E_{3n}$ we obtain by standard way from the condition of orthogonality of natural modes.

The consequential solution of free beam vibration with the flexible bedding of the second end is in the form

$$w(x,t) = \sum_{n=1}^{\infty} W_n(x).T_n(t) \qquad (10)$$

Only for completeness we show, that if we would assume the solution of free cantilever beam vibration with free end in form (index *h*)

$$w^h(x,t) = \sum_{n=1}^{\infty} W_n^h(x).\left(A_n.\sin\Omega^h t + B_n.\cos\Omega^h t\right) \qquad (11a)$$

and the solution of free cantilever beam vibration with flexible bedding of the end (index *d*)

$$w^d(x,t) = \sum_{n=1}^{\infty} W_n^d(x).\left(C_n.\sin\Omega^d t + D_n.\cos\Omega^d t\right) \qquad (11b)$$

than for the transition from positive value of deflection of the end of beam ($x_0=l$) to negative value in time $t=\tau$, the values of constants *C, D* would be:

$$D_n = \frac{\sum_{n=1}^{\infty}\left(A_n.\sin\Omega^h\tau + B_n.\cos\Omega^h\tau\right).\int_0^l W_n^h(x)W_n^d(x)dx}{\int_0^l \left(W_n^d(x)\right)^2 dx} \qquad (12a)$$

$$C_n = \frac{\sum_{n=1}^{\infty}\left(A_n.\cos\Omega^h\tau - B_n.\sin\Omega^h\tau\right).\Omega^h.\int_0^l W_n^h(x)W_n^d(x)dx}{\Omega^d.\int_0^l \left(W_n^d(x)\right)^2 dx} \qquad (12b)$$

and for transition from negative value of deflection of the end of beam ($x_0=l$) to positive value in time $t=\tau^*$, the values of constants *A, B* would be:

$$B_n = \frac{\sum_{n=1}^{\infty}\left(C_n.\sin\Omega^d\tau^* + D_n.\cos\Omega^d\tau^*\right).\int_0^l W_n^h(x)W_n^d(x)dx}{\int_0^l \left(W_n^h(x)\right)^2 dx} \qquad (13a)$$

$$A_n = \frac{\sum_{n=1}^{\infty}\left(C_n.\cos\Omega^d\tau^* - D_n.\sin\Omega^d\tau^*\right)\Omega^d.\int_0^l W_n^h(x)W_n^d(x)dx}{\Omega^h.\int_0^l \left(W_n^h(x)\right)^2 dx} \tag{13b}$$

**3. Solution by finite element method**

Transition from analytical solution to FEM (discretization of continuum) is possible after we define the matrixes of stiffness **K** and mass **M**. As noted previously, solution is for object-lesson without inner and outer damping at the moment and so there is no need to define the matrix of damping **B**. It's evident that it's beam element, which have 2 degrees of freedom, translation and rotation of element. It would be possible to deduce the matrix of stiffness of beam element $\mathbf{K}_e$ and the matrix of mass of beam element $\mathbf{M}_e$ by means of cubical function for beam element type by some of known method e.g. [4],.

$$\mathbf{K_e} = EJ_e\begin{bmatrix} \frac{12}{l_e^3} & \frac{6}{l_e^2} & -\frac{12}{l_e^3} & \frac{6}{l_e^2} \\ & \frac{4}{l_e} & -\frac{6}{l_e^2} & \frac{2}{l_e} \\ & sym & \frac{12}{l_e^3} & -\frac{6}{l_e^2} \\ & & & \frac{4}{l_e} \end{bmatrix}, \quad \mathbf{M}_e = \frac{\rho S_e l_e}{420}\begin{bmatrix} 156 & 22l_e & 54 & -13l_e \\ & 4l_e^2 & 13l_e & -3l_e^2 \\ & sym & 156 & -22l_e \\ & & & 4l_e^2 \end{bmatrix} \tag{14}$$

where $E$ is the Young's modulus and $J_e$ is the area moment of inertia of beam element and $l_e$ is length of element. If we reconsider that soled problem is without damping effect, it is possible to write the equation of motional in form

$$\mathbf{M\ddot{w}} + \mathbf{Kw} = \mathbf{f} \tag{15}$$

where displacement and rotary vector of nodal points and external forces vector is in form

$\mathbf{w} = \begin{Bmatrix} w_1 \\ \varphi_1 \\ \vdots \\ w_n \\ \varphi_n \end{Bmatrix}$ and $\mathbf{f} = \begin{Bmatrix} F_1 \\ M_1 \\ \vdots \\ F_n \\ M_n \end{Bmatrix}$, where $n$ is number of beam elements and $F_i$ and $M_i$ are external forces and external moments.

For the discrete model the spring substituting the backstop, features as an external force at the end point and it's only in the case when the end point of the beam gains negative values. The dependency of this force on displacement of the end point can be graphically expressed as

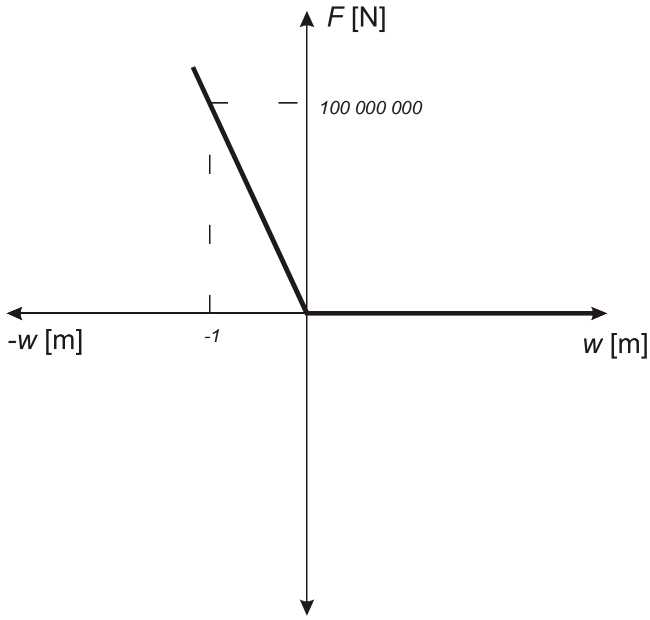


Fig. 2

It means that external force vector **f**, if we don't consider excitation, will be zero vector and in the case of transition of the end point to negative deflection it will have values $\mathbf{f} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -K.w_n \\ 0 \end{Bmatrix}$,

where constant $K$ represents already mentioned spring stiffness.

**4. Analysis achievement results**

A steel beam was chosen, as a concrete example, with the length 0,04 m with constant cross section area 0,008m x 0,000381m. The spring stiffness is $K$ = 100.10$^6$ kg/s$^2$. The beam was cantilevered at the one end (Fig. 1). The initial conditions were identical for analytical as well as for numerical solution, namely the initial velocity of the beam was zero along all length and the bending axis of initial displacement was derived from static force $F_o$ = 2 N acting at the free end of the beam. The solution of continuously distributed parameters as well as FEM was realized by software MATLAB and also the beam was modeled in software ANSYS and the results were compared. The beam was discretized by 20 elements.

On the graphical dependency (Fig.3), the beam deflection in time at the end point $x=l$ is showed (green), which has the contact with the spring and also the deflection in $x= l/2$ (red). This dependency (Fig. 3) of deflection in time is the result of vibration of the beam with continuously distributed parameters.

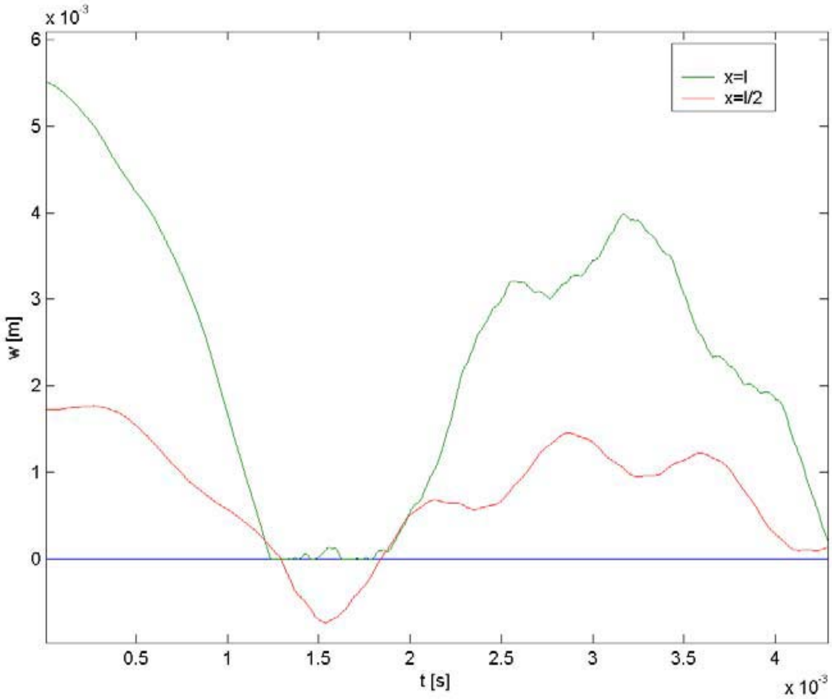


Fig. 3

From showed plot it's possible to watch the path of the end point of the free part of the beam, painted with green color, which is coming in the contact with the spring, where for its great stiffness against the beam stiffness, is coming to minimal overshoot to negative values of the beam deflection, which maximal value is up to $2{,}5\text{x}10^{-7}$ m. By red color, the deflection in the middle part of the beam $x = {}^{l}/_{2}$ is represented, where it is coming to overshoot and consequently to spring back to the positive values of deflection. It is interesting to remark that in the moment of contact with the spring in the time range between $1{,}25\text{x}10^{-3}$s and $1{,}79\text{x}10^{-3}$s it is coming to local rebounds from the spring, even though the rest of the beam with its inertia, permanently continues in motion direct down.

On Fig. 4 the dependency of the beam deflection solved by equation (1) and also time dependency of the beam deflection obtained by finite element method – equation (15) are showed.

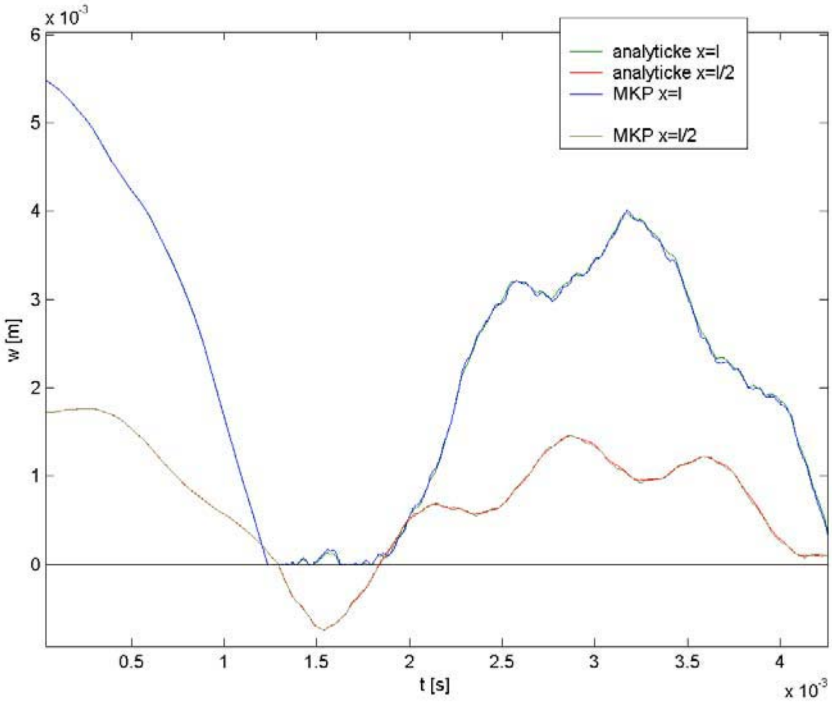


Fig. 4

From Fig 4 is clear that analytical solution and FEM solution achieve good agreement what prove also difference of deflections $\Delta w$ in time $3{,}17 \mathrm{x} 10^{-3}$ s, what is the region with maximal re-bounce of the end point of beam (blue and green curve). Value of this difference is $\Delta w = 3 \mathrm{x} 10^{-5}$ m. The cantilever beam with the same parameters was simulated also in ANSYS, with nonlinear spring connected to the ground applied on the free end of the beam. The characteristic of this spring is evident from figure 2. Time dependence of beam deflection in the end point $x=l$ (blue curve) and the deflection in the beam point $x= {}^{l}/_{2}$ (yellow curve) are shown on from figure 5.

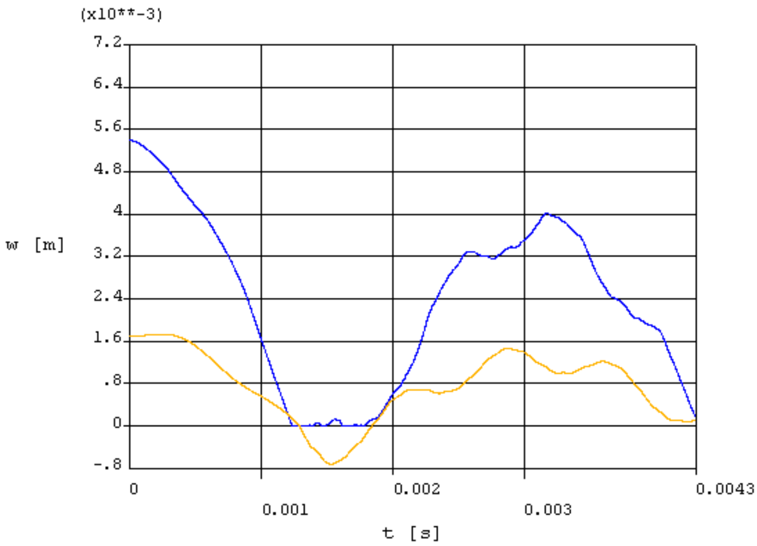


Fig. 5

On the Fig. 5 we can see a good agreement of the results of free vibration of cantilever beam with the backstop at the free end, which was replaced by spring with the meaning stiffness, solved as a vibration of the beam with continuously distributed parameters and the vibration of the beam solved by finite element method.

## 5. Conclusion

In present paper undamped vibration of the cantilever beam with the backstop on the free end was analyzed. The problem was solved with model of the beam with continuously distributed parameters, the principle of the separation of the beam vibration in two cases was used. The first case relates with the vibration with the free end where the necessary assumption for this case was the positive value of deflection at the free end. The second case of vibration occurred in time domain when the value of deflection at the free end was negative namely when the problem of vibration of the cantilever beam with the flexible bedding at the other end was solved. Because for demonstration we didn't consider external excitation, only homogenous solution of vibration in both cases of beam was applied, where they "transferred the initial conditions of vibration" to each other. This model of the beam vibration was compared with the model solved as vibration with lumped parameters. The deflections in time were compared, where high agreement in the solution of both models was showed. This result was necessary to achieve, because for today's solution status of vibration of the beam, the analytical model gives us reliable results but for solution with for e.g. more complicated cross section area over length and with continuously areas of backstop, already the analytical solution grind to considerably complicated forms and because discretized model have agreement in results, it is possible to solve these other cases with it.

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## 6. Literature

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