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ANALYTICAL SOLUTION OF GASEOUS FLOW IN A RECTANGULAR MICROCHANNEL WITH SECOND-ORDER SLIP FLOW BOUNDARY CONDITIONS

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Summary: *This paper deals with the analytical solution of gaseous slip flow in a rectangular microchannel. The flow is supposed to be steady, laminar, incompressible and hydrodynamically fully developed. The velocity slip at microchannel walls is expressed by the second-order velocity slip boundary conditions. The results derived using the Fourier method are compared with results obtained by the authors considering the first-order velocity slip boundary conditions.*

1. Introduction

Problems of flow in very narrow channels and microdevices have been studied increasingly in recent decades. This topic plays an important role for example in biological systems and also in a number of industrial devices such as heat exchangers, nuclear reactors or microturbines. Depending on the value of the Knudsen number Kn, the character of microflow in such objects can be divided into four flow regimes, (Kandlikar et al., 2006) or (Karniadakis et al., 2005). The slip flow regime, which occurs in flows with $10^{-3} < Kn < 10^{-1}$, is particularly interesting because it generally leads to analytical or semi-analytical models which allow us to calculate velocities, flow rates or temperature distributions for laminar and fully developed microflows. The Navier-Stokes equations remain applicable for the mathematical description of the slip flow regime, but a velocity slip and a temperature jump have to be taken into account at the channel walls. Let us note that the Knudsen number can be calculated either as the ratio of the molecular mean free path λ and the hydraulic diameter D_h or as the function of the Reynolds and Mach numbers, $Kn = \lambda/D_h = Ma/Re\sqrt{\pi\gamma/2}$, where γ is the specific heat ratio.

In order to express the velocity slip and temperature jump, the appropriate boundary conditions have to be prescribed at the microchannel walls. From (Kennard, 1938) it is known that Kundt and Wartburg in 1875 and Maxwell in 1879 were probably the first who mentioned the velocity slip and the temperature jump at the wall. For a gas flow in the direction s parallel to the wall, the first-order velocity slip boundary condition has the general form, (Kandlikar et al., 2006),

$$u_{slip} = u_s - u_{wall} = \frac{2-\sigma}{\sigma}\lambda\frac{\partial u_s}{\partial n}\Big|_w + \frac{3}{4}\frac{\eta}{\rho T}\frac{\partial T}{\partial s}\Big|_w, \qquad (1)$$

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where n is the normal to the wall. The tangential momentum accommodation coefficient is often chosen as $\sigma = 1$ and the second right-hand side term in (1) is often neglected by many authors. The general form of the second-order slip flow boundary condition is, (Kandlikar et al., 2006),

$$u_{slip} = u_s - u_{wall} = K_1 \lambda \frac{\partial u_s}{\partial n}\Big|_w + K_2 \lambda^2 \frac{\partial^2 u_s}{\partial n^2}\Big|_w, \tag{2}$$

where K_1 and K_2 are constants.

There is a number of studies focused on the analytical solution of flow in microchannels of various types. For example, in (Dongari et al., 2007), the problem of compressibility of gaseous flow between two parallel plates is studied analytically. Numerical solution of the same problem is given in (Asako et al., 2003). Analytical solution of three-dimensional fully developed laminar slip flow in rectangular microchannels is given in (Ebert and Sparrow, 1965), (Morini and Spiga, 1998). Analytical determination of temperature field and Nusselt number between two parallel plates, including axial heat transfer, temperature jump and viscous dissipation, is studied by Ho-Eyoul Jeong and Jae-Tack Jeong in paper (Jeong and Jeong, 2006). The works (Spiga and Morini, 1996), (Morini, 2000) are devoted to the analytical solution of temperature field and Nusselt number computation in three-dimensional rectangular microchannels. The flow is supposed to be steady, laminar, incompressible, fully hydrodynamically and thermally developed. Let us note that further examples of laminar flow and heat transfer in various microchannels and microtubes are given in (Kandlikar et al., 2006).

The gaseous flow and heat transfer in the microchannel with the fist-order slip flow boundary conditions is solved analytically by the authors of this study in (Klášterka et al., 2009). The analytical solution of steady, laminar, incompressible and fully developed flow is derived using the Fourier method and it is compared with numerical results obtained using the finite difference scheme. In this study, the second-order velocity slip boundary conditions are adopted in order to derive the analytical solution of the gaseous slip flow in the rectangular microchannel. The flow is assumed to be steady, laminar, incompressible and hydrodynamically fully developed. The analytical solution is derived using the Fourier method and the obtained results are compared with results derived by the authors for the same case with first-order boundary conditions that are presented in (Klášterka et al., 2009).

2. Mathematical formulation of the problem

Let us consider a steady laminar flow of a viscous incompressible fluid in a long microchannel with a rectangular cross-section. The microchannel dimensions are illustrated in fig. 1, where $L = 5 \cdot 10^{-3}$m is the microchannel length and the rectangle sides are considered as $2h = 10^{-6}$m and $2b = 2 \cdot 10^{-5}$m.

The incompressible fluid flow can be described by the non-linear system of the Navier-Stokes equations, (Hoffman and Chiang, 2000). Because we suppose the fully developed flow, we can assume $\partial u / \partial x = 0$. Furthermore, the cross-sectional components v, w of the velocity vector can be considered as very small compared to the longitudinal velocity u. Thus, the non-linear system of the Navier-Stokes equations reduce to

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{\eta} \frac{dp}{dx}, \tag{3}$$

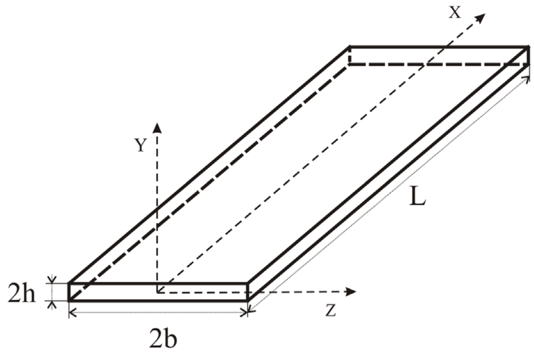

Figure 1: Geometry of the microchannel with the rectangular cross-section

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0, \tag{4}$$

which means that $p = p(x)$ and $u = u(y, z)$. The flow is considered to be axisymmetric and therefore, for the channel axis $y = 0$, $z = 0$, we can write the boundary conditions

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0, \quad \left(\frac{\partial u}{\partial z}\right)_{z=0} = 0. \tag{5}$$

In this work, the second order slip flow boundary conditions are prescribed at the microchannel walls. Their general form can be written as the combination of first and second derivatives of the velocity at the wall

$$u(h, z) = -KnD_h \left(\frac{\partial u}{\partial y}\right)_{y=h} + K_2 Kn^2 D_h^2 \left(\frac{\partial^2 u}{\partial y^2}\right)_{y=h}, \tag{6}$$

$$u(y, b) = -KnD_h \left(\frac{\partial u}{\partial z}\right)_{z=b} + K_2 Kn^2 D_h^2 \left(\frac{\partial^2 u}{\partial z^2}\right)_{z=b}, \tag{7}$$

where D_h is the hydraulic diameter

$$D_h = \frac{4bh}{b+h}. \tag{8}$$

Relating the coordinates x, y, z to D_h, the velocity u to the average velocity U_{avg} and the static pressure p to the reference pressure $p_{ref} = \rho U_{avg}^2$, we obtain the dimensionless form of the problem. Let us note that the average velocity is defined as

$$U_{avg} = \frac{1}{bh} \int_0^h \int_0^b u(y, z)\, dz\, dy. \tag{9}$$

From now, we will consider all the quantities as dimensionless. We can rewrite the equation (3) in the dimensionless form

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = Re \frac{dp}{dx}, \tag{10}$$

the dimensionless form of the velocity slip boundary conditions (6), (7) is

$$u(h, z) = -Kn \left(\frac{\partial u}{\partial y}\right)_{y=h} + K_2 Kn^2 \left(\frac{\partial^2 u}{\partial y^2}\right)_{y=h}, \tag{11}$$

$$u(y,b) = -Kn\left(\frac{\partial u}{\partial z}\right)_{z=b} + K_2 Kn^2 \left(\frac{\partial^2 u}{\partial z^2}\right)_{z=b} \tag{12}$$

and the boundary conditions expressing the symmetry remain unchanged, so the equations (5) still hold for dimensionless quantities. In the equation (10), we consider the Reynolds number as $Re = U_{avg}\rho D_h/\eta$. Values of the constant K_2 in (11) and (12) are discussed in (Kandlikar et al., 2006), in this study we choose $K_2 = -9/8$.

3. Analytical solution of incompressible fluid flow

In this section, the analytical solution of incompressible slip flow in the microchannel is derived. Generally, we expect the solution in the form

$$u(y,z) = u^{(1)}(z) + u^{(2)}(y,z). \tag{13}$$

and after substituting (13) into (10) we get two differential equations

$$\frac{d^2u^{(1)}(z)}{dz^2} = Re\frac{dp}{dx}, \tag{14}$$

$$\frac{\partial^2 u^2(y,z)}{\partial y^2} + \frac{\partial^2 u^{(2)}(y,z)}{\partial z^2} = 0. \tag{15}$$

We can express the general solution of equation (14) as

$$u^{(1)}(z) = \frac{Re}{2}\frac{dp}{dx}z^2 + C_1 z + C_2 \tag{16}$$

and the solution of the equation (15) is expected to be a product of two functions $f(y)$ and $g(z)$

$$u^{(2)}(y,z) = f(y)g(z). \tag{17}$$

Substituting (17) into (15) we get

$$\frac{1}{f(y)}\frac{d^2f(y)}{dy^2} = -\frac{1}{g(z)}\frac{d^2g(z)}{dz^2} = \kappa^2, \tag{18}$$

where κ is the unknown constant. Thus, the treatment of the partial differential equation (15) is transformed to the solution of two ordinary differential equations

$$\frac{d^2f(y)}{dy^2} - \kappa^2 f(y) = 0, \quad \frac{d^2g(z)}{dz^2} + \kappa^2 g(z) = 0 \tag{19}$$

generally having the solution

$$f(y) = A_1 e^{\kappa y} + A_2 e^{-\kappa y} \quad \text{and} \quad g(z) = B_1\cos(\kappa z) + B_2\sin(\kappa z), \tag{20}$$

and therefore according to (17) we get

$$u^{(2)}(y,z) = \left(A_1 e^{\kappa y} + A_2 e^{-\kappa y}\right)\left[B_1\cos(\kappa z) + B_2\sin(\kappa z)\right]. \tag{21}$$

Now, we can rewrite the solution (13) as

$$u(y,z) = \frac{Re}{2}\frac{dp}{dx}z^2 + C_1 z + C_2 + \left(A_1 e^{\kappa y} + A_2 e^{-\kappa y}\right)\left[B_1\cos(\kappa z) + B_2\sin(\kappa z)\right], \quad (22)$$

which must satisfy the boundary conditions (5), (11) and (12). The symmetry conditions (5) yield

$$C_1 = 0, \quad A_1 = A_2, \quad B_2 = 0. \quad (23)$$

Afterwards, the solution (22) reduces to

$$u(y,z) = \frac{Re}{2}\frac{dp}{dx}z^2 + C_2 + A\cosh(\kappa y)\cos(\kappa z), \quad (24)$$

where $A = 2A_1B_1$. To derive the remaining constants A, C_2, κ we will use the boundary conditions (11), (12), so we get

$$\begin{aligned}\frac{Re}{2}\frac{dp}{dx}z^2 + C_2 + A\cosh(\kappa h)\cos(\kappa z) &= -\kappa Kn A\sinh(\kappa h)\cos(\kappa z) + \\ &+ K_2 A Kn^2\kappa^2\cosh(\kappa h)\cos(\kappa z), \quad (25)\end{aligned}$$

$$\begin{aligned}\frac{Re}{2}\frac{dp}{dx}b^2 + C_2 + A\cosh(\kappa y)\cos(\kappa b) &= -Kn\left[Re\frac{dp}{dx}b - A\kappa\cosh(\kappa y)\sin(\kappa b)\right] + \\ &+ K_2 Kn^2\left[Re\frac{dp}{dx} - A\kappa^2\cosh(\kappa y)\cos(\kappa b)\right]. \quad (26)\end{aligned}$$

In order to be the equation (26) fulfilled for every $y \in (0,h)$, following conditions have to be satisfied

$$C_2 = -\frac{Re}{2}\frac{dp}{dx}b^2\left(1 + \frac{2Kn}{b} - 2K_2\frac{Kn^2}{b^2}\right), \quad (27)$$

$$\tan(\kappa b) = \frac{1 + K_2 Kn^2\kappa^2}{Kn\kappa}. \quad (28)$$

The transcendent equation (28) has an infinite number of roots $\kappa b = \kappa_i b$, $i = 1, \ldots, \infty$, and therefore we can write the solution (24) as

$$u(y,z) = \frac{Re}{2}\frac{dp}{dx}\left[z^2 - b^2\left(1 + \frac{2Kn}{b} - \frac{2K_2Kn^2}{b^2}\right)\right] + \sum_{i=1}^{\infty} A_i\cosh(\kappa_i y)\cos(\kappa_i z). \quad (29)$$

The last step is to determine the constants A_i using (25) and (27) that result in

$$\begin{aligned}\sum_{i=1}^{\infty} A_i\left[\cosh(\kappa_i h)(K_2Kn^2\kappa_i^2 - 1)\right. &- \left.\kappa Kn\sinh(\kappa_i h)\right]\cos(\kappa_i z) = \\ &= -\frac{Re}{2}\frac{dp}{dx}b^2\left(1 + \frac{2Kn}{b} - \frac{2K_2Kn^2}{b^2} - \frac{z^2}{b^2}\right). \quad (30)\end{aligned}$$

Multiplying this equation by $\cos(\kappa_j z)dz$, where j is any given value of i, and integrating over the interval $(0,b)$, we get

$$A_i = \frac{Re\frac{dp}{dx}\frac{b}{\kappa_i^2}\left[-\cos(\kappa_i b) + \left(Kn\kappa_i + \frac{1}{\kappa_i b} - \frac{K_2\kappa_i Kn^2}{b}\right)\sin(\kappa_i b)\right]}{\left[\cosh(\kappa_i h) + Kn\kappa_i\sinh(\kappa_i h) - K_2\kappa_i^2Kn^2\cosh(\kappa_i h)\right]\left[\frac{b}{2} + \frac{\sin(2\kappa_i b)}{4\kappa_i}\right]}. \quad (31)$$

4. Analytical results

Analytically obtained velocity distribution in the rectangular microchannel will be shown in this section. The dimensionless sizes of the microchannel are considered to be $h = 0.2625$, $b = 5.25$ and $L = 2625$. We consider a pressure driven flow of argon characterized by following parameters: $\gamma = 1.67$, $\rho = 1.35\,\mathrm{kg\,m^{-3}}$, $p_1 = 202650\,\mathrm{Pa}$, $p_2 = 25000\,\mathrm{Pa}$, $\eta = 2.588 \cdot 10^{-5}\,\mathrm{Pa\,s}$. This results in dimensionless numbers $Re = 0.015$ and $Kn = 0.0326$.

In fig. 2, the three-dimensional profile of the dimensionless velocity u in the y-z plane is shown. The velocity slip at the channel walls (for $y = h$, $z = b$) is easily seen from this figure. It is no surprise that the maximum velocity is reached in the middle of the channel (for $y = 0$, $z = 0$).

The comparison of velocity profiles obtained using the first- and second-order velocity slip boundary conditions in given $y-$ and z-cuts is made in fig. 3. The y-cuts are considered for $y = 0$, $y = h/2$ and $y = h$. Similarly, the z-cuts are made for $z = 0$, $z = b/2$ and $z = b$. This comparison shows that the velocity profiles for first- and second-order boundary conditions are similar with small diferences in maximum values of velocity. Let us note that for $K_2 = 0$ we obtain identical results as in the case of the application of first-order boundary conditions, (Klášterka et al., 2009).

5. Conclusion

This article deals with the analytical solution of gaseous flow in the microchannel with rectangular cross-section. The flow is assumed to be steady, laminar, incompressible and fully developed. The velocity distribution is derived analytically using the Fourier method. The main

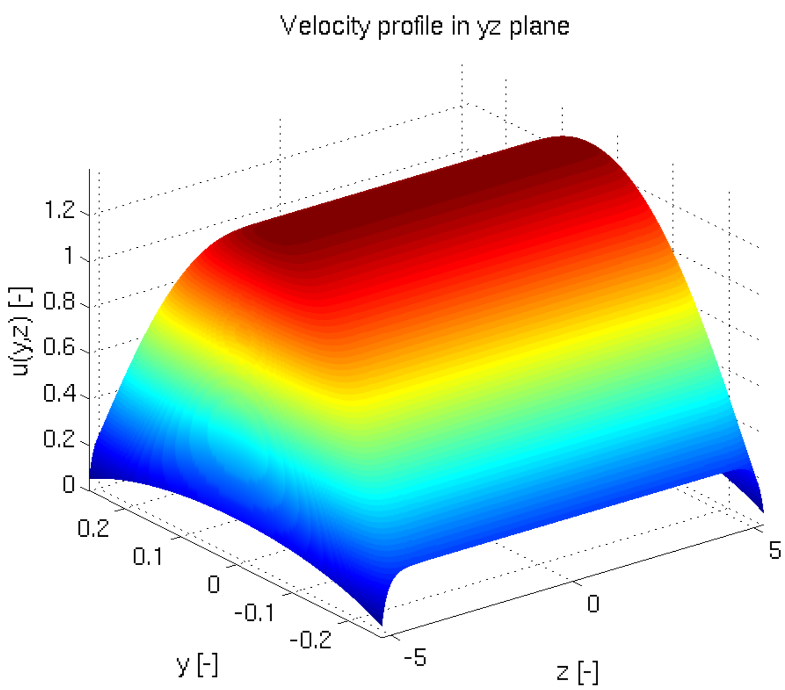

Figure 2: Profile of the dimensionless velocity in the $y - z$ plane.

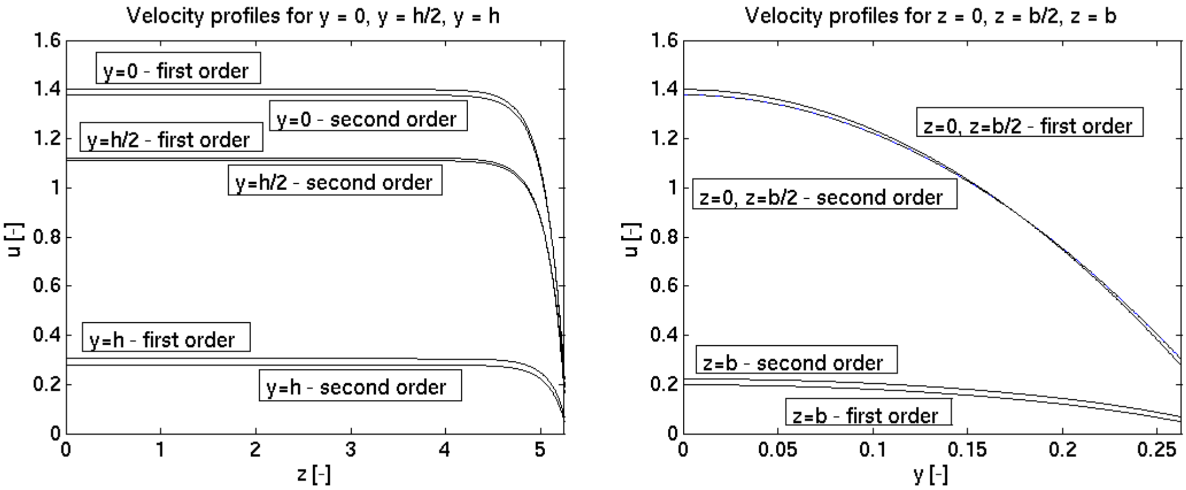

Figure 3: Dimensionless velocity profiles obtained analytically using first- and second-order slip boundary conditions.

objective of this study is the application of second-order slip flow boundary conditions at the microchannel walls. Comparison of the results valid for the second-order slip boundary conditions with the solution for the first-order boundary conditions presented by the authors in (Klášterka et al., 2009) shows the similarity and only small differences between both analytical solutions. The solutions with first- and second-order velocity slip boundary conditions are identical when considering $K_2 = 0$.

In future works, authors want to continue in the analysis of microflow effects. Particularly, the analytical solution of laminar incompressible flow in the inlet part of the rectangular microchannel using the Oseen flow model will be derived and the second-order temperature jump boundary conditions will be analyzed.

6. Acknowledgment

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