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VIBRATIONS OF ROTORS WITH FLEXIBLE DISKS

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Summary: *This paper deals with the methodology of the modelling of rotating shafts with flexible disks. Rotating shafts are modelled as one dimensional continuum on the basis of the Bernouli-Euler theory, which assumes that the shaft cross section remains a flat plane and is perpendicular to the centerline during vibration. Disks are modelled as three dimensional continuum by means of the finite element method. The presented approach allows the effects of the rotation such as centrifugal and gyroscopic effects to be introduced. The two possible approaches to the coupling of the shaft and disk subsystems are proposed. The elementary numerical example is presented. The results based on the new modelling methodology are compared with the results based on the original methodology using the assumption of a rigid disk.*

1. Introduction

The field of the dynamics of rotating systems has a long tradition and deals with many different types of problems. The typical mechanical systems in rotor dynamics are rotating shafts (or shaft systems) of different shapes joined with special, mostly axi-symmetric, bodies, which can be bladed disks, geared wheels, flywheels etc. In the recent years analyzed problems, designed rotating systems and operation conditions are becoming more and more complex. These facts lead also to more complex mathematical models, for which it is sometimes needed to develop new modelling approaches.

The vibration analysis of rotating systems is commonly performed with the assumption of ideal rigid disks. However, there are cases in which the vibrations of disks become important and the rigid body assumption is too rough for detailed dynamic analysis. The assumption is not correct for example in the case of the high frequency excitation and therefore it is necessary to care about special approaches to the modelling of rotating shafts with flexible disks.

Classic monographs, like Krämer (1993), describe the modelling of rotors considering disks as rigid bodies with their mass and inertia moments. Rotating shaft are modelled usually on the basis of Bernouli-Euler or Timoshenko theories. This theories assume that the shaft cross section remains a flat plane and is perpendicular to the centerline during vibration. Effects of rotation are presented in the form of the gyroscopic matrix. Special elastic shaft finite element

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based on these theories is introduced for example in Yamamoto & Ishida (2001) or in Kang et al. (2000), but rigid disks are still supossed. The specific area of rotor dynamics is the investigation of fan vibrations, see e.g. Zeman & Hlaváč (2003), where assumption of rigid disk was also considered.

Many publications are dedicated to the dynamic analysis of thin rotating disks. Mignolet et al. (1996) employed perturbation techniques to estimate free vibration characteristics of a centrally clamped disk. The cyclic symmetry approach in connection with Ritz method were used in Tomioka et al. (1996), but there were taken into account only centrifugal effects and Coriolis effects were neglected. Chatelet et al. (2005) published the methodology based on the finite element method and cyclic symmetry approach for the modelling of thin flexible rotating structures considering centrifugal stiffening and gyroscopic effects. Another application of finite elements can be found in Jang et al. (2005). Again it is goaled to the thin flexible disks using membrane theory. One of the newest and most comprehensive monographs intended to dynamics of rotating systems is the monograph of Genta (2005), where the issue of the rotating disks is described in more detail. Started from the classical membrane theory for thin disks there are shown some possible approaches for modelling general three dimensional rotors and disks considering gyroscopic and centrifugal effects.

In all mentioned publications the flexible disk is modelled separately and uncoupled of the shaft subsystem or the disk is supposed to be rigid. There are no difficulty in modelling rotors using standard commercial FEM tools, but such codes usually do not take into account the all effects caused by the rotation. This paper deals with the methodology of the modelling of rotating shafts with flexible disks. Rotating shafts are modelled as one dimensional continuum on the basis of the Bernouli-Euler theory. Disks are modelled as three dimensional continuum by means of the finite element method. The presented approach allows the effects of the rotation such as centrifugal and gyroscopic effects to be introduced.

2. Mathematical model of a system with ideally rigid coupling between the disk and the shaft

For the sake of the mathematical model derivation it will be supposed that the whole system consists of two subsystems — disk subsystem (subscript D) and shaft subsystem (subscript S). The generalization to the more complex disk-shaft systems is straightforward. It is supposed that the subsystems are rotating with constant angular velocity ω around their X-axis.

According to the derivation in Šašek et al. (2006) the disk can be discretized using solid 3D finite elements. The conservative mathematical model of the uncoupled disk subsystem is of the form

$$\boldsymbol{M}_D\ddot{\boldsymbol{q}}_D(t) + \omega\boldsymbol{G}_D\dot{\boldsymbol{q}}_D(t) + (\boldsymbol{K}_{sD} - \omega^2\boldsymbol{K}_{dD})\boldsymbol{q}_D(t) = \omega^2\boldsymbol{f}_D, \tag{1}$$

where $\boldsymbol{M}_D$ is the mass matrix, term $\omega\boldsymbol{G}_D$ expresses gyroscopic effects, $\boldsymbol{K}_{sD}$ is the static stiffness matrix, $\boldsymbol{K}_{dD}$ is the dynamic stiffness matrix and $\omega^2\boldsymbol{f}_D$ is the centrifugal load vector. The matrices are rectangular of the n_D-th order and except gyroscopic matrix they are symmetrical ones. The gyroscopic matrix is skew-symmetrical. The motion equations of the disk are written in the configuration space defined by vector

$$\boldsymbol{q}_D = [\ \dots\ u_j^{(D)}\ v_j^{(D)}\ w_j^{(D)}\ \ \dots]^T\ \in\ \mathbb{R}^{n_D} \tag{2}$$

of nodal displacements (see Fig.1) with respect to the rotating coordinate system x, y, z.

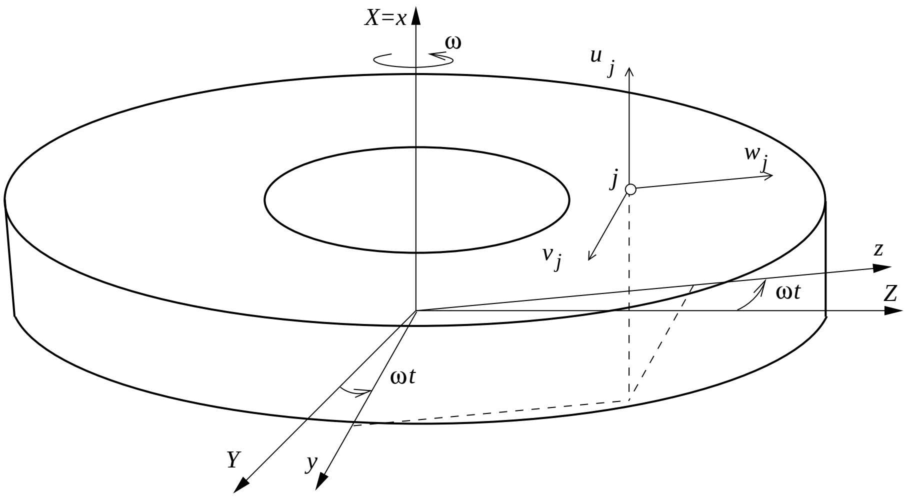

Fig.1 Scheme of the disk and its coordinate systems.

The shaft subsystem is modelled as an one dimensional continuum on assumption of the undeformable cross-section that is still perpendicular to the shaft center-line. The shaft is discretized using shaft finite elements (see Fig.2) with two nodes. The displacements of each shaft finite element in node "i" are described by six generalized coordinates — three displacements u_i, v_i, w_i and three rotations φ_i, ϑ_i, ψ_i. The shaft conservative mathematical model in the configuration space defined by the vector

$$\boldsymbol{q}_S = [\,\dots\; u_i\; v_i\; w_i\; \varphi_i\; \vartheta_i\; \psi_i\; \dots]^T \;\in\; \mathbb{R}^{n_S} \tag{3}$$

is of the form

$$\boldsymbol{M}_S\ddot{\boldsymbol{q}}_S(t) + \omega\boldsymbol{G}_S\dot{\boldsymbol{q}}_S(t) + (\boldsymbol{K}_{sS} - \omega^2\boldsymbol{K}_{dS} + \boldsymbol{K}_B)\boldsymbol{q}_S(t) = \boldsymbol{0}, \tag{4}$$

where mass matrix $\boldsymbol{M}_S$, static stiffness matrix $\boldsymbol{K}_{sS}$ and dynamic stiffness matrix $\boldsymbol{K}_{dS}$ are symmetrical and gyroscopic matrix $\omega\boldsymbol{G}_S$ is skew-symmetrical. Rolling-element bearings are described in this mathematical model by symmetrical stiffness matrix $\boldsymbol{K}_B$. Particular forms of

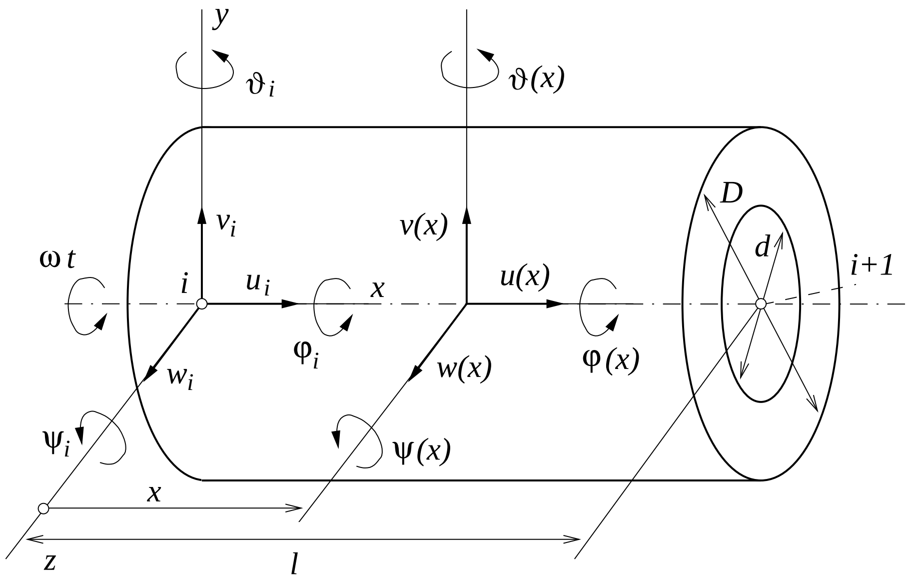

Fig.2 Scheme of the shaft finite element.

the bearing stiffness matrix in non-rotating coordinate system can be found e.g. in Slavík et al. (1997) or in Zeman & Hajžman (2005).

In order to derive the equations of motion of the disk-shaft system with ideally rigid coupling between the subsystems it can be supposed that the disk is mounted in the defined set of shaft nodes. Then the displacements of the disk nodes on the inner surface can be expressed by the displacements of shaft node ”i” that is defined as one of the mounting points (Fig.3). It holds

$$\boldsymbol{q}_j^{(D)} = \boldsymbol{T}_{ji}(\alpha_j)\boldsymbol{q}_i^{(S)} \tag{5}$$

for the transformation between the displacements of the disk node $\boldsymbol{q}_j$ and the shaft node $\boldsymbol{q}_i$. Expressing the transformation matrix $\boldsymbol{T}_{ji}(\alpha)_j$ statement (5) can be rewritten as

$$\begin{bmatrix} u_j^{(D)} \\ v_j^{(D)} \\ w_j^{(D)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & r\sin\alpha_j & -r\cos\alpha_j \\ 0 & 1 & 0 & -r\sin\alpha_j & 0 & \xi \\ 0 & 0 & 1 & r\cos\alpha_j & -\xi & 0 \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ w_j \\ \varphi_i \\ \vartheta_i \\ \psi \end{bmatrix}. \tag{6}$$

The vectors of generalized coordinates can be partitioned with respect to the coupling of the nodes as

$$\boldsymbol{q}_D = \begin{bmatrix} \boldsymbol{q}_{D1} \\ \boldsymbol{q}_{D2} \end{bmatrix} \quad \text{and} \quad \boldsymbol{q}_S = \begin{bmatrix} \boldsymbol{q}_{S1} \\ \boldsymbol{q}_{S2} \end{bmatrix}. \tag{7}$$

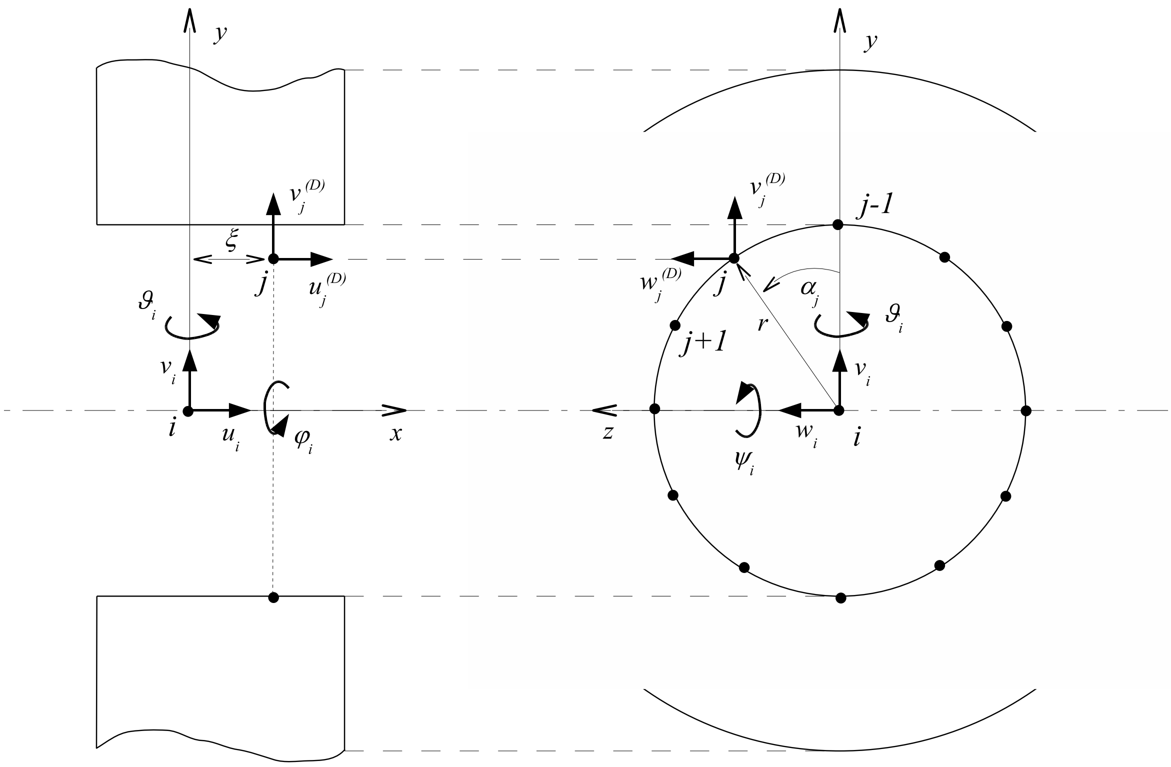

Fig.3 Shaft and disk nodes and their displacements in the rotating coordinate system.

In the same manner each matrix or vector in the equations (1) and (4) can be rearranged as it is shown for the disk mass matrix

$$\boldsymbol{M}_D = \begin{bmatrix} \boldsymbol{M}_{D1} & \boldsymbol{M}_{D3} \\ \boldsymbol{M}_{D3}^T & \boldsymbol{M}_{D2} \end{bmatrix}. \tag{8}$$

The displacements of the disk nodes coupled with the shaft nodes are expressed in terms of shaft generalized coordinates

$$\boldsymbol{q}_{D2} = \boldsymbol{T}\boldsymbol{q}_{S1}, \tag{9}$$

where $\boldsymbol{T}$ is the rectangular global transformation matrix composed of matrices $\boldsymbol{T}_{ji}(\alpha)$. After introducing this transformation in equations (1) and (4) it yields

$$\boldsymbol{M}_{D1}\boldsymbol{q}_{D1} + \boldsymbol{M}_{D3}\boldsymbol{q}_{D2} + \omega\boldsymbol{G}_{D1}\boldsymbol{q}_{D1} + \omega\boldsymbol{G}_{D3}\boldsymbol{q}_{D2} + \boldsymbol{K}_{D1}(\omega)\boldsymbol{q}_{D1} + \boldsymbol{K}_{D3}(\omega)\boldsymbol{q}_{D2} = \omega^2\boldsymbol{f}_{D1}, \tag{10}$$

$$\boldsymbol{M}_{D3}^T\boldsymbol{q}_{D1} + \boldsymbol{M}_{D2}\boldsymbol{T}\boldsymbol{q}_{S1} - \omega\boldsymbol{G}_{D3}^T\boldsymbol{q}_{D1} + \omega\boldsymbol{G}_{D2}\boldsymbol{T}\boldsymbol{q}_{S1} + \boldsymbol{K}_{D3}^T(\omega)\boldsymbol{q}_{D1} + \boldsymbol{K}_{D2}(\omega)\boldsymbol{T}\boldsymbol{q}_{S1} = \omega^2\boldsymbol{f}_{D2}, \tag{11}$$

$$\boldsymbol{M}_{S1}\boldsymbol{q}_{S1} + \boldsymbol{M}_{S3}\boldsymbol{q}_{S2} + \omega\boldsymbol{G}_{S1}\boldsymbol{q}_{S1} + \omega\boldsymbol{G}_{S3}\boldsymbol{q}_{S2} + \boldsymbol{K}_{S1}\boldsymbol{q}_{S1} + \boldsymbol{K}_{S3}\boldsymbol{q}_{S2} = \boldsymbol{0}, \tag{12}$$

$$\boldsymbol{M}_{S3}^T\boldsymbol{q}_{S1} + \boldsymbol{M}_{S2}\boldsymbol{q}_{S1} - \omega\boldsymbol{G}_{S3}^T\boldsymbol{q}_{S1} + \omega\boldsymbol{G}_{S2}\boldsymbol{q}_{S1} + \boldsymbol{K}_{S3}^T\boldsymbol{q}_{S1} + \boldsymbol{K}_{S2}\boldsymbol{q}_{S2} = \boldsymbol{0}, \tag{13}$$

where the total disk stiffness matrix was denoted $\boldsymbol{K}_D(\omega) = (\boldsymbol{K}_{sD} - \omega^2\boldsymbol{K}_{dD})$ and the shaft stiffness matrix including bearing couplings was denoted $\boldsymbol{K}_S = (\boldsymbol{K}_{sS} - \omega^2\boldsymbol{K}_{dS} + \boldsymbol{K}_B)$. In order to obtain the full set of equations of motion of the coupled disk-shaft system equation (11) can be multiplied by matrix $\boldsymbol{T}^T$ and add to equation (12). Then the mathematical model of the disk-shaft system is of the form

$$\begin{aligned}
&\begin{bmatrix} \boldsymbol{M}_{D1} & \boldsymbol{M}_{D3}\boldsymbol{T} & \boldsymbol{0} \\ \boldsymbol{T}^T\boldsymbol{M}_{D3}^T & \boldsymbol{T}^T\boldsymbol{M}_{D2}\boldsymbol{T} + \boldsymbol{M}_{S1} & \boldsymbol{M}_{S3} \\ \boldsymbol{0} & \boldsymbol{M}_{S3}^T & \boldsymbol{M}_{S2} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}}_{D1} \\ \ddot{\boldsymbol{q}}_{S1} \\ \ddot{\boldsymbol{q}}_{S2} \end{bmatrix} + \\
&+\omega \begin{bmatrix} \boldsymbol{G}_{D1} & \boldsymbol{G}_{D3}\boldsymbol{T} & \boldsymbol{0} \\ -\boldsymbol{T}^T\boldsymbol{G}_{D3}^T & \boldsymbol{T}^T\boldsymbol{G}_{D2}\boldsymbol{T} + \boldsymbol{G}_{S1} & \boldsymbol{G}_{S3} \\ \boldsymbol{0} & -\boldsymbol{G}_{S3}^T & \boldsymbol{G}_{S2} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{q}}_{D1} \\ \dot{\boldsymbol{q}}_{S1} \\ \dot{\boldsymbol{q}}_{S2} \end{bmatrix} + \\
&+ \begin{bmatrix} \boldsymbol{K}_{D1}(\omega) & \boldsymbol{K}_{D3}(\omega)\boldsymbol{T} & \boldsymbol{0} \\ \boldsymbol{T}^T\boldsymbol{K}_{D3}(\omega)^T & \boldsymbol{T}^T\boldsymbol{K}_{D2}(\omega)\boldsymbol{T} + \boldsymbol{K}_{S1} & \boldsymbol{K}_{S3} \\ \boldsymbol{0} & \boldsymbol{K}_{S3}^T & \boldsymbol{K}_{S2} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{D1} \\ \boldsymbol{q}_{S1} \\ \boldsymbol{q}_{S2} \end{bmatrix} = \begin{bmatrix} \omega^2\boldsymbol{f}_{D1} \\ \omega^2\boldsymbol{T}^T\boldsymbol{f}_{D1} \\ \boldsymbol{0} \end{bmatrix}.
\end{aligned} \tag{14}$$

Generally the corresponding coupling nodes of the shaft and of the disk can be chosen with respect to the discretization of both bodies by finite elements. For purpose to make a realistic model of the rotor it should be suitable to fix different disk nodes in more than one shaft nodes.

3. Mathematical model of a system with flexible coupling between the disk and the shaft

The second possibility of the disk-shaft system modelling is introducing the flexible coupling between the subsystems. This methodology can be used e.g. for representing the key fitting between shaft and disk, that is usual design solution in some engineering applications. The conservative mathematical model of disk and shaft subsystems mutually joined by flexible coupling is of the form

$$\boldsymbol{M}_D\ddot{\boldsymbol{q}}_D(t) + \omega\boldsymbol{G}_D\dot{\boldsymbol{q}}_D(t) + (\boldsymbol{K}_{sD} - \omega^2\boldsymbol{K}_{dD})\boldsymbol{q}_D(t) = \omega^2\boldsymbol{f}_D + \boldsymbol{f}_D^C, \tag{15}$$

$$\boldsymbol{M}_S\ddot{\boldsymbol{q}}_S(t) + \omega\boldsymbol{G}_S\dot{\boldsymbol{q}}_S(t) + (\boldsymbol{K}_{sS} - \omega^2\boldsymbol{K}_{dS} + \boldsymbol{K}_B)\boldsymbol{q}_S(t) = \boldsymbol{f}_S^C, \tag{16}$$

where all matrices and vectors except vectors $\boldsymbol{f}_D^C$ and $\boldsymbol{f}_S^C$ are explained in the previous section of the paper. These vectors represent the coupling forces between particular subsystems. The coupling forces are acting in the chosen shaft nodes, where the disk is mounted on, and in the chosen disk nodes, that lie on the inner circumference of the disk body (see Fig.3).

The global coupling force vector $\boldsymbol{f}_C$ in global configuration space of the disk-shaft system

$$\boldsymbol{q} = \left[\, \boldsymbol{q}_D^T \; \boldsymbol{q}_S^T \,\right]^T \tag{17}$$

can be calculated by differentiating the potential (strain) energy

$$\boldsymbol{f}_C = \begin{bmatrix} \boldsymbol{f}_D^C \\ \boldsymbol{f}_S^C \end{bmatrix} = -\frac{\partial E_p^C}{\partial \boldsymbol{q}}. \tag{18}$$

If the disk-shaft coupling is realized using n_i shaft nodes and n_j disk nodes for each shaft node the global coupling force vector can be rewritten in the form

$$\boldsymbol{f}_C = -\boldsymbol{K}_C\boldsymbol{q} = -\sum_{i=1}^{n_i}\sum_{j=1}^{n_j}\boldsymbol{K}_{i,j}^C\boldsymbol{q}, \tag{19}$$

where stiffness matrices $\boldsymbol{K}_{i,j}$ corresponding to the particular coupling between i-th and j-th nodes are calculated as

$$\frac{\partial E_{i,j}^C}{\partial \boldsymbol{q}} = \boldsymbol{K}_{i,j}^C\boldsymbol{q}. \tag{20}$$

The coupling is characterized by three stiffnesses k_t in tangent direction to the shaft cross-section, k_r in radial direction and k_{ax} in axial direction. These stiffnesses are used for each coupling between i-th and j-th nodes. The mathematical model of the whole system is

$$\begin{aligned} &\begin{bmatrix} \boldsymbol{M}_D & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M}_S \end{bmatrix}\begin{bmatrix} \ddot{\boldsymbol{q}}_D \\ \ddot{\boldsymbol{q}}_S \end{bmatrix} + \omega\begin{bmatrix} \boldsymbol{G}_D & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{G}_S \end{bmatrix}\begin{bmatrix} \dot{\boldsymbol{q}}_D \\ \dot{\boldsymbol{q}}_S \end{bmatrix} + \\ &\qquad + \left(\begin{bmatrix} \boldsymbol{K}_D(\omega) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K}_S \end{bmatrix} + \boldsymbol{K}_C\right)\begin{bmatrix} \boldsymbol{q}_D \\ \boldsymbol{q}_S \end{bmatrix} = \begin{bmatrix} \omega^2\boldsymbol{f}_D \\ \boldsymbol{0} \end{bmatrix}. \end{aligned} \tag{21}$$

The advantage of this approach is the possibility of the usage of the modal synthesis method, that allows to reduce degrees of freedom of the model in the course of model creation. The modal analysis of the conservative models of uncoupled and non-rotating subsystems

$$\boldsymbol{M}_D\ddot{\boldsymbol{q}}_D + \boldsymbol{K}_{sD}\boldsymbol{q}_D = \boldsymbol{0}, \quad \boldsymbol{M}_S\ddot{\boldsymbol{q}}_S + (\boldsymbol{K}_{sS} + \boldsymbol{K}_B)\,\boldsymbol{q}_S = \boldsymbol{0} \tag{22}$$

is performed and master eigenvectors of the subsystems are arranged in the rectangular modal matrices ${}^m\boldsymbol{V}_D \in \mathbb{R}^{n_D,m_D}$ and ${}^m\boldsymbol{V}_S \in \mathbb{R}^{n_S,m_S}$. The modal transformations

$$\boldsymbol{q}_D(t) = {}^m\boldsymbol{V}_D\boldsymbol{x}_D(t) \quad \text{and} \quad \boldsymbol{q}_S(t) = {}^m\boldsymbol{V}_S\boldsymbol{x}_S(t) \tag{23}$$

are then introduced in the model (21) and after multiplying from left by

$$\boldsymbol{V}^T = \begin{bmatrix} {}^m\boldsymbol{V}_D^T & \boldsymbol{0} \\ \boldsymbol{0} & {}^m\boldsymbol{V}_S^T \end{bmatrix} \tag{24}$$

the reduced mathematical model of the disk-shaft system is

$$\begin{bmatrix} \ddot{\boldsymbol{x}}_D \\ \ddot{\boldsymbol{x}}_S \end{bmatrix} + \omega \begin{bmatrix} {}^m\boldsymbol{V}_D^T \boldsymbol{G}_D\, {}^m\boldsymbol{V}_D & \boldsymbol{0} \\ \boldsymbol{0} & {}^m\boldsymbol{V}_S^T \boldsymbol{G}_S\, {}^m\boldsymbol{V}_S \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{x}}_D \\ \dot{\boldsymbol{x}}_S \end{bmatrix} +$$
$$+ \left(\begin{bmatrix} {}^m\boldsymbol{\Lambda}_D - \omega\, {}^m\boldsymbol{V}_D^T \boldsymbol{K}_{dD}\, {}^m\boldsymbol{V}_D & \boldsymbol{0} \\ \boldsymbol{0} & {}^m\boldsymbol{\Lambda}_S - \omega\, {}^m\boldsymbol{V}_S^T \boldsymbol{K}_{dS}\, {}^m\boldsymbol{V}_S \end{bmatrix} + \boldsymbol{V}^T \boldsymbol{K}_C \boldsymbol{V} \right) \begin{bmatrix} \boldsymbol{x}_D \\ \boldsymbol{x}_S \end{bmatrix} = \quad (25)$$
$$= \begin{bmatrix} \omega^2\, {}^m\boldsymbol{V}_D^T \boldsymbol{f}_D \\ \boldsymbol{0} \end{bmatrix}.$$

This model is created in the new configuration space

$$\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{x}_D(t) \\ \boldsymbol{x}_S(t) \end{bmatrix} \in \mathbb{R}^{m_D+m_S}, \quad m_D \ll n_D,\ m_S \ll n_S. \tag{26}$$

4. Application of the methodology to the test disk-shaft-bearing system

The modelling methodology will be illustrated by means of a test rotor (disk-shaft-bearing system) shown in Fig.4. The shaft radius is $r = 0.025$ m, the disk radius $R = 0.08$ m, the disk width $h = 0.04$ m and the shaft lengths $a = b = 0.14$ m. The shaft were discretized using 16 shaft 1D elements and the disk were discretized using 576 solid elements. The isotropic bearings (stiffness $k_B = 10^9$ N/m) were considered in the outside nodes of the shaft (left bearing — radial and axial direction, right bearing — radial direction). Standard steel material properties were considered. The original in-house software was created in MATLAB system based on the developed modelling methodology.

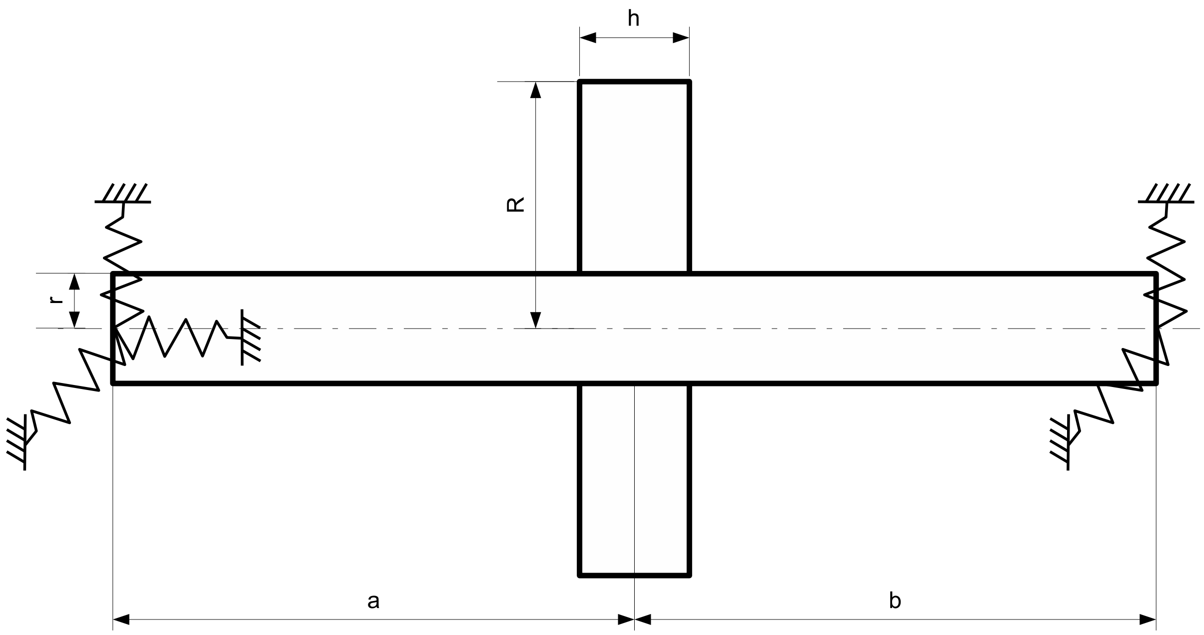

Fig.4 Scheme of the test rotor.

In this paper the results of the modal analysis of the test rotor for $\omega = 0$ and for flexible coupling between the disk and shaft subsystems are compared with the reference model of the test rotor. This reference model is based on the original methodology using the representation

Tab.1 Eigenfrequencies [Hz] of the test rotor calculated using different mathematical models. The numbers in brackets denote the number of the corresponding eigenmode.

Reference model	**Flexible coupling**		Note
	Stiff	Soft	
0 (1)	0 (1)	0 (1)	free rotation around X-axis
651 (2,3)	708 (2,3)	659 (2,3)	bending
1410 (4)	1402 (4)	1373 (4)	axial
2222 (5,6)	2226 (5,6)	1719 (5,6)	bending
5758 (7)	5878 (8)	5773 (12)	torsional
5912 (8)	4564 (7)	1834 (7)	torsional
6083 (9,10)	6291 (11,12)	4256 (8,9)	bending
6248 (11,12)	5939 (9,10)	4799 (10,11)	bending
-	7065 (13)	6414 (14)	bending of the disk
-	7072 (14)	6424 (15)	bending of the disk
9760 (13)	7432 (15)	5803 (13)	axial
11997 (14)	10254 (16)	9993 (18)	axial
12627 (15,16)	11599 (19,20)	10612 (21,22)	bending
12810 (17,18)	11443 (17,18)	7658 (16,17)	bending
-	12899 (21)	10410 (19)	radial deformation of the disk
-	12900 (22)	10416 (20)	radial deformation of the disk
17558 (19)	13919 (23)	11835 (23)	torsional
-	14278 (24,25)	14142 (24,25)	bending of the disk

of the disk as a rigid body with calculated inertia properties mounted in a chosen shaft node, see e.g. Slavik et al. (1997). Two different flexible couplings, stiff and soft, characterized by different local stiffnesses k_t, k_r, $k_a x$ were considered. These stiffnesses were chosen with respect to the ratio of the global coupling stiffnesses and the shaft stiffnesses that can be analytically expressed (Slavík et al, 1997).

The comparison of eigenfrequencies [Hz] obtained by means of three models (reference, stiff flexible coupling, soft flexible coupling) is shown in Tab.1. The numbers in brackets denote the number of corresponding eigenmodes. The character of the eigenmodes of vibration is described in the last column.

The first eigenfrequency is zero because the rotor can freely rotate around its axis of rotation. From the several eigenfrequencies corresponding to bending eigenmodes obtained using reference and stiff coupling models it can be seen the effect of the bending stiffening, because the disk in reference model is fixed only in one shaft node and the flexible disk is joined with more adjacent shaft nodes. The same effect cannot be seen in comparison with soft flexible coupling because the appropriate eigenmodes are characterized also by the deformation of the discrete disk-shaft coupling and therefore the system is more ”soft”. This is also the case axial and torsional eigenmodes, where the eigenfrequencies of the models based on the flexible coupling are lower than reference eigenfrequencies. The differences between the corresponding eigenfrequencies obtained using stiff and soft coupling models are caused by the soft disk-shaft coupling that is dominantly deformed. However the most important result of this analysis is the presence of the eigenmodes characterized by the pure flexible disk vibration (bending or radial deformation) that cannot be catched by the original reference model. These eigenmodes can be excited by high-frequency excitation and the proposed new modelling methodology can be more accurate than the original one based on the assumption of rigid disks.

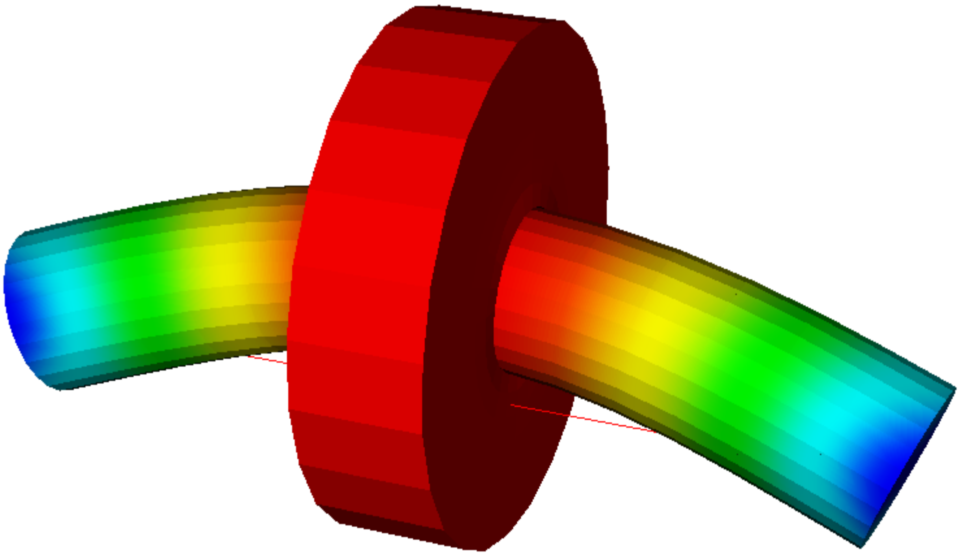

Fig.5 The second eigenmode of the test rotor (for stiff flexible coupling).

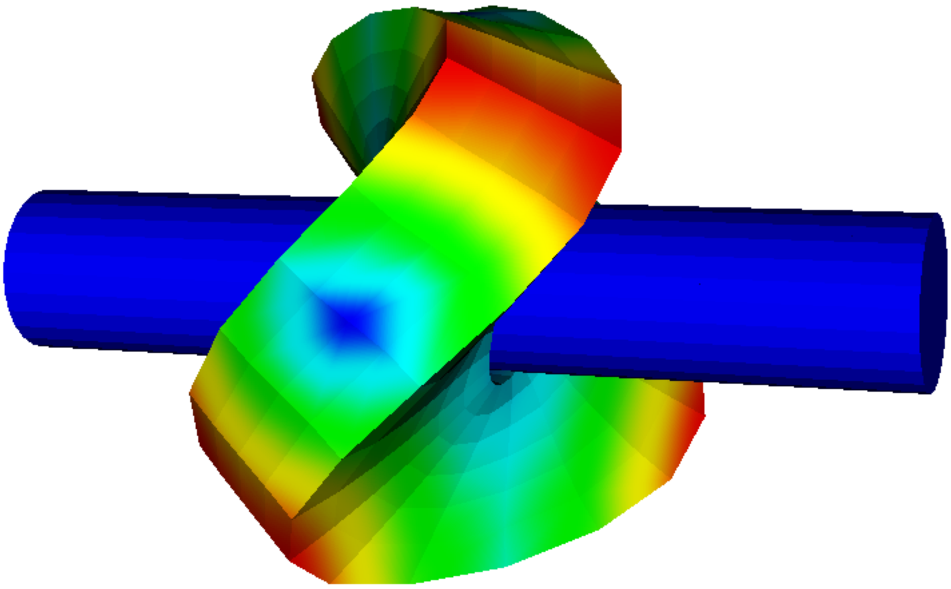

Fig.6 The thirteenth eigenmode of the test rotor (for stiff flexible coupling).

5. Conclusions

The modelling methodology of the flexible rotors (disk-shaft-bearing systems) was presented in the paper. In comparison with previous works the new methodology for the flexible disk and flexible shaft modelling is introduced. The work was motivated by the missing commercial tools intended for the modal analysis of rotating structures, such as disks, turbines, fly-wheels, rotors etc. The methodology is based on the finite element modelling and it allows to consider effects caused by the rotation of the structure (gyroscopic and centrifugal effects). The flexible shaft is modelled as 1D continuum, the disk as 3D continuum and two types of the disk-shaft coupling (ideally rigid and flexible) are developed.

The introduced methodology was applied to the simple test rotor. The comparison of the modal analysis results ($\omega = 0$) for flexible disk-shaft coupling and for original reference model were shown. The reference model is characterized by the assumption of a rigid disk. The advantage of the new methodology is mainly in the consideration of the flexible disk for high-frequency excitation.

The methodology can be used e.g. for the mathematical modelling of bladed disks in turbine dynamics or for the modelling of the geared wheels and disks in gearboxes and similar transmission systems. It can be easily generalized for damped system and for various types of excitation. The introduced mathematical model is suitable for the analysis of the effects of rotation on the flexible rotating systems.

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