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VORTICITY DECOMPOSITION: METHODS AND PHYSICAL ASPECTS

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Summary: *A brief survey dealing with vorticity-decomposition methods is presented. A particular emphasis is put on their physical aspects and practical applicability.*

1. Introduction

The idea of vorticity (i.e. $\nabla \times \mathbf{u}$) decomposition has its own history, though much shorter than the decomposition of motion. It is almost fifty years old according to Astarita (1979), Wedgewood (1999) and the references therein. The present paper provides a very brief survey dealing with different vorticity-decomposition techniques including a novel approach proposed by Kolář (2004, 2007a). A particular emphasis is put on the underlying physical reasoning, similarities and differences between different concepts including practical applicability of these schemes.

2. Methods

The Giesekus-Harnoy-Drouot decomposition (this terminology is adopted following Wedgewood 1999), first treated by Giesekus (1962), leads to the objective vorticity tensor obtained with respect to the principal axes of the strain-rate tensor. A quantity is called objective if it fulfils frame indifference (i.e. both translational and rotational independence by remaining invariant under translational and rotational coordinate changes), see e.g. Leigh (1968). Astarita (1979) proposed this measure for a flow classification scheme while others for the description of complex inelastic fluids.

Wedgewood (1999) derived a new vorticity decomposition into two parts, the so-called *deformational* vorticity and the *rigid* vorticity. His analysis employs the cross product of a particle's velocity and acceleration, $\mathbf{u} \times \mathrm{D}\mathbf{u}/\mathrm{D}t$, and leads to the evolution equation for the objective *deformational* vorticity. The solution of the 'Wedgewood equation,' which depends on both space and time derivatives of the velocity-gradient tensor, is proposed for a flow classification scheme and to develop objective constitutive equations for the description of complex rheological fluids (viscoelastic fluids). It should be emphasized that the application of the Wedgewood criterial quantity $\mathbf{u} \times \mathrm{D}\mathbf{u}/\mathrm{D}t$ results in a necessity of knowing the temporal changes (time derivatives) of experimentally and/or numerically determined velocity-gradient fields. On the other hand, the Wedgewood procedure provides an objective portion of the vorticity tensor similarly as the above mentioned well-known Giesekus-Harnoy-Drouot decomposition.

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Recall that the velocity-gradient tensor $\nabla \mathbf{u}$ can be decomposed in the conventional manner as $\nabla \mathbf{u} = \mathbf{S} + \boldsymbol{\Omega}$, its symmetric and antisymmetric parts representing the strain-rate tensor $\mathbf{S}$ and vorticity tensor $\boldsymbol{\Omega}$, respectively. For the local flow field near a point, Wedgewood (1999) adopted the assumption that $\mathbf{u} \times \mathrm{D}\mathbf{u}/\mathrm{D}t$ must vanish — on average — along three orthogonal axes (the same results can be obtained by volume averaging) to derive the following equation for the *deformational* vorticity tensor $\boldsymbol{\omega}_\mathrm{D}$ while decomposing the vorticity tensor $2\boldsymbol{\Omega} = \boldsymbol{\omega}_\mathrm{D} + \boldsymbol{\omega}_\mathrm{R}$ where $\boldsymbol{\omega}_\mathrm{R}$ is the *rigid* vorticity tensor (notation $\boldsymbol{\omega}_\mathrm{D}$ and $\boldsymbol{\omega}_\mathrm{R}$ is retained following Wedgewood)

$$\left(2\mathbf{S} - \boldsymbol{\omega}_\mathrm{D}\right)\left\{\frac{\mathcal{D}}{\mathcal{D}t}\left(2\mathbf{S} + \boldsymbol{\omega}_\mathrm{D}\right)\right\} - \left\{\frac{\mathcal{D}}{\mathcal{D}t}\left(2\mathbf{S} - \boldsymbol{\omega}_\mathrm{D}\right)\right\}\left(2\mathbf{S} + \boldsymbol{\omega}_\mathrm{D}\right) - \boldsymbol{\omega}_\mathrm{D}^3 + \boldsymbol{\omega}_\mathrm{D}\left(2\mathbf{S} - \boldsymbol{\omega}_\mathrm{D}\right)\mathbf{S} + \mathbf{S}\left(2\mathbf{S} + \boldsymbol{\omega}_\mathrm{D}\right)\boldsymbol{\omega}_\mathrm{D} = 0 \tag{1}$$

where the Jaumann derivative $\frac{\mathcal{D}}{\mathcal{D}t}$ is employed ($\mathbf{T}$ denotes an arbitrary second-order tensor)

$$\frac{\mathcal{D}}{\mathcal{D}t}\mathbf{T} = \frac{\mathrm{D}}{\mathrm{D}t}\mathbf{T} + \left(\boldsymbol{\Omega}\mathbf{T} - \mathbf{T}\boldsymbol{\Omega}\right). \tag{2}$$

The Jaumann derivative reflects the temporal rate of change relative to a corotating frame (the rotation of this local reference frame is given by the vorticity tensor $\boldsymbol{\Omega}$).

The 'Wedgewood equation' (1) requires $\mathbf{S}$ and $\boldsymbol{\omega}_\mathrm{D}$ to be differentiable in both space and time. Wedgewood (1999) inferred from (1) the objectivity of $\boldsymbol{\omega}_\mathrm{D}$. Finally, he formulated a flow classification and general objective constitutive equations based on invariants of $\mathbf{S}$ and $\boldsymbol{\omega}_\mathrm{D}$, and on the so-called *rigid*-rotational derivative quite similar to the Jaumann derivative (formally obtainable by substituting vorticity tensor $\boldsymbol{\Omega}$ by the *rigid* vorticity tensor $\boldsymbol{\omega}_\mathrm{R}$).

Kolář (2004, 2007a) directly decomposed the relative motion near a point through the analysis of a "frozen" flow field at a given instant in time. However, unlike the Wedgewood procedure, the velocity-gradient tensor $\nabla \mathbf{u}$ is decomposed as a whole rather than the vorticity tensor itself. The outcome of this effort, the triple decomposition of motion (TDM) — based on the extraction of a so-called "effective" pure shearing motion — has been motivated by the fact that vorticity cannot distinguish between pure shearing motions and the actual swirling motion of a vortex. In the corresponding triple decomposition of $\nabla \mathbf{u}$, conventionally decomposed as $\nabla \mathbf{u} = \mathbf{S} + \boldsymbol{\Omega}$, the strain-rate tensor $\mathbf{S}$ and vorticity tensor $\boldsymbol{\Omega}$ are cut down in magnitudes to "share" their portions through the third term $(\nabla \mathbf{u})_\mathrm{SH}$ associated with a pure shearing motion. In terms of the residual portions of $\mathbf{S}$ and $\boldsymbol{\Omega}$, it reads

$$\nabla \mathbf{u} = \mathbf{S}_\mathrm{RES} + \boldsymbol{\Omega}_\mathrm{RES} + (\nabla \mathbf{u})_\mathrm{SH}. \tag{3}$$

The third term of the triple decomposition denoted as $(\nabla \mathbf{u})_\mathrm{SH}$ is described by the "purely asymmetric tensor form" $(\nabla \mathbf{u})_\mathrm{SH}$ its components $u_{i,j}$ fulfilling *in a suitable reference frame*

$$u_{i,j} = 0 \quad OR \quad u_{j,i} = 0 \quad \text{(for all } i, j\text{)}. \tag{4}$$

This term is responsible for a specific portion of vorticity labelled "*shear* vorticity" and for a specific portion of strain rate labelled "*shear* strain rate" while the remaining portions of **S** and **Ω** are labelled "*residual* strain rate" and "*residual* vorticity".

The triple decomposition of motion is closely associated with the so-called *basic reference frame* (BRF) where it is performed. In this frame, **(i)** an *effective* pure shearing motion is shown "in a clearly visible manner" described by the form (4) under the definition condition that **(ii)** the effect of extraction of a "shear tensor" is maximized within the following — quite natural and straightforward — decomposition scheme applicable to an arbitrary reference frame

$$\nabla \mathbf{u} \equiv \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} = \begin{pmatrix} residual \\ tensor \end{pmatrix} + \begin{pmatrix} shear \\ tensor \end{pmatrix} \tag{5a}$$

where the residual tensor is given by

$$\begin{pmatrix} residual \\ tensor \end{pmatrix} = \begin{pmatrix} u_x & (\operatorname{sgn} u_y)\,\mathrm{MIN}(|u_y|, |v_x|) & \bullet \\ (\operatorname{sgn} v_x)\,\mathrm{MIN}(|u_y|, |v_x|) & v_y & \bullet \\ \bullet & \bullet & w_z \end{pmatrix}. \tag{5b}$$

In (5a, b) the following simplified notation is employed: u, v, w are velocity components, subscripts x, y, z stand for partial derivatives. The remaining two non-specified pairs of off-diagonal elements of the residual tensor in (5b) are constructed strictly analogously as the specified one, each pair being either symmetric or antisymmetric.

The effect of extraction of the shear tensor is maximized where the absolute tensor value of the residual tensor is minimized by changing the reference frame under an orthogonal transformation. This extremal condition guarantees that a pure shearing motion — if considered separately — is recognized as a third elementary part of the TDM.

The qualitative model of three elementary motions of the TDM is depicted in Fig. 1. The deformable fluid element in Fig. 1 consists of discrete undeformable material points in terms of which the local rate of deformation is described through their relative motion. The material point represents — in the present context — "much less than a fluid element" and generally allows translation and rotation only. A pure shearing motion in Fig. 1 is not a mere combination of an irrotational straining motion with a rigid-body rotation as in the case of the double decomposition. This fact can be easily checked through the rotational change of material points which remains zero for a pure shearing motion within the proposed qualitative model of the TDM. The rotational change of material points is just the quantity reflecting the actual swirling motion of a vortex: note that both an irrotational straining and a pure shearing motion do not contribute to this rotational change (at least according to the present approach).

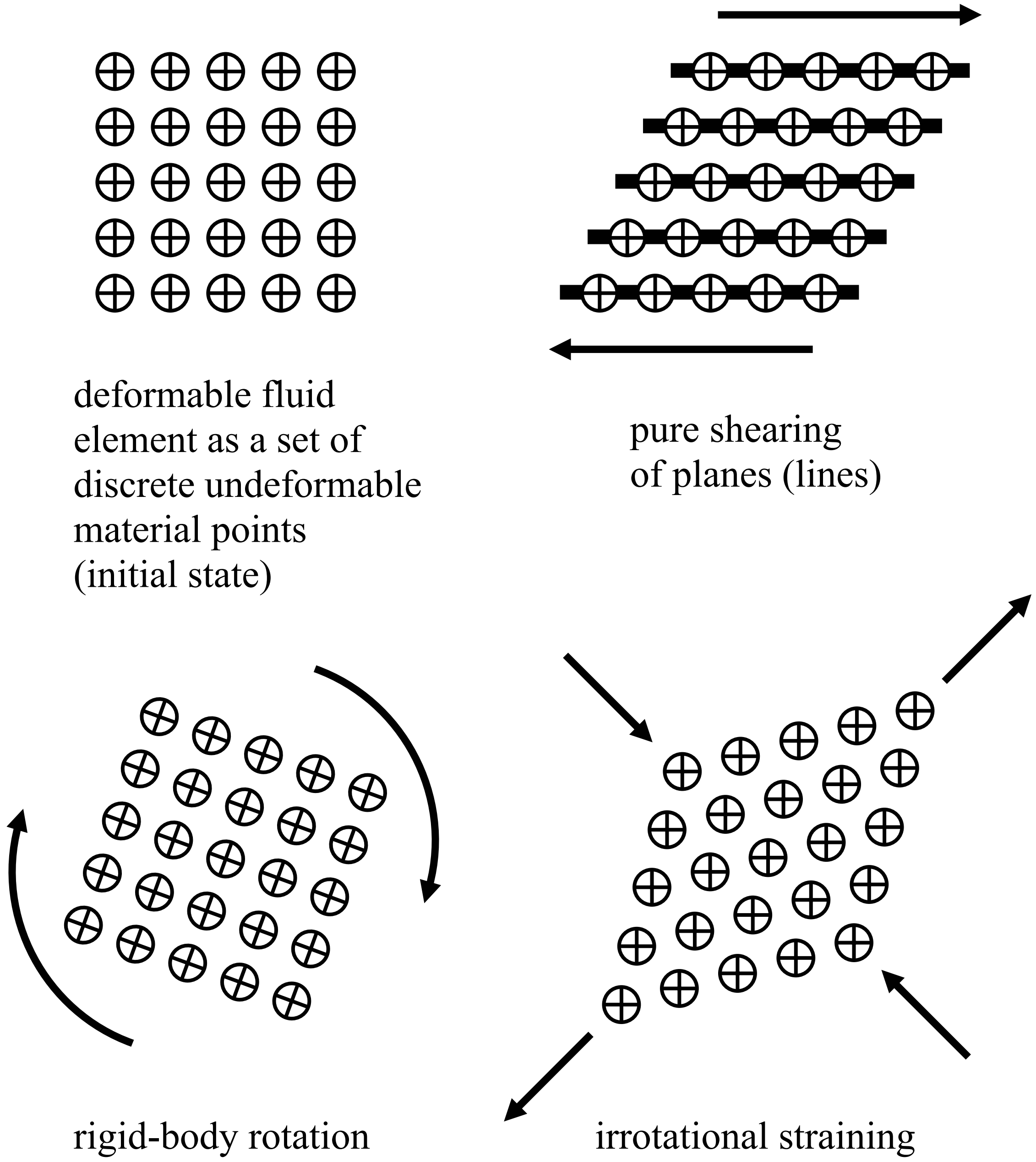

Fig.1 Qualitative model of three elementary motions of the TDM.

For further details, quantitative TDM evaluation algorithm, discussion, and particularly for the qualitative description of flow kinematics near a point adopted in the frame of the TDM see Kolář (2007a).

The proposed novel decomposition technique results in two additive vorticity parts (and, analogously, in two additive strain-rate parts) of distinct nature, namely the *shear* component and the *residual* one. The *residual* vorticity obtained after the extraction of an "effective" pure shearing motion represents a direct kinematic measure of the actual swirling motion of a vortex as it can be related to (twice) the angular velocity of material points, see Fig. 1. More conventionally, the *residual* vorticity in 2D can be directly interpreted in terms of (twice) the *least-absolute-value* angular velocity of all line segments, within the flow plane, going through the given point and perpendicular to the vorticity direction.

The above mentioned interpretations of the *residual* vorticity are obviously good arguments for using this measure in vortex identification. Consequently, a new vortex-identification method is proposed and applied to typical vortical shear flows, turbulent jets and wakes, see Kolář /et al. (2004, 2007a, 2007b, based on data 1997, 2000, 2003, 2006).

3. Concluding remark

Different methods of vorticity decomposition are based on different physical grounds, and naturally provide different sets of new kinematic variables. Practical applications of these kinematic variables may range from flow classification schemes, through constitutive equations for complex rheological fluids, to vortex identification and the description of turbulent and/or complex vortical flows.

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