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IS THE LOGARITHMIC TIME DERIVATIVE SIMPLY THE ZAREMBA-JAUMANN DERIVATIVE?

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Summary: *The paper raises a question whether the logarithmic time derivative, expressed in specific coordinate system, is the Zaremba-Jaumann time derivative, and if not why. In fact, it has been already proved that the Z-J derivative represents the geometrically consistent linearization of tensor fields in terms of the covariant derivative in the space of right Cauchy-Green deformation tensors* $\mathbf{C}$. *This is the space of symmetric, positive-definite* 3×3 *matrices of real numbers* $Sym^+(3,\mathbb{R}) \cong GL^+(3,\mathbb{R})/SO(3,\mathbb{R})$*, which has a natural geometry of a Riemannian (globally) symmetric space of constant curvature, with the covariant derivative based on its Riemannian metric. Since in this geometry matrix exponentials stand for geodesics (that is a generalization of straight lines), the logarithmic strain* $\log(\mathbf{C})$ *can be interpreted simply as the change of coordinates in* $Sym^+(3,\mathbb{R})$*, called the normal coordinates. There are some indications that the Z-J time derivative expressed in this normal coordinate system might be the logarithmic time derivative.*

1. Introduction

According to Norris, the main development in the past 20 years in solid mechanics is the proof of Xiao et al. (1997a) that the relation

$$\overset{\circ}{\overline{\log(\mathbf{V})}} = d \tag{1}$$

between every possible left stretch $\mathbf{V}$ (from the polar decomposition of the deformation gradient $\mathbf{F} = \mathbf{VR}$) and the stretching $\mathbf{d}$ (the symmetric velocity gradient) holds true if and only if the corotational and objective time derivative, marked by '$\circ$', is the logarithmic one. Furthermore, only $\log(\mathbf{V})$ enjoys this property. This opinion can be found at the webpage *http://imechanica.org/node/1646* together with a discussion about the derivative of logarithmic strain, which in a way illustrates pretty well the state of understanding of the theory of finite deformations.

As a substantial ingredient, the logarithmic time derivative especially enters the two subjects: the work-conjugacy in Eulerian setting and the rate-type constitutive relations via hypoelasticity. The concept of conjugate Lagrangian strain E and stress S with material time derivative was introduced by Hill (1968): He defines the stress power per unite volume via

$$\pi^{ref} \equiv J\,\sigma : d = S : \dot{E}\,, \tag{2}$$

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where J denotes the Jacobian, and σ the Cauchy stress. The relation (1) enables then to claim that the Kirchhoff stress $\tau = J\sigma$ and the Eulerian strain $\log(\mathbf{V})$ with the logarithmic time derivative form the conjugate pair (Xiao et al., 1997c, 1998; Norris, 2008)).

Hypoelasticity, as a direct extension by Truesdell (1955a,b) of the Hooke's law to finite deformations, establishes a linear relationship between the objective rate (increment) of the Kirchhoff stress and the rate (increment) of deformation via a forth-order stress-dependent moduli tensor:

$$\overset{\circ}{\tau} = \mathbb{L}(\tau) : d \tag{3}$$

Due to its straightforward Eulerian rate form, hypoelasticity enters as a basic constituent into Eulerian rate-type formulations of inelastic material behaviours, in particular, metal plasticity. Without demanding any further requirements, equation (3) seems to permit the use of any objective rate, where the objectivity insures that any superimposed rigid rotating motion has no effect. This however proved not to be the case, and the choice of a particular objective stress rate is the most important part of the hypoelasticity theory, and not only of this theory (Naghdi et al., 1961; Guo, 1963; Durban et al., 1977; Matolcsi et al., 2007) to cite just a few.

Truesdell used the Lie derivative of the Kirchhoff stress, known as the Truesdell stress rate of the Cauchy stress (Marsden et al., 1993). Slightly later, since in addition it meets the requirement of Prager (1960) that vanishing of the stress rate implies the stationary behaviour of the stress invariants, the Zaremba-Jaumann stress rate was employed (see Biot (1965) for example)

$$\overset{\circ}{\tau}^{\mathrm{ZJ}} = \dot{\tau} - w\,\tau + \tau\, w\,, \tag{4}$$

where $w = \frac{1}{2}(\nabla\, v - (\nabla\, v)^{\mathrm{T}})$ is the vorticity tensor. However, when the material is subjected to finite simple shear deformation, Dienes discovered unexpected spurious phenomena known as shear oscillations. Other stress rates were therefore suggested and shown to be possible alternatives by means of the reasonable simple shear responses. These rates were either non-corotational, based on Lie derivative, such as the Cotter-Rivlin and Oldroyd stress rates (even though neither satisfies Prager's criterion), or corotational, such as the Green-Naghdi stress rate, stress rates based on the twirl tensor of Eulerian or Lagrangian triads as the spin, and finally the logarithmic one (Liu et al., 1999; Meyers et al., 2000; Lin, 2003). The corotational (or rotated) rates are defined in terms of the time-dependent spin tensor Ω, replacing w in (4), which measure the rate of change of tensors, as seen by an observer in a rotating frame specified by a rotating tensor Q, such that $\Omega = \dot{Q}Q$.

However reasonable response to a particular mode of deformation the stress rate offers, this can not serve as a criterion to draw decisive conclusion about the right choice. In fact, Simo and Pister in Simo et all. (1984) proved that, except for the logarithmic rate, none of such rate equations is exactly integrable to really define an elastic relation. That is, they generate a path-dependent stress-deformation via integration, and so they specity material behaviour, which is incompatible with hyperelasticity (Green-elasticity) and even with elasticity in general, since Cauchy-elasticity is in fact hyperelasticity (Casey, 2005). That is why many people try to avoid rate form equations of elasticity (particularly Truesdell's hypoelasticity) and favour approaches such as hyperelasticity (which can be derived from a potential) with some kind of linearization, in spite of some objections to this approach in elastoplasticity, highlighted in Xiao et al. (2007).

Still, among all possible objective corotational stress rates there is one and only one, namely the ***logarithmic stress rate*** (Xiao et al., 1997a, 1998; Bruhns et al., 2002)

$$\overset{\circ}{\tau}^{\log} = \dot{\tau} - \Omega^{\log}\,\tau + \tau\,\Omega^{\log}\,, \tag{5}$$

for which the hypoelastic equation with an initial natural (stress-free) state is exactly integrable (Xiao et al., 1997b, 1999a,b, 2002).

In our paper, we show first that a deformation process can be described as a curve in the space of symmetric, positive-definite 3×3 real matrices Sym^+ and introduce here a Riemannian metric via the stress power. Then, we shall describe the Riemannian geometry of Sym^+ to apply it further in the analysis of deformation of continua. Finally, we introduce and discuss normal coordinates in Sym^+, especially as far as the Zaremba-Jaumann time derivative is concerned.

2. Deformation process as a curve in the space of symmetric, positive-definite matrices

In this section we shall briefly summarize basic facts about deformation process (see Fiala (2008) for more). Let a continuous body $\mathcal{B}$ occupy a region of the three-dimensional Euclidean point space $\mathbb{E}^3$, considered here as the Riemannian manifold $\mathcal{E}^3$. That is, as a set of points with no privileged coordinate system, endowed with a Riemannian metric, which enters the manifold via an inner product of any two vectors emanating from the common point.

Globally, a deformation is represented by a diffeomorphism $\Phi : \mathcal{B} \to \mathcal{E}^3$ (i.e. a one-to-one map, which is differentiable together with its inverse) and a deformation process by a time-dependent diffeomorphism $\Phi : I \times \mathcal{B} \to \mathcal{E}^3$. However, within continuum mechanics one adopts a local point of view to describe a deformation process in terms of a time-dependent deformation field, expressed by means of the deformation gradient $\mathbf{F}$ – a linearized diffeomorphism Φ, and its transpose $\mathbf{F}^{\mathrm{T}}$. We shall consider here the *right Cauchy-Green deformation field* $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$, represented by a symmetric, positive-definite matrix, even though other deformation fields, such as the left Cauchy-Green, the Piola, or the Almansi might be also possible.

We shall adopt the following convention: In general, 2-tensors will be labelled in italic, but their specific representation as linear mappings in bold. Covariant 2-tensors will be denoted by $\flat$, contravariant by $\sharp$, and mixed will be without superscript. Furthermore, we shall denote $\partial C_t^\flat := \frac{\partial}{\partial t} C_t^\flat$.

STARTING POINT: *From the viewpoint of finite deformations, a deformation process can be represented by a trajectory* $C^\flat : I \to \mathcal{M}$ *in the space* $\mathcal{M} = \mathrm{Met}(\mathcal{B})$ *of all (covariant) deformation tensors on the reference configuration* $\mathcal{B}$*, or equivalently by a trajectory* $\mathbf{C} : I \to Sym^+$*, in the set of all symmetric, positive-definite real matrices. If the initial configuration is unstrained with an initial condition* $C_0^\flat = G$*, resp* $\mathbf{C}_0 = \mathbf{I}$*, where* $\mathbf{I}$ *stands for the identity matrix.*

Since both $C^\flat$ and G belong to the same tensor space, we can subtract G from $C^\flat$ to find a relative deformation (strain) – the Green-St. Venant strain tensor $\frac{1}{2}(C^\flat - G)$, resp $\frac{1}{2}(\mathbf{C} - \mathbf{I})$.

To every diffeomorphism $\Phi : \mathcal{B} \to \mathcal{S}$ there is one-to-one mapping between corresponding spaces of tensors – ***push-forward*** Φ_* and its inverse – ***pull-back*** Φ^*. We can write Fiala (2008):

(1) $C^\flat = \Phi^*(g)$ (or $\mathbf{C}^\flat = \mathbf{\Phi}^*(\mathbf{g})$), where g is Riemannian metric on actual configuration $\mathcal{S}$, and $\mathbf{C}^\flat = \mathbf{G}\mathbf{C}$, where G is Riemannian metric on referential configuration $\mathcal{B}$

(2) $\partial C_t^\flat = 2\Phi_t^*(d^\flat)$, where $d^\flat$ is covariant form of the symmetric velocity gradient d, that is $2d = \nabla v_t + (\nabla v_t)^{\mathrm{T}} = (v_t^i|_j + v_{t\,j}|^i)\,\partial x^i \otimes dx^j$, where ∇v_t stands for covariant derivative; $\mathbf{D} := \mathbf{\Phi}^*(\mathbf{d}) = \mathbf{\Phi}^*(\mathbf{g}^\sharp \mathbf{d}^\flat) = \mathbf{\Phi}^*(\mathbf{g}^\sharp)\mathbf{\Phi}^*(\mathbf{d}^\flat) = \mathbf{B}^\sharp \frac{1}{2}\,\partial\mathbf{C}^\flat = \frac{1}{2}\,\mathbf{C}^{-1}\partial\mathbf{C}$, where $\partial\mathbf{C} = \partial(\mathbf{F}^{\mathrm{T}}\mathbf{F})$

(3) $B^\sharp = \Phi^*(g^\sharp)$ (or $\mathbf{B}^\sharp = \mathbf{B}\mathbf{G}^{-1}$) is the contravariant *Piola deformation tensor*, and $\mathbf{B} = \mathbf{C}^{-1}$

The following proposition relates tangent vectors $\partial C^\flat$ along curves C_t in $\mathcal{M}$ through $C^\flat$ to small deformations. Let us consider a tangent space $T_{C^\flat}\mathcal{M}$ to the manifold $\mathcal{M}$ at a point $C^\flat$.

By definition, it consists of all *deformation rates* $\partial C^\flat$ starting at $C^\flat$, i.e. of tangent vectors to all curves $C_t^\flat$ through the point $C^\flat$. Similarly, denote by $T_{\mathbf{C}}Sym^+$ the corresponding tangent space of Sym^+ at $\mathbf{C}$.

PROPOSITION 1: *Within small deformations, a deformation process superposed on the initially strained body, which is characterized by the initial deformation field $C^\flat$, is represented by a trajectory in the linear vector space $T_{C^\flat}\mathcal{M}$ – the tangent space to the manifold $\mathcal{M}$ at point $C^\flat$, or equivalently by a trajectory in $T_{\mathbf{C}}Sym^+ \approx sym$ – the vector space of symmetric matrices.*

For proof see Fiala (2008). The equivalent form then follows from the fact that Sym^+ is open in sym (see next section), and so the tangent space $T_{\mathbf{C}}Sym^+ \cong sym$. That is, the vector space of all tangent vectors emanating from a common footpoint $\mathbf{C} \in Sym^+$ is again sym.

PROPOSITION 2: *One can naturally introduce Riemannian metric on $\mathcal{M}$ to become a manifold with Riemannian geometry of Sym^+ – the space of symmetric, positive-definite matrices.*

Proof. Let us consider the power of internal forces (*stress power*)

$$\begin{aligned} \frac{\delta E_i}{\delta t} &:= \int_{\mathcal{S}} (\sigma : d)\, dv \equiv \int_{\mathcal{S}} \sigma_j^{\ i}\, d_i^{\ j}\ dv = \int_{\mathcal{S}} g^{ik} g^{jl} \sigma_{kl}\, d_{ij}\ dv = \\ &= \int_{\mathcal{B}} B_t^{ik} B_t^{jl} K_{kl}\, \tfrac{1}{2}\, \partial C_{t\,ij}\, dV = \int_{\mathcal{B}} \Omega_{C_t^\flat}\left(\tfrac{1}{\rho_{\mathcal{B}}} K^\flat, \tfrac{1}{2}\,\partial C_t^\flat\right) dm = \omega_{C_t^\flat}\left(\tfrac{1}{\rho_{\mathcal{B}}} K^\flat, \tfrac{1}{2}\,\partial C_t^\flat\right), \end{aligned} \tag{6}$$

where we employed relations $\partial C_t^\flat = 2\Phi_t^*(d^\flat)$, $JdV = \Phi_t^*(dv)$, and $dV = (\det G)^{\frac{1}{2}} dX$ for volume element. The symbol σ as usual stands for the Cauchy stress field, and $\frac{1}{\rho_{\mathcal{B}}} K^\flat$ represents the *convective stress* related to mass, instead of volume element. From the mathematical point of view, the space $\mathcal{M}$ forms an infinite-dimensional manifold, but its geometry factorises into identical geometries of individual spaces $\mathcal{M}_X$, made up of all metric tensors at the point $X \in \mathcal{B}$, with Riemannian metric

$$\Omega_{C_t^\flat}(D^\flat, H^\flat) = B_t^{ik} B_t^{jl} D_{kl} H_{ij} = \mathrm{tr}\,(\mathbf{B}_t^\sharp \mathbf{D}^\flat \mathbf{B}_t^\sharp \mathbf{H}^\flat) = \mathrm{tr}\,(\mathbf{C}_t^{-1}\mathbf{D}\mathbf{C}_t^{-1}\mathbf{H}) \tag{7}$$

for $D^\flat, H^\flat \in T_{C_t^\flat(X)}\mathcal{M}_X$. In the next section we shall show that this is a natural Riemannian metric on Sym^+, making it *Riemannian (globally) symmetric space* (cf. (21)).

A Riemannian metric on the space $\mathcal{M}_X$ can also be introduced via the inner product of vectors $\partial C^\flat$ to curves $C_t^\flat$ passing through the point $C^\flat \in \mathcal{M}_X$. Since the inner product $g(u, v) = g_{ij} u^i v^j = u_i v^i$ of two vectors from $\mathcal{S}$ naturally extends to the scalar product of symmetric covariant 2-tensors via

$$(g^\sharp \otimes g^\sharp)(d^\flat, h^\flat) = g^{ik} g^{jl}\, d_{kl}\, h_{ij} = d_j^{\ i}\, h_i^{\ j} = d : h, \tag{8}$$

its corresponding counterpart in reference configuration $\mathcal{B}$ is again (7), thanks to the relation $\partial C^\flat = 2\Phi_t^*(d^\flat)$. In order to extend this inner product to fields, i.e. to the whole $\mathcal{M}$, it remains yet to integrate all the contributions from particular points X through the space $\mathcal{B}$. In principle, we have two opportunities: integrate by volume or mass element, but unlike the introduction of Riemannian metric via stress power, now the choice is not a priory obvious.

Considering Sym^+ as the Riemannian manifold rather than a subset of the vector space of all symmetric matrices enables us to analyse in a geometrically consistent way a deformation process by means of geometrical tools of Riemannian geometry. Surprisingly, we thus arrive at the Zaremba-Jaumann time derivative, logarithmic strain, and possibly at the logarithmic time derivative. Before analysing deformation process, we therefore introduce in detail some geometrical facts about Sym^+ in the next section. This approach to mechanics of continua was initiated by Rougée (1997) (see also Rougée (2006)), and modified by Fiala (2004), where some results based on integration by volume element can be found.

3. Riemannian geometry of the space of symmetric, positive-definite matrices $Sym^+(n)$

The purpose of this section is to give a self-contained exposition of the natural intrinsic Riemannian geometry of the space of symmetric, positive-definite matrices $Sym^+(n)$ (Bhatia, R., 1997, 2007).

First, let us consider a nonsingular matrix $\mathbf{F} \in GL^+(n)$ and its polar decomposition $\mathbf{F} = \mathbf{RU}$, for which $\mathbf{U} \in Sym^+(n)$, and $\mathbf{R} \in SO(n)$ – the group of orthogonal matrices with determinant one, satisfying $\mathbf{R}^{\mathrm{T}} = \mathbf{R}^{-1}$. We denote by $GL^+(n)/SO(n)$ the space of all left cosets of $GL^+(n)$

$$GL^+(n)/SO(n) := \{SO(n)\mathbf{F} \,|\, \mathbf{F} \in GL^+(n)\}, \tag{9}$$

and by $[\mathbf{F}]$ the particular left coset corresponding to $\mathbf{F}$. Note that $[\mathbf{I}] \equiv SO(n)$ for the identity matrix $\mathbf{I}$. From uniqueness of polar decomposition it follows that each left coset contains a unique symmetric, positive-definite matrix given by $\mathbf{U} = (\mathbf{F}^{\mathrm{T}}\mathbf{F})^{\frac{1}{2}} \in [\mathbf{F}] \equiv [\mathbf{U}]$, and so we can identify the space $Sym^+(n)$ with the left coset space

$$Sym^+(n) \cong GL^+(n)/SO(n) = \mathbb{R}^+ \times SL(n)/SO(n). \tag{10}$$

This space further splits into two irreducible factors – the group of positive real numbers with group operation of number multiplication $\mathbb{R}^+$and the space of symmetric, positive-definite matrices of determinant one.

Based on this identification, we can derive the geometrical properties of $Sym^+(n)$, which is no longer a group, but belongs to the family of so-called *Riemannian (globally) symmetric spaces*. These are Riemannian manifolds with *(global) central symmetry* through every point, called involutive isometry (cf. (29)), which reverses all geodesics through this point. Geometry of Riemannian symmetric spaces is then characterized by a high degree of symmetry, so that they constitute a natural generalization of the Euclidean space. In fact, the space $Sym^+(n)$ is *homogeneous* with respect to the group of translations $GL(n)$, cf. (16) – i.e. its geometry in the vicinity of any point is the same, and it is *isotropic* with respect to the group of rotation $SO(n)$, cf. (24) – i.e. its geometry is also the same in all directions. Thus the geometry of $Sym^+(n)$ is the same at every point, as in the Euclidean space.

Before introducing the Riemannian metric, we shall specify these symmetries. We can introduce the right action of $GL(n)$ on $GL^+(n)/SO(n)$, based on the right group operations in $GL(n)$. For $\mathbf{A} \in GL(n)$ and $[\mathbf{F}] \in GL^+(n)/SO(n)$ it is given by

$$r_{\mathbf{A}}([\mathbf{F}]) := [\mathbf{FA}], \tag{11}$$

that is by

$$r_{\mathbf{A}}([(\mathbf{F}^{\mathrm{T}}\mathbf{F})^{\frac{1}{2}}]) := [(\mathbf{A}^{\mathrm{T}}(\mathbf{F}^{\mathrm{T}}\mathbf{F})\mathbf{A})^{\frac{1}{2}}]. \tag{12}$$

Because any symmetric, positive-definite matrix has exactly one square root, the mapping $f : GL(n) \to Sym^+(n)$ given by $f(\mathbf{F}) = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ is, when restricted to $Sym^+(n) \subset GL(n)$, one-to-one. Then the transformations $\rho_{\mathbf{A}} \equiv f \circ r_{\mathbf{A}} \circ f^{-1}$ form a group isomorphic to the group of transformations $r_{\mathbf{A}}$, and the group of movements induced by $GL(n)$ acts in $Sym^{+}(n)$ on the right by the formula

$$\rho_{\mathbf{A}}(\mathbf{U}) := \mathbf{A}^{\mathrm{T}}\mathbf{U}\mathbf{A}\,. \tag{13}$$

For identity matrix $\mathbf{I}$ and any $\mathbf{R} \in SO(n)$

$$\rho_{\mathbf{R}}(\mathbf{I}) = \mathbf{I}\,, \tag{14}$$

and so $SO(n)$ specifies a subgroup of rotations around $\mathbf{I}$ from all the movements of $GL(n)$. Note that we can start with $\mathbf{F} = \mathbf{VR}$, then employ the right coset space and the left group operation to obtain $\lambda_{\mathbf{A}}(\mathbf{V}) := \mathbf{AVA}^{\mathrm{T}}$ – the left action of $GL(n)$ on $GL^{+}(n)/SO(n)$. Both approaches are however equivalent.

Now, we introduce on $Sym^{+}(n)$ a Riemannian metric and prove, that described the symmetries are in fact isometries. First, the space $M(n)$ can be identified with $\mathbb{R}^{n^2}$ and as such, both $M(n)$ and $Sym^{+}(n)$ inherit a natural topological and differentiable structure. Moreover, the usual scalar product $(\,.\,,.)$ induces in $M(n)$ the corresponding inner product (Frobenius product) given by

$$\langle \mathbf{A}, \mathbf{B} \rangle_{\mathbf{I}} \equiv (vec[\mathbf{A}], vec[\mathbf{B}]) = \mathrm{tr}\,(\mathbf{A}^{\mathrm{T}}\mathbf{B})\,, \tag{15}$$

where $\mathrm{tr}\,(.)$ denotes the trace operator, and by $vec[\mathbf{A}]$ we denote the n^2-column vector, which is obtained by stacking the columns of $\mathbf{A}$ in one row. This way the subspace $sym(n)$ becomes an inner product vector space. Since $Sym^{+}(n)$ is open in $sym(n)$, actually an open convex cone (for n = 2, see picture in Moakher (2005)), the tangent space $T_{\mathbf{C}}Sym^{+} \cong sym(n)$. That is, the vector space of all tangent vectors emanating from the common footpoint $\mathbf{C} \in Sym^{+}(n)$ is again $sym(n)$.

To any translation by $\mathbf{A} \in GL^{+}(n)$ given by

$$\rho_{\mathbf{A}}(\mathbf{U}) := \mathbf{A}^{\mathrm{T}}\mathbf{U}\mathbf{A}\,, \tag{16}$$

there corresponds mutually inverse mappings of vectors – the *push-forward* operation $[\,\rho_{\mathbf{A}}]_*$ and the *pull-back* operation $[\,\rho_{\mathbf{A}}]^*$ between corresponding tangent spaces

$$[\,\rho_{\mathbf{A}}]_* : T_{\mathbf{U}}\,Sym^{+} \to\; T_{\rho_{\mathbf{A}}(\mathbf{U})}Sym^{+} \tag{17}$$

$$[\,\rho_{\mathbf{A}}]^* : T_{\rho_{\mathbf{A}}(\mathbf{U})}Sym^{+} \to\; T_{\mathbf{U}}\,Sym^{+}, \tag{18}$$

which satisfy

$$[\,\rho_{\mathbf{A}}]_*(\mathbf{H}) = \mathbf{A}^{\mathrm{T}}\mathbf{H}\mathbf{A} \tag{19}$$

$$[\,\rho_{\mathbf{A}}]^*(\mathbf{G}) = \mathbf{A}^{-\mathrm{T}}\mathbf{G}\mathbf{A}^{-1}. \tag{20}$$

This can be readily derived from transformation of a curve $l(t)$ in $Sym^{+}(n)$ through the point $l(0) = \mathbf{U}$, with a tangent vector $\partial l(0) = \mathbf{H}$.

Now, the Riemannian metric at $\mathbf{C} \equiv \mathbf{U}^2 \in Sym^{+}(n)$ (since $\mathbf{U}^2 = \rho_{\mathbf{U}}(\mathbf{I})$) given by

$$\Omega_{\mathbf{C}}(\mathbf{D},\mathbf{H}) := \langle [\,\rho_{\mathbf{U}}]^*(\mathbf{D}), [\,\rho_{\mathbf{U}}]^*(\mathbf{H}) \rangle_{\mathbf{I}} = \mathrm{tr}\,(\mathbf{C}^{-1}\mathbf{D}\mathbf{C}^{-1}\mathbf{H})\,, \tag{21}$$

is expressed in terms of inner product of any two vectors $\mathbf{D}, \mathbf{H} \in T_{\mathbf{C}}Sym^{+} \cong sym(n)$ emanating from the common footpoint $\mathbf{C} \in Sym^{+}(n)$ (cf. (7)). This metric is natural, because is obtained by pushing forward the usual trace norm for matrices at $\mathbf{I}$ to the whole $Sym^{+}(n)$

$$\Omega_{\rho_{\mathbf{U}}(\mathbf{I})} = [\,\rho_{\mathbf{U}}]_*\,(\,\Omega_{\mathbf{I}}), \tag{22}$$

and hence is necessarily invariant with respect to the group of movements $GL^+(n)$ acting on $Sym^+(n)$

$$\Omega_{\mathbf{C}}(\mathbf{D},\mathbf{H}) = \Omega_{\rho_{\mathbf{A}}(\mathbf{C})}([\,\rho_{\mathbf{A}}]_*(\mathbf{D}),[\,\rho_{\mathbf{A}}]_*(\mathbf{H}))\,. \tag{23}$$

For the rotation $\theta_{\mathbf{R}}$ with amplitude $\mathbf{R} \in SO(n)$ around $\mathbf{I}$ we have $\theta_{\mathbf{R}} \equiv \rho_{\mathbf{R}}$, and around a general point $\mathbf{P} = \mathbf{U}^2$ we obtain, via composition with translations

$$\theta^{\mathbf{P}}_{\mathbf{R}} = \rho_{\mathbf{U}} \circ \theta_{\mathbf{R}} \circ \rho_{\mathbf{U}^{-1}} = \rho_{\mathbf{U}^{-1}\mathbf{R}\mathbf{U}}. \tag{24}$$

The assertion about homogenity and isotropy of $Sym^+(n)$ is thus proved.

Moreover, the metric is also invariant with respect to the central symmetry $\sigma_{\mathbf{I}}$ through $\mathbf{I}$

$$\Omega_{\mathbf{C}}(\mathbf{D},\mathbf{H}) = \Omega_{\sigma_{\mathbf{I}}(\mathbf{C})}([\,\sigma_{\mathbf{I}}]_*(\mathbf{D}),[\,\sigma_{\mathbf{I}}]_*(\mathbf{H})) \tag{25}$$

defined by

$$\sigma_{\mathbf{I}}(\mathbf{C}\,) := \mathbf{C}^{-1}, \tag{26}$$

where

$$[\,\sigma_{\mathbf{I}}]_*(\mathbf{H}) = -\mathbf{C}^{-1}\mathbf{H}\mathbf{C}^{-1} \tag{27}$$

$$[\,\sigma_{\mathbf{I}}]^*(\mathbf{G}) = -\mathbf{C}\mathbf{G}\mathbf{C}\,. \tag{28}$$

This can be proved again from transformation of a curve $l(t)$ in $Sym^+(n)$ through the point $l(0) = \mathbf{C}$, with a tangent vector $\partial l(0) = \mathbf{H}$, making use of the identity $l(t)\,\sigma_{\mathbf{I}}(l(t)) = \mathbf{I}$. The central symmetry with respect to an arbitrary point $\mathbf{P} = \mathbf{U}^2$, leaving it invariant, is obtained via composition with translations:

$$\sigma_{\mathbf{P}}(\mathbf{Q}) = \rho_{\mathbf{U}} \circ \sigma_{\mathbf{I}} \circ \rho_{\mathbf{U}^{-1}}(\mathbf{Q}) = \mathbf{P}\mathbf{Q}^{-1}\mathbf{P}\,. \tag{29}$$

This high degree of symmetry is closely related to the fact that $Sym^+(n)$ has constant negative curvature: For more about $Sym^+(n)$ from the viewpoint of symmetric spaces see Jost (2002) chap. 5.4, and from the viewpoint of spaces of non-positive curvature see Bridson et al. (1999), chap II.10.

To specify the Riemannian *covariant derivative* (Ohara et al., 1996) note that

$$\begin{aligned} 2\Omega_{\mathbf{C}}(\nabla_{\mathbf{G}}\mathbf{H},\mathbf{D}) = {}& \delta_{\mathbf{G}}\Omega_{\mathbf{C}}(\mathbf{H},\mathbf{D}) + \delta_{\mathbf{H}}\Omega_{\mathbf{C}}(\mathbf{D},\mathbf{G}) - \delta_{\mathbf{D}}\Omega_{\mathbf{C}}(\mathbf{G},\mathbf{H}) + \\ & + \Omega_{\mathbf{C}}([\mathbf{G},\mathbf{H}],\mathbf{D}) + \Omega_{\mathbf{C}}([\mathbf{D},\mathbf{G}],\mathbf{H}) - \Omega_{\mathbf{C}}([\mathbf{H},\mathbf{D}],\mathbf{G})\,. \end{aligned} \tag{30}$$

Because for matrices $[\mathbf{H},\mathbf{G}] := \mathbf{H}\mathbf{G} - \mathbf{G}\mathbf{H}$, and due to $\mathrm{tr}(\mathbf{A}\mathbf{B}) = \mathrm{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}})$

$$\Omega_{\mathbf{C}}([\mathbf{G},\mathbf{H}],\mathbf{D}) = \Omega_{\mathbf{C}}([\mathbf{D},\mathbf{G}],\mathbf{H}) = \Omega_{\mathbf{C}}([\mathbf{H},\mathbf{D}],\mathbf{G}) = 0 \tag{31}$$

Next, since

$$\delta_{\mathbf{G}}\Omega_{\mathbf{C}}(\mathbf{H},\mathbf{D}) = -\frac{\partial}{\partial t}\,\Omega_{(\mathbf{C}+t\mathbf{G})}(\mathbf{H},\mathbf{D})\bigg|_{t=0} = -2\mathrm{tr}\,(\mathbf{C}^{-1}\mathbf{G}\mathbf{C}^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{D}), \tag{32}$$

and due to $\mathrm{tr}(\mathbf{A}\mathbf{B}) = \mathrm{tr}(\mathbf{B}\mathbf{A})$ and due to

$$\delta_{\mathbf{G}}\Omega_{\mathbf{C}}(\mathbf{H},\mathbf{D}) = \delta_{\mathbf{H}}\Omega_{\mathbf{C}}(\mathbf{D},\mathbf{G}) = \delta_{\mathbf{D}}\Omega_{\mathbf{C}}(\mathbf{G},\mathbf{H}) \tag{33}$$

one obtains

$$\Omega_{\mathbf{C}}(\nabla_{\mathbf{G}}\mathbf{H},\mathbf{D}) = -\mathrm{tr}\,(\mathbf{C}^{-1}\mathbf{G}\mathbf{C}^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{D})\,. \tag{34}$$

Also note that

$$\Omega_{\mathbf{C}}(\mathbf{H},\mathbf{D}) = -\delta_{\mathbf{H}}\delta_{\mathbf{D}}S(\mathbf{C}) = -\frac{\partial^2}{\partial s\,\partial t}\,S(\mathbf{C}+t\mathbf{H}+s\mathbf{D})\Big|_{t=s=0} \tag{35}$$

where $\delta_{\mathbf{G}}$ is directional derivative, and $S(\mathbf{C}) := \log(\det\mathbf{C}) = \mathrm{tr}(\log\mathbf{C})$.

Let $\mathbf{E}_\alpha$, α=1, …, $\frac{1}{2}n(n+1)$ denote a coordinate frame, then since Riamannian connections are symmetric or torsion-free, i.e. $\Gamma(\mathbf{E}_\alpha,\mathbf{E}_\beta) := \nabla_{\mathbf{E}_\alpha}\mathbf{E}_\beta = \nabla_{\mathbf{E}_\beta}\mathbf{E}_\alpha$, for the *Christoffel symbols* from (34) one gets

$$\Gamma_{\mathbf{C}}(\mathbf{E}_\alpha,\mathbf{E}_\beta) = -\frac{1}{2}\left(\mathbf{E}_\alpha\mathbf{C}^{-1}\mathbf{E}_\beta + \mathbf{E}_\beta\mathbf{C}^{-1}\mathbf{E}_\alpha\right). \tag{36}$$

For general vector fields $\mathbf{G}$, $\mathbf{K}$ then

$$\nabla_{\mathbf{G}}\mathbf{K} := \delta_{\mathbf{G}}\mathbf{K} + \Gamma_{\mathbf{C}}(\mathbf{G},\mathbf{K})\,, \tag{37}$$

where $\delta_{\mathbf{G}}\mathbf{K}$ denotes derivation of the field $\mathbf{K}$ in the $\mathbf{G}$ direction, performed in the ambient space sym.

Since the corresponding covector $\mathbf{P} := \mathbf{\Omega}_{\mathbf{C}}\mathbf{K} = \mathbf{C}^{-1}\mathbf{K}\mathbf{C}^{-1}$ (cf. (21)), and because the Riemannian metric $\mathbf{\Omega}_{\mathbf{C}}$ is covariant constant, i.e.

$$\nabla_{\mathbf{G}}\left(\mathbf{C}^{-1}\mathbf{K}\mathbf{C}^{-1}\right) = \mathbf{C}^{-1}\left(\nabla_{\mathbf{G}}\mathbf{K}\right)\mathbf{C}^{-1}\,, \tag{38}$$

for covariant derivative of the covector field we deduce from (37)

$$\nabla_{\mathbf{G}}\mathbf{P} := \delta_{\mathbf{G}}\mathbf{P} + \Gamma^*_{\mathbf{C}}(\mathbf{G},\mathbf{P})\,, \tag{39}$$

where now

$$\Gamma^*_{\mathbf{C}}(\mathbf{G},\mathbf{P}) = \frac{1}{2}\left(\mathbf{P}\mathbf{G}\mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{G}\mathbf{P}\right). \tag{40}$$

The Riemannian metric determines via the covariant derivative the equation for *geodesics*. As locally the shortest curves between two points, they represent in Riemannian geometry a generalization of straight lines. A curve $\mathbf{C}(t)$ is the geodesic, if for its tangent vectors $\partial\mathbf{C}(t)$

$$\nabla_{\partial\mathbf{C}}\,\partial\mathbf{C} = \partial^2\mathbf{C} - (\partial\mathbf{C})\mathbf{C}^{-1}\partial\mathbf{C} = \mathbf{C}\,\partial(\mathbf{C}^{-1}\partial\mathbf{C}) = 0\,. \tag{41}$$

The equation reduces to

$$\partial\mathbf{C} = \mathbf{C}\mathbf{D}\,, \tag{42}$$

where $\mathbf{D}$ is any constant matrix in $sym(n)$. The solution for initial conditions $\mathbf{C}(0) \equiv \mathbf{C}_0 = \mathbf{U}^2 \in Sym^+(n)$ and $\partial\mathbf{C}(0) \equiv \partial\mathbf{C}_0 = \mathbf{U}\mathbf{D}\mathbf{U} \in sym(n)$ is then given by

$$\mathbf{C}(t) = \mathbf{U}\exp(t\,\mathbf{D})\mathbf{U} = \mathbf{C}_0\exp\left(t\,\mathbf{C}_0^{-1}\partial\mathbf{C}_0\right), \tag{43}$$

where the last equality holds true because the matrix exponential is an analytic function. Note that this geodesic can be obtained by means of translation by $\mathbf{U}$

$$\mathbf{C}(t) = \rho_{\mathbf{U}}(\exp\,(t\,\mathbf{D})) \tag{44}$$

from the geodesic $\exp\,(t\,\mathbf{D})$ through $\mathbf{I}$, with the initial vector $\mathbf{D}$. Moreover, any two points in $Sym^+(n)$ (i.e. symmetric positive-definite matrices) can be joined by exactly one geodesic, there is exactly one geodesic for any initial vector from sym, and these geodesics can be further extended without limits (Hopf - Rinow theorem about geodesic completeness).

4. Continuum mechanics via Riemannian geometry of Sym^+

From the viewpoint of Sym^+, the deformation process is a ***curve***, and so the deformation at a particular time, represented by the deformation tensor $\mathbf{C}_t$, is a ***point***. The rate of deformation $\partial\mathbf{C}_t$ at $\mathbf{C}_t$ is a ***vector*** attached to this particular point $\mathbf{C}_t$, as well as the convective stress tensor $\mathbf{K}$, corresponding to $K^\flat = \Phi_t^*(J\sigma^\flat)$. On the other hand (cf. (6)), due to the first term in the third row of

$$\begin{aligned} \frac{\delta E_i}{\delta t} &:= \int_{\mathcal{B}} \Omega_{C_t^\flat}\left(\tfrac{1}{\rho_{\mathcal{B}}}K^\flat, \tfrac{1}{2}\,\partial C_t^\flat\right) dm = \omega_{C_t^\flat}\left(\tfrac{1}{\rho_{\mathcal{B}}}K^\flat, \tfrac{1}{2}\,\partial C_t^\flat\right) \\ &:= \int_{\mathcal{S}} \sigma^{ij} d_{ij}\ dv = \int_{\mathcal{B}} P^{ij}\tfrac{1}{2}\,\partial C_{t\,ij}\, dV = \\ &= \int_{\mathcal{B}} \langle \tfrac{1}{\rho_{\mathcal{B}}}P^\sharp, \tfrac{1}{2}\,\partial C_t^\flat\rangle_{T_{C_t^\flat(X)}\mathcal{M}}\, dm = \left\langle\!\!\left\langle \tfrac{1}{\rho_{\mathcal{B}}}P^\sharp, \tfrac{1}{2}\,\partial C_t^\flat \right\rangle\!\!\right\rangle_{T_{C_t^\flat}\mathcal{M}}, \end{aligned} \tag{45}$$

the *second Piola-Kirchhoff stress* tensor $\mathbf{P} := \mathbf{\Omega}_{\mathbf{C}_t}\mathbf{K} = \mathbf{C}_t^{-1}\mathbf{K}\mathbf{C}_t^{-1}$, corresponding to $P^\sharp = \Phi_t^*(J\sigma^\sharp)$, is a ***covector*** (i.e. covariant vector), also attached to this particular point $\mathbf{C}_t$. In fact, (see (7))

$$\langle P^\sharp, \partial C_t^\flat\rangle_{T_{C_t^\flat(X)}\mathcal{M}} = \mathrm{tr}\,(\mathbf{C}_t^{-1}\mathbf{K}\mathbf{C}_t^{-1}\partial\mathbf{C}_t) = \mathrm{tr}\,(\mathbf{P}\partial\mathbf{C}_t) \equiv \langle \mathbf{P}, \partial\mathbf{C}_t\rangle_{T_{\mathbf{C}_t}Sym^+}\,. \tag{46}$$

PROPOSITION 3: *The Kirchhoff stress rate is given by the Zaremba-Jaumann time derivative.*
Due to the above, in order to obtain a geometrically consistent stress rate, we have to linearize in Sym^+ the ***vector/covector field*** over the curve representing a deformation process. Since Sym^+ in curved, we have to resort to the covariant derivative, based on the metric Ω, with respect to the deformation rate $\partial C_t^\flat$. This automatically guarantees an objectivity of the time derivative introduced. Rather surprisingly, what we obtain proves to be the Zaremba-Jaumann stress rate (Fiala, 2008).

In fact, for a time-dependent covector or vector field $\mathbf{\Theta} \in Sym^+$ we set (cf. Fiala (2008))

$$\frac{D}{dt}\mathbf{\Theta} := \frac{\partial\mathbf{\Theta}}{\partial t} + \nabla_{\partial\mathbf{C}}\,\mathbf{\Theta}\,, \tag{47}$$

and so from (37) for a time-dependent *vector field* $\mathbf{K}$ along a trajectory $\mathbf{C}_t$ in Sym^+, we get

$$\frac{D}{dt}\mathbf{K} := \frac{\partial}{\partial t}\mathbf{K} + \nabla_{\partial\mathbf{C}_t}\mathbf{K} = \frac{d}{dt}\mathbf{K} - \frac{1}{2}\left\{(\partial\mathbf{C}_t)\,\mathbf{C}_t^{-1}\mathbf{K} + \mathbf{K}\,\mathbf{C}_t^{-1}\,(\partial\mathbf{C}_t)\right\}, \tag{48}$$

For a *covector field* $\mathbf{P} := \mathbf{\Omega}_{\mathbf{C}}\mathbf{K} = \mathbf{C}^{-1}\mathbf{K}\mathbf{C}^{-1}$, analogically from (39) we obtain

$$\frac{D}{dt}\mathbf{P} := \frac{\partial}{\partial t}\mathbf{P} + \nabla_{\partial\mathbf{C}_t}\mathbf{P} = \frac{d}{dt}\mathbf{P} + \frac{1}{2}\left\{\mathbf{P}\,(\partial\mathbf{C}_t)\,\mathbf{C}_t^{-1} + \mathbf{C}_t^{-1}(\partial C_t)\,\mathbf{P}\right\}. \tag{49}$$

For their corresponding mixed 2-tensor on $\mathcal{E}^3$ – the Kirchhoff stress $\boldsymbol{\tau} = \mathbf{F}^{-\mathrm{T}}\,\mathbf{K}\mathbf{F}^{-1} = \mathbf{F}\mathbf{P}\mathbf{F}^{\mathrm{T}}$, the stress rate will be given by

$$\frac{D}{dt}\boldsymbol{\tau} := \mathbf{F}^{-\mathrm{T}}\left(\frac{D}{dt}\mathbf{K}\right)\mathbf{F}^{-1} = \mathbf{F}\left(\frac{D}{dt}\mathbf{P}\right)\mathbf{F}^{\mathrm{T}}, \tag{50}$$

which results in the Zaremba-Jaumann time derivative (Fiala, 2008)

$$\frac{D}{dt}\boldsymbol{\tau} = \dot{\boldsymbol{\tau}} - \boldsymbol{w}\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{w} \equiv \mathring{\boldsymbol{\tau}}^{\mathbf{ZJ}}. \tag{51}$$

Unlike Fiala (2008), where we used the pull-back operation Φ^*, the relations (50) look a bit cumbersome. But this is due to our starting from mixed 2-tensors, which combine metric tensors from both reference G and actual g configurations, and so mixed tensors are less natural than the covariant and contravariant 2-tensors. In fact, we may write $\boldsymbol{\tau} = \mathbf{g}^{-1}\Phi^*(\mathbf{GK}) = \Phi^*(\mathbf{PG}^{-1})\mathbf{g}$, and correspondingly for (50).

Note an important fact, that the proof of proposition establishes *the stress rate of the second Piola-Kirchhoff* $\mathbf{P}$ and of *the convected* $\mathbf{K}$ *stress mixed tensors* to find out that it is not the simple time derivative $\frac{\partial}{\partial t}$, nor the material time derivative $\frac{d}{dt}$, but it is the *derivative based on covariant derivative* $\frac{D}{dt}$, which in actual configuration corresponds to the Zaremba-Jaumann time derivative of the Kirchhoff stress. Even though this might seem contentious in view of the problems mentioned in connection with its use in hypoelasticity, the following considerations based on the Hopf – Rinow theorem about geodesic completeness of Sym^+ might support this time derivative.

THEOREM (Hopf - Rinow) (see Bridson et al. (1999); Jost (2002)): *$Sym^+(n)$ is geodesically complete. That is, for every $\mathbf{C}_0 \in Sym^+(n)$, the exponential map*

$$\mathrm{Exp}_{\mathbf{C}_0}(.) := \mathbf{C}_0 \exp(\mathbf{C}_0^{-1}(.)) \tag{52}$$

maps the entire tangent space $T_{\mathbf{C}_0}Sym^+(n) \approx sym(n)$ onto $Sym^+(n)$ bijectively (i.e. one-to-one) and differentiably.

Among others, the theorem says that any two symmetric, positive matrices can be connected by exactly one, unlimited geodesic (straight line).

In our context, the generalized exponential map (52) (cf. (43)) of the theorem adds up an increment of deformation $\mathbf{H} \equiv \partial\mathbf{C}_0 \in T_{\mathbf{C}_0}Sym^+$ to the deformation $\mathbf{C}_0 \in Sym^+$, so that the resulting deformation $\mathbf{C}_1(\mathbf{H}) = \mathrm{Exp}_{\mathbf{C}_0}(\mathbf{H})$ stays in the space of deformations Sym^+. This will not be the case if we set just $\mathbf{C}_1(\mathbf{H}) = \mathbf{C}_0 + \mathbf{H}$.

PROPOSITION 4: *Resulting deformation $\mathbf{C}_1(\mathbf{H})$ from adding an increment of deformation $\mathbf{H}$ to the deformation $\mathbf{C}_0$ is given by*

$$\begin{aligned} \mathbf{H} \longmapsto \ \mathbf{C}_1(\mathbf{H}) \equiv \mathrm{Exp}_{\mathbf{C}_0}(\mathbf{H}) :&= \mathbf{C}_0 \exp(\mathbf{C}_0^{-1}\,\mathbf{H}) \\ &= \mathbf{C}_0 + \mathbf{H} + \tfrac{1}{2!}\,\mathbf{HC}_0^{-1}\,\mathbf{H} + \tfrac{1}{3!}\,\mathbf{HC}_0^{-1}\,\mathbf{HC}_0^{-1}\,\mathbf{H} + \dots \end{aligned} \tag{53}$$

Thanks to its properties, $\mathrm{Exp}_{\mathbf{C}_0}(.)$ has an inverse – the generalized logarithmic map $\mathrm{Log}_{\mathbf{C}_0}(.)$, in our context with the meaning of a generalized logarithmic strain:

PROPOSITION 5: *The* ***generalized logarithmic strain*** $\mathbf{H}(\mathbf{C})$ *is a result of mapping of a deformation* $\mathbf{C}$ *with respect to the deformation* $\mathbf{C}_0$ by

$$\mathbf{C} \longmapsto \ \mathbf{H}(\mathbf{C}) \equiv \mathrm{Log}_{\mathbf{C}_0}(\mathbf{C}) := \mathbf{C}_0 \log\left(\mathbf{C}_0^{-1}\mathbf{C}\right), \tag{54}$$

so that one gets a vector from $T_{\mathbf{C}_0}Sym^+$, which specifies a geodesic line connecting two states of deformation $\mathbf{C}_0$ and $\mathbf{C}$.

Given the fact that $\mathbf{D} := \Phi^*(\mathbf{d}) = \frac{1}{2}\mathbf{C}^{-1}\partial\mathbf{C}$ (see point 3 after STARTING POINT), then $2\mathbf{D}_t = t\log\left(\mathbf{C}_0^{-1}\mathbf{C}\right)$ along the geodesic between $\mathbf{C}_0$ and $\mathbf{C}$, where $t \in \langle 0, 1\rangle$, and whose initial vector $\partial\mathbf{C}_0 = \mathbf{H}(\mathbf{C}) = 2\mathbf{C}_0\mathbf{D}$ (cf. (42)).

COROLLARY: *The logarithmic strain tensor* $\log(\mathbf{C})$ *is a vector that determines a geodesic line connecting the undeformed state* $\mathbf{I}$ *with the deformed one with* $\mathbf{C}$.

Bruhns et al. (2002) discusses the exact integrability of the hypoelasticity equation to conclude that, in addition to employing logarithmic strain and logarithmic derivative, this is only possible when a ***natural stress-free initial configuration*** is prescribed at the same time. Propositions 5 and 1 suggest why this should happen and how to possibly modify the logarithmic strain. This form of remedy stems from the geometry of Sym^+, which is the same both at $\mathbf{I}$ and $\mathbf{C}$. Hence the only difference is the footpoint, but not the approach itself. Still, one question remains. Namely, what the logarithmic rate in this case should look like.

5. Logarithmic time derivative and normal coordinates

Thanks to the Hopf - Rinow theorem, we can introduce in Sym^+ a global system of coordinates based on geodesics, which are called *normal (Riemannian) coordinates*. Using the generalized exponential map, and choosing an arbitrary basis for the vector space sym, one defines the normal coordinates of a point $\mathbf{C}$ to be the components of the vector $\mathbf{H} \in T_{\mathbf{C}_0}Sym^+ \approx sym$, for which $\mathrm{Exp}_{\mathbf{C}_0}(\mathbf{H}) = \mathbf{C}$. In other words, if we denote by $x^\alpha(\mathbf{H}) = h^\alpha$ coordinates of $\mathbf{H} = h^\alpha \mathbf{E}_\alpha \in sym$, where the set $\{\mathbf{E}_\alpha | \alpha = 1, \ldots, \frac{1}{2}n(n+1)\}$ forms in sym a basis, then components of vector $\mathbf{H}(\mathbf{C}) \equiv \mathrm{Log}_{\mathbf{C}_0}(\mathbf{C}) = \mathbf{C}_0 \log(\mathbf{C}_0^{-1}\mathbf{C})$ constitute normal coordinates $\mathbf{C}^{\log}$ of point $\mathbf{C}$, so that normal coordinates of point $\mathbf{C}_0$ are all zero's.

Now, the geodesics through point $\mathbf{C}_0$ in normal coordinate system are all of the form $\mathbf{C}_t^{\log} = t\partial\mathbf{C}_0$, and so it can be proved that the Christoffel symbols $\Gamma_{\mathbf{C}_0}^{\log}(\mathbf{E}_\alpha, \mathbf{E}_\beta)$ at this point are all zero's. As a result, the covariant derivative $\nabla_{\partial\mathbf{C}_0}^{\log}$ at $\mathbf{C}_0$, expressed in normal coordinate system, equals to the partial derivative (see Kobayashi et al. (1963) for example), i.e.

$$\nabla_{\partial\mathbf{C}_0}^{\log}(.)\Big|_{t=0} = \partial(.)|_{t=0}\,. \tag{55}$$

Moreover due to (66) with (63) and (61), at the start of the deformation process also

$$\partial\mathbf{C}_t^{\log}\Big|_{t=0} = (\mathrm{Log}_{\mathbf{C}_0}(\mathbf{C}_0))_*(\partial\mathbf{C}_0) = \partial\mathbf{C}_0 \tag{56}$$

and then

$$\frac{D}{dt}\mathbf{C}_t^{\log}\Big|_{t=0} = \nabla_{\partial\mathbf{C}_0}^{\log}(\mathbf{C}_t^{\log})\Big|_{t=0} = \partial\mathbf{C}_0, \tag{57}$$

where we have omitted the first term with partial time derivative, since points in Sym^+ are stationary. Due to (54), after multiplying (57) by $\frac{1}{2}\mathbf{C}_0^{-1}$ on the left

$$\frac{1}{2}\frac{D}{dt}\log(\mathbf{C}_0^{-1}\mathbf{C}_t)\Big|_{t=0} = \frac{1}{2}\,\mathbf{C}_0^{-1}\partial\mathbf{C}_0 = \mathbf{D}_0. \tag{58}$$

Considering that $\mathbf{d} = \mathbf{F}\mathbf{D}\mathbf{F}^{-1}$ and that $\mathbf{F}\mathbf{C}\mathbf{F}^{-1} = \mathbf{b} = \mathbf{V}^2$, where $\mathbf{b}$ is the left Cauchy-Green deformation tensor (Fiala, 2008), and so also $\mathbf{F}\log(\mathbf{C}_0^{-1}\mathbf{C}_t)\mathbf{F}^{-1} = \log(\mathbf{b}_0^{-1}\mathbf{b}_t)$, we obtain (cf. transformation (50) and (51))

$$\frac{1}{2}\frac{D}{dt}\log(\mathbf{b}_0^{-1}\mathbf{b}_t)\Big|_{t=0} \equiv \frac{1}{2}\left[\log(\mathbf{b}_0^{-1}\mathbf{b}_t)\right]_{\log}^{\circ\,\mathbf{ZJ}}\Big|_{t=0} = \mathbf{d}_0, \tag{59}$$

and for undeformed initial state, i.e. $\mathbf{b}_0 = \mathbf{I}$ and $\log(\mathbf{b}_0^{-1}\mathbf{b}_t) = 2\mathbf{V}_t$, we obtain

$$\left[\mathbf{V}_t^{\log}\right]_{\log}^{\circ\,\mathbf{ZJ}}\Big|_{t=0} \equiv \left[\log(\mathbf{V}_t)\right]_{\log}^{\circ\,\mathbf{ZJ}}\Big|_{t=0} = \mathbf{d}_0. \tag{60}$$

CONCLUSION: *We can thus conclude that in normal coordinates the Zaremba-Jaumann time derivative of the right stretch* $\mathbf{V}_t$ *at the onset of deformation, when starting from the undeformed initial state, is equal to the stretching* $\mathbf{d}$*, as is the case when using the logarithmic time derivative in* (1). *A question still remains, whether this holds true at all times, and if not why.*

Finally, a change of coordinates of $\mathbf{C}$ from standard one to that of $\mathbf{C}^{\log}$ via the transformation $\mathbf{C}^{\log} = \mathrm{Log}_{\mathbf{C}_0}(\mathbf{C})$ induces a corresponding transformation of coordinates of vectors

$$\mathbf{D}^{\log} = \frac{\delta \mathrm{Log}_{\mathbf{C}_0}(\mathbf{C})}{\delta \mathbf{C}}\,\mathbf{D} := \delta_{\mathbf{D}} \mathrm{Log}_{\mathbf{C}_0}(\mathbf{C}) \equiv [\mathrm{Log}_{\mathbf{C}_0}(\mathbf{C})]_*(\mathbf{D}), \tag{61}$$

where $\frac{\delta \log(\mathbf{C})}{\delta \mathbf{C}}$ is an analogue of the deformation gradient $\mathbf{F}$ for $\Phi : \mathcal{B} \to \mathcal{S}$. In the same way we thus obtain the push-forward $[\mathrm{Log}_{\mathbf{C}_0}(\mathbf{C})]_*$ and the pull-back $[\mathrm{Log}_{\mathbf{C}_0}(\mathbf{C})]^*$ operations, where

$$[\mathrm{Log}_{\mathbf{C}_0}(\mathbf{C})]^* = [\mathrm{Log}_{\mathbf{C}_0}(\mathbf{C})]_*^{-1} = [\mathrm{Exp}_{\mathbf{C}_0}(\mathbf{H})]_* \,. \tag{62}$$

Since

$$\delta_{\mathbf{D}} \mathrm{Log}_{\mathbf{C}_0}(\mathbf{C}) = \mathbf{C}_0 \delta_{\mathbf{C}_0^{-1}\mathbf{D}} \log(\mathbf{C}_0^{-1}\mathbf{C}), \tag{63}$$

it suffices to derive the formula (61) just for $\mathbf{C}_0 = \mathbf{I}$.

Let now $\mathbf{C} = \mathbf{R}\boldsymbol{\Lambda}\mathbf{R}^{\mathrm{T}} = \sum \lambda_i P_i$ be the spectral decomposition of $\mathbf{C}$, where $\boldsymbol{\Lambda}$ is the corresponding diagonal matrix with diagonal entries λ_i and P_i corresponding projectors, then (Bhatia, R., 1997, 2007)

$$[\log(\mathbf{C})]_*(\mathbf{D}) := \delta_{\mathbf{D}} \log(\mathbf{C}) = \mathbf{R}\left[\log^{[1]}(\boldsymbol{\Lambda}) \circ (\mathbf{R}^{\mathrm{T}}\mathbf{D}\mathbf{R})\right]\mathbf{R}^{\mathrm{T}} \tag{64}$$

$$= \sum_i \sum_j \log^{[1]}(\lambda_i, \lambda_j) P_i \mathbf{D} P_j \tag{65}$$

$$\text{with } [\log(\mathbf{I})]_* = \mathbf{Id}\,, \tag{66}$$

where the *Hadamard (or Schur) product* of two matrices $\mathbf{A}$ and $\mathbf{B}$ is defined to be the matrix $\mathbf{A} \circ \mathbf{B}$ whose (i, j)-entry is $A_i^j B_i^j$, and the 3×3 symmetric matrix $\log^{[1]}(\boldsymbol{\Lambda})$ has numbers

$$\log^{[1]}(\lambda_i, \lambda_j) = \frac{\log(\lambda_i) - \log(\lambda_j)}{\lambda_i - \lambda_j} \quad \text{if } i \neq j$$

$$\log^{[1]}(\lambda_i, \lambda_i) = \log'(\lambda_i) = \frac{1}{\lambda_i} \tag{67}$$

as its (i, j)-entries. The matrix $\log^{[1]}(\boldsymbol{\Lambda})$ is called *Loewner matrix*, and it was introduced by the mathematician of Czech origin, Karl Loewner (1893–1968), who emigrated to America in 1939, as early as 1934. The relation (64-65), known as the *Daleckii-Krein formula*, was presented by Daleckii et al. (1951) and later rediscovered in mechanical literature starting from mid-eighties. For a comprehensive survey of this literature, see (Xiao et al., 1998; Norris, 2008).

6. Conclusion

In addition to the conclusion of the previous section, I would like to point out the fact that the geometrical approach to deformation via geometry of Sym^+ looks remarkably compact and self-consistent. It provides a natural way to the linearization of deformation process and to the incremental approach within finite deformations, though with some unusual conclusions -

see Proposition 4 about adding up a deformation increment, and (49) and (48) about the second Piola-Kirchhoff and the convective stress rates. Due to the high degree of symmetry in the space Sym^+ we need not distinguish between initially unstrained and initially strained states, and this might turn out essential for extending hypoelasticity to strained initial configurations. Moreover, the generalized logarithmic map puts in one-to-one correspondence small deformations with the finite ones (see Proposition 1 and 5), which might provide geometrical support for the nice achievements of the hypoelasticity based on logarithmic strain and logarithmic time derivative.

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