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DYNAMIC BEHAVIOUR OF A BEAM COUPLED WITH A PRESTRESSED STRING – INFLUENCE OF DAMPING

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Summary: *The beam with an axial force is coupled by an elastic layer of Winkler type with the pretensiled string. It is subjected to a row of moving forces. The theoretical model corresponds to a prestressed bridge. The concrete bridges of this type are the most spread types appearing on both the road and railway bridges of small and medium spans. The governing equations form a coupled set of partial differential equations that are solved using the Fourier and Laplace integral transformations for the undamped case. The numerical solution is examined for the damped case.*

1. Introduction

Many years standing effort has been devoted to damp the dynamic effects of both the highway and railway vehicles crossing bridges. For that purpose, a significant amount of systems have been developed, e.g. elastic supports of bridges, triangular falsework systems with controlled damping, double systems with two beams or two strings connecting together with an elastic layer, etc. They are briefly described in [1] and cited in details in [2] and [3]. The double systems of double beams and double strings were firstly introduced by the papers of Kawanazoe et al. [4] and Oniszczuk [5]. However, each of the systems mentioned above shows its technical or economic of effectiveness only in some specific conditions.

On the other hand, the prestressed bridges, world wide used for highway as well as railway bridges of small and medium spans, form naturally a double system with two elements: beam and pretensiled strings. That is the reason, why an idea arose — to bind both the elements with an elastic layer and dampers to diminish the dynamic response.

The aim of the present paper is to show the effect of the double system beam and string and put a question on the effectiveness of damping between the beam and the string. The presented paper extends the topic presented at the conference IM2006 by the influence of damping in the elastic layer.

2. Theory

The Figure 1 represents the theoretical model of a beam, pretensiled string and an elastic layer subjected to a row of axle forces. The axle forces F_n, $n = 1, \ldots, N$ move with a constant speed c. The simple supported beam and string provide the span l. The beam is subjected to an axial

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force N_1, while the string is tensed by a force N_2, $N_1 = -N_2$. An elastic layer of Winkler type binds together both the carrying elements, its characteristic is k [N/mm] and its viscous damping ω_{d1} or ω_{d2}, respectively.

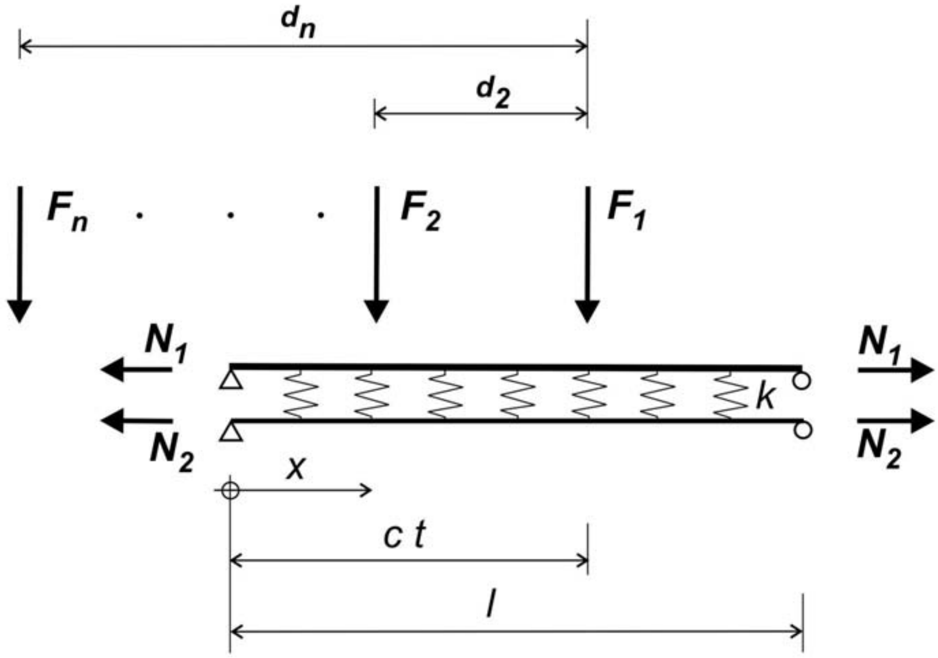

Fig. 1. Theoretical model

A system of partial differential equations describes the dynamic behaviour of the Bernoulli-Euler beam and string:

$$EI\frac{\partial^4 v_1(x,t)}{\partial x^4} - N_1\frac{\partial^2 v_1(x,t)}{\partial x^2} + \mu_1\frac{\partial^2 v_1(x,t)}{\partial t^2} + k\left(v_1(x,t) - v_2(x,t)\right) + \\ + 2\mu_1\omega_{d1}\left(\frac{\partial v_1(x,t)}{\partial t} - \frac{\partial v_2(x,t)}{\partial t}\right) = \sum_{n=1}^{N} F_n\varepsilon_n(t)\delta(x - x_n) \quad (1)$$

$$-N_1\frac{\partial^2 v_2(x,t)}{\partial x^2} + \mu_2\frac{\partial^2 v_2(x,t)}{\partial t^2} + k\left(v_2(x,t) - v_1(x,t)\right) + \\ + 2\mu_2\omega_{d2}\left(\frac{\partial v_2(x,t)}{\partial t} - \frac{\partial v_1(x,t)}{\partial t}\right) = 0 \quad (2)$$

where

$v_i(x,t),\ i = 1,2$ – vertical deflection of the beam ($i = 1$) and the string ($i = 2$), respectively, at place x and time t,

E, I – modulus of elasticity and inertial cross-section moment, respectively, of the beam,

μ_i – mass of the beam ($i = 1$) and of the string ($i = 2$), respectively, per unit length,

$\delta(x)$ – Dirac delta function,

$\varepsilon_n(t) = h(t - t_n) - h(t - T_n)$ – the function describing the position of the n-th force with respect to the beam, $h(t)$ is the Heaviside unit function, $h(t) = 0$ for $t < 0$ and $h(t) = 1$ for $t \geq 0$,

d_n – distance of the n-th force from the first one, $d_1 = 0$,

$t_n = d_n/c$, $T_n = (d_n + l)/c$, c is the moving speed,

$x_n = ct - d_n$.

The boundary and initial conditions are supposed to be zero:

$$v_1(0,t) = v_1''(0,t) = v_1(l,t) = v_1''(0,t) = v_2(0,t) = v_2(l,t) = 0 \tag{3}$$

$$v_i(x,0) = \dot{v}_i(x,0) = 0, \quad i = 1,2 \tag{4}$$

where the primes and dots denote the derivatives with respect to x or t, respectively.

Let us introduce remaining symbols which are used further:

$$\begin{aligned}
\omega_j^2 &= \frac{EI\pi^4 j^4}{l^4\mu_1}, & \omega &= \frac{\pi c}{l}, \\
\omega_{1j}^2 &= \frac{EI\pi^4 j^4}{l^4\mu_1} + \frac{\pi^2 N_1 j^2}{l^2\mu_1}, & \omega_{2j}^2 &= \frac{\pi^2 N_2 j^2}{l^2\mu_2}, \\
\omega_{ik}^2 &= \frac{k}{\mu_k}, & \omega_{ijk}^2 &= \omega_{ij}^2 + \omega_{ik}^2; \quad \text{for } i = 1,2 \\
\Omega_{1,2}^2 &= \frac{1}{2}(\omega_{1jk}^2 + \omega_{2jk}^2) \mp \sqrt{\frac{1}{4}(\omega_{1jk}^2 - \omega_{2jk}^2)2 + \omega_{1k}^2\omega_{2k}^2}
\end{aligned} \tag{5}$$

3. Solution

For the solution of eqns (1) and (2), the Fourier integral transformation method can be applied in the variable x and the Laplace transform in the variable t. If the solution $v_i(x,t)$ is written in the form of

$$v_i(x,t) = \sum_{j=1}^{\infty} \frac{\mu_i}{V_{i,j}} V_i(j,t) \sin\left(\frac{j\pi x}{l}\right), \quad V_{i,j} = \int_0^l \sin^2\left(\frac{j\pi x}{l}\right)\mu_i \mathrm{d}x = \frac{\mu_i l}{2} \tag{6}$$

and the Fourier coefficients using their Laplace transform

$$V_i(j,t) = \frac{1}{2i\pi}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \mathrm{e}^{st} V_i^\star(j,p)\mathrm{d}s, \tag{7}$$

one can get following relation for the Laplace transform $V_i^\star(j,p)$:

$$V_1^\star(j,p) = \frac{1}{\mu_1}\frac{j\omega(p^2 + j^2\omega^2)^{-1}\left(p^2 + 2\omega_{d2}p + \omega_{2jk}^2\right)\left(1 - (-1)^j \mathrm{e}^{-\frac{Lp}{c}}\right)\sum_{n=1}^N \mathrm{e}^{-\frac{pd_n}{c}} F_n}{\left(p^2 + 2\omega_{d1}p + \omega_{1jk}^2\right)\left(p^2 + 2\omega_{d2}p + \omega_{2jk}^2\right) - (\omega_{1k}^2 + 2p\omega_{d1})(\omega_{2k}^2 + 2p\omega_{d2})}, \tag{8}$$

$$V_2^\star(j,p) = \frac{1}{\mu_1}\frac{j\omega(p^2 + j^2\omega^2)^{-1}(\omega_{2k}^2 + 2p\omega_{d2})\left(1 - (-1)^j \mathrm{e}^{-\frac{Lp}{c}}\right)\sum_{n=1}^N \mathrm{e}^{-\frac{pd_n}{c}} F_n}{\left(p^2 + 2\omega_{d1}p + \omega_{1jk}^2\right)\left(p^2 + 2\omega_{d2}p + \omega_{2jk}^2\right) - (\omega_{1k}^2 + 2p\omega_{d1})(\omega_{2k}^2 + 2p\omega_{d2})}. \tag{9}$$

The explicit inverse Laplace transform of the expressions (8,9) for the undamepd case ($\omega_{d2} = \omega_{d1} = 0$) can be found e.g. in [6]. For the damped case is the explicit expression so complicated that such a formula brings no advantage.

4. Numerical solution

To obtain a system of equations which feasible for the numerical solution we will assume the solution $v_i(x,t)$ to have a form of Fourier expansion:

$$v_i(x,t) = \sum_{j=0}^{\nu} q_{i,j}(t) \sin\left(\frac{j\pi x}{l}\right). \tag{10}$$

Here we take as an advantage that $v_j(x) = \sin\left(\frac{j\pi x}{l}\right)$ is the j-th natural mode of the Bernoulli-Euler beam. Introducing (10) into (1) and (2), multiplying by the term $\sin\left(\frac{k\pi x}{L}\right)$ and integrating both sides for $x \in (0,l)$ we obtain:

$$q''_{1j}(t) + 2\omega_{1d}(q'_{1j}(t) - q'_{2j}(t)) + \omega^2_{1jk}q_{1j}(t) - \omega^2_{1k}q_{2j}(t) = \\ = \frac{\mu_1 l}{2}\sum_{n=1}^{N} \sin\left(\frac{j\pi\,(ct-d_n)}{l}\right) F_n h\,(ct-d_n)\,h\,(L-ct+d_n) \tag{11}$$

$$q''_{2j}(t) + 2\omega_{2d}(q'_{2j}(t) - q'_{1j}(t)) + \omega^2_{2jk}q_{2j}(t) - \omega^2_{2k}q_{1j}(t) = 0 \tag{12}$$

The system of two ordinary differential equations is to be solved numerically for selected number ν of terms of expansion (10) and fixed coordinate $x = l/2$. The deflection value $v_i(l/2,t)$ obtained using (10) will be further normalized by the term (F is the total weight of the bridge)

$$v_0 = \frac{2F}{\mu_1 l}\frac{\omega^2_{21k}}{\omega^2_{11k}\omega^2_{21k} - \omega^2_{1k}\omega^2_{2k}} \tag{13}$$

giving

$$v_{i0}(x,t) = \frac{1}{v_0}\sum_{j=0}^{\nu} q_{i,j}(t)\sin\left(\frac{j\pi x}{l}\right). \tag{14}$$

5. Numerical results

The deflection-time histories were calculated for several hundreds of cases in 3 series of different bridges. The eqn (14) was used for the damped series, explicit inverse Laplace transformation of eqns. (8), (9) for undamped cases.

The following case is demonstrated here as an example: The concrete bridge with parameters summarized in the Table 1 is subjected to a row of 10 vehicles with axle loads $F_1 = F_3 = 1.6 \times 10^5$ N, $F_2 = F_4 = 4.8 \times 10^5$ N in distances $d_1 = 0$, $d_2 = 3$ m, $d_3 = 12$ m, $d_4 = 15$ m (valid for Czech standard highways) and with the gaps of 9 m between the vehicles, at low velocity 5 km/h and at speed 70 km/h. The responses of the beam midspans (in dimensionless form) are depicted in Figure 2 for undamped case (first row) and for the damped case with $\omega_{d1} = \omega_{d2} = 0.2$ in the second row.

Tab. 1. Properties of the bridge under study

$l = 30$m,	$E = 3.2 \times 10^4$ N mm^{-2},	$I = 1.347 \times 10^{12}$ mm^4,
$\mu_1 = 1.1 \times 10^{-2}$ Ns^{2}mm^{-2},	$F = 4.8 \times 10^5$ N,	$N_2 = 5.6 \times 10^6$ N,
$k = 100$ Ns^{2}mm^{-2},	$\mu_2 = 0.002$ Ns^{2}mm^{-2},	$\omega_{d1,2} = 0$ or 0.2 s^{-1}

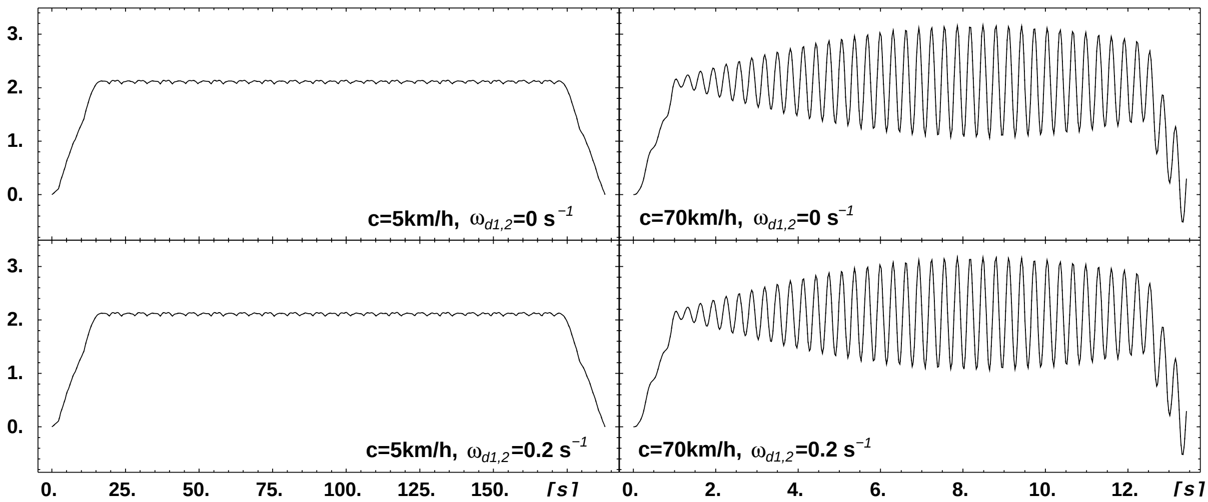

Fig. 2. Deflection-time history at 5 km/h (left column) and at 70 km/h (right column) for the undamped case (first row) and the damped case with $\omega_{d1} = \omega_{d2} = 0.2$ (second row).

As it can be seen from the Figure 2, as well as from the values in the Table 1, the damping coefficient has very little influence on the overall behaviour. This fact can be justified by the small difference between maximal deflections $v_{10}(l/2, t)$ and $v_{20}(l/2, t)$ even for the undamped case. These dimensionless values are 3.16593 and 3.17447 respectively, for both $\omega_{d1,2} = 0$ and $c = 70$km/h and 3.17033 and 3.17858 respectively for $\omega_{d1} = \omega_{d2} = 0.2$. The lines of $v_{10}(l/2, t)$ and $v_{20}(l/2, t)$ almost coincide in both cases.

The influence of the individual parameters k, μ_2 and velocity c on the total deflections $v_{10}(l/2, t)$ and $v_{20}(l/2, t)$ can be examined by computing the correlation coefficients between maximal response and various values of the input parameters for a large number of combination of parameters. These correlation coefficients are shown in the Table 2.

Tab. 2. Correlation coefficients of the maximal deflections $v_{10}(l/2, t)$ and $v_{02}(l/2, t)$ and input parameters k, μ_2 and c.

	k	μ_2	c	ω_{d1}	ω_{d2}	$v_1(l/2, t)$	$v_2(l/2, t)$
$v_{10}(l/2, t)$	0.02701	0.08099	0.57417	0.01333	-0.00387	1.00000	0.24892
$v_{20}(l/2, t)$	0.39730	0.01804	0.08769	-0.00229	0.00104	0.24891	1.00000

The Figure 2 and Table 1 symbolize the great effect of the speed of moving forces on the beam deflection $v_{01}(l/2, t)$, correlation coefficient 0.57, whereas the characteristic k of the Winkler type binding between the beam and string has the major influence on the behaviour of the string deflection $v_{02}(l/2, t)$ – correlation coefficient 0.4. On the other hand, influence of the deflection of the string to the behaviour of the beam is relatively small (correlation coefficient 0.25).

It was derived in [2] and [3], that the eqns (1) and (2) depend on 6 dimensionless parameters. Then, it is difficult, particularly in practice, to develop the materials corresponding to the severe conditions for the parameters. Further on, the results in [2] and [3] show the ranges of parameters, which provide a low response of the beam subjected to a moving force.

6. Conclusions

The dynamic behaviour of the system beam and string bounded together by an elastic layer is analysed. A set of partial differential equations describes the problem and is solved using the integral transformation methods and numerically. The equations are governed by several input parameters and, only in rear cases, the response of the beam with an elastic layer may be substantially lower than that one without the layer.

The major effect on the beam deflection has the speed of moving forces, whereas the characteristic k of the elastic layer between the beam and string has the major influence on the behaviour of the string deflection. On the other hand, influence of the deflection of the string on the behaviour of the beam is relatively small.

7. Acknowledgment

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8. References

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