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SOUND SPEED IN THE MIXTURE WATER – AIR

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Summary: *Analytical derivation of the sound speed is presented in this report. The derivation is shown on the example of water – air mixture, but the results are applicable for any fluid – gas mixture.*

1. Introduction

The sound speed (fluid celerity) is usually assumed to be about 1500 m/s and it is also assumed that this value is little variable. It is possible to compute more accurate value using equations, which are published by IAPWS (The International Association for the Properties of Water and Steam). This speed is valid only for water. In practise, there is also some volume of air included with the water. This air causes noticeable decrement of the sound speed. In this contribution, the air in the form of bubbles of not absorbed air is assumed.

2. Nomenclature

v_s	$[m \cdot s^{-1}]$	Sound speed in mixture
v_v	$[m \cdot s^{-1}]$	Sound speed in water
A_t	[J]	Technical work
c_p	$[J\ kg^{-1} \cdot K^{-1}]$	Specific heat of air – constant pressure
c_v	$[J\ kg^{-1} \cdot K^{-1}]$	Specific heat of air – constant volume
c_w	$[J\ kg^{-1} \cdot K^{-1}]$	Specific heat of water
I	[J]	Enthalpy
K_s	$[m \cdot s^{-1}]$	Bulk modulus of mixture
K_v	[Pa]	Bulk modulus of water
K_{vz}	[Pa]	Bulk modulus of air
M_v	[-]	Mass ratio of water
M_{vz}	[-]	Mass ratio of air
m_v	[kg]	Mass of water
m_{vz}	[kg]	Mass of air
O_v	[-]	Volume ratio of water
O_{vz}	[-]	Volume ratio of air
p	[Pa]	Pressure

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Q	[J]	Heat
r	$[J\ kg^{-1} \cdot K^{-1}]$	Gas constant
T	[K]	Temperature
V_v	$[m^3]$	Water volume
V_{vz}	$[m^3]$	Air volume
κ	[-]	Adiabatic constant
ρ_s	$[kg\ m^{-3}]$	Mixture density
ρ_v	$[kg\ m^{-3}]$	Water density

3. Properties of water and air

$v_v = 1450\ m \cdot s^{-1}$ is independent on the pressure
$c_w = 4200\ J\ kg^{-1} \cdot K^{-1}$
$\rho_v = 1000\ kg \cdot m^{-3}$
T = 293 K
κ = 1,4
r = 287
Air is uniformly spread out in the water.

4. Adiabatic behaviour – constant air mass

At first we assume adiabatic behaviour of the air.

We can compute sound speed of the mixture as a square root of the ratio between bulk modulus and density.

$$v_s^2 = \frac{K_s}{\rho_s} \tag{1}$$

where

$$\rho_s = \frac{m_s}{V_s} \tag{2}$$

We separate total mass and total volume to the water content and air content. Air volume is given by state equation.

$$\rho_s = \frac{m_v + m_{vz}}{V_v + V_{vz}} = \frac{m_v + m_{vz}}{\dfrac{m_v}{\rho_v} + \dfrac{m_{vz} \cdot r \cdot T}{p}} \tag{3}$$

If we divide numerator and denominator with total mass we get resultant relationship for density of the mixture ($M_v + M_{vz} = 1$).

$$\rho_s = \frac{M_v + M_{vz}}{\dfrac{M_v}{\rho_v} + \dfrac{M_{vz} \cdot r \cdot T}{p}} = \frac{\rho_v \cdot p}{M_v \cdot p + M_{vz} \cdot r \cdot T \cdot \rho_v} = \frac{\rho_v \cdot p}{(1 - M_{vz}) \cdot p + M_{vz} \cdot r \cdot T \cdot \rho_v} \tag{4}$$

With similar procedure we obtain bulk modulus, see following equation:

$$\frac{1}{K_s} = \frac{O_v}{K_v} + \frac{O_{vz}}{K_{vz}} \tag{5}$$

We assume adiabatic behaviour and so $K_{vz} = \kappa \cdot p$, $K_v = v_v^2 \cdot \rho_v$ and we express volume ratios by volumes.

$$K_s = \frac{v_v^2 \cdot \rho_v \cdot \kappa \cdot p}{v_v^2 \cdot \rho_v \cdot O_{vz} + \kappa \cdot p \cdot O_v} = \frac{v_v^2 \cdot \rho_v \cdot \kappa \cdot p}{v_v^2 \cdot \rho_v \cdot \frac{V_{vz}}{V_v + V_{vz}} + \kappa \cdot p \cdot \frac{V_v}{V_v + V_{vz}}} \tag{6}$$

$$K_s = \frac{(V_v + V_{vz}) \cdot v_v^2 \cdot \rho_v \cdot \kappa \cdot p}{v_v^2 \cdot \rho_v \cdot V_{vz} + \kappa \cdot p \cdot V_v} \tag{7}$$

Now, we can specify volumes and again divide numerator and denominator with total mass.

$$K_s = \frac{\left(\frac{m_v}{\rho_v} + \frac{m_{vz} \cdot r \cdot T}{p}\right) \cdot v_v^2 \cdot \rho_v \cdot \kappa \cdot p}{v_v^2 \cdot \rho_v \cdot \frac{m_{vz} \cdot r \cdot T}{p} + \kappa \cdot p \cdot \frac{m_v}{\rho_v}} = \frac{\left(\frac{M_v}{\rho_v} + \frac{M_{vz} \cdot r \cdot T}{p}\right) \cdot v_v^2 \cdot \rho_v \cdot \kappa \cdot p}{v_v^2 \cdot \rho_v \cdot \frac{M_{vz} \cdot r \cdot T}{p} + \kappa \cdot p \cdot \frac{M_v}{\rho_v}} \tag{8}$$

We obtain result after a few modifications.

$$K_s = \frac{\left[(1 - M_{vz}) \cdot p + M_{vz} \cdot r \cdot T \cdot \rho_v\right] \cdot v_v^2 \cdot \kappa \cdot p \cdot \rho_v}{v_v^2 \cdot \rho_v^2 \cdot M_{vz} \cdot r \cdot T + \kappa \cdot p^2 \cdot (1 - M_{vz})} \tag{9}$$

Resultant sound speed of the air is then:

$$v_s = \sqrt{\frac{K_s}{\rho_s}} \tag{10}$$

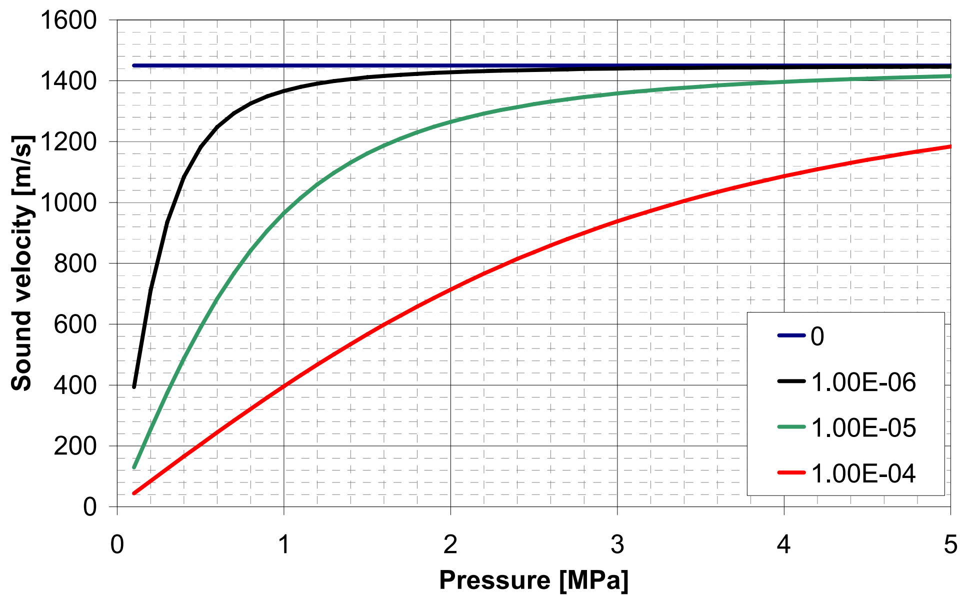

Figure 1 Sound speed dependence on the pressure for different air mass ratios

5. Adiabatic behaviour – constant pressure

In this case is it advantageous to compute density with volume ratios.

$$\rho_s = \frac{m_v + m_{vz}}{V_v + V_{vz}} = \frac{\rho_v \cdot V_v + \rho_{vz} \cdot V_{vz}}{V_v + V_{vz}} \tag{11}$$

We divide numerator and denominator with total volume ($O_v + O_{vz} = 1$, it is valid for constant pressure) and air density is expressed by the equation of state.

$$\rho_s = \frac{\rho_v \cdot O_v + \rho_{vz} \cdot O_{vz}}{O_v + O_{vz}} = \rho_v \cdot O_v + \frac{p}{r \cdot T} \cdot O_{vz} \tag{12}$$

Bulk modulus from equation (5):

$$K_s = \frac{v_v^2 \cdot \rho_v \cdot \kappa \cdot p}{v_v^2 \cdot \rho_v \cdot O_{vz} + \kappa \cdot p \cdot O_v} \tag{13}$$

and sound speed in the mixture is again:

$$v_s = \sqrt{\frac{K_s}{\rho_s}} \tag{14}$$

For completeness, translational equations between volume ratio and mass ratio of the air:

$$O_{vz} = \frac{M_{vz} \cdot \rho_v \cdot r \cdot T}{p - M_{vz} \cdot (p - \rho_v \cdot r \cdot T)} \tag{15}$$

$$M_{vz} = \frac{p \cdot O_{vz}}{p \cdot O_{vz} + \rho_v \cdot r \cdot T(1 - O_{vz})} \tag{16}$$

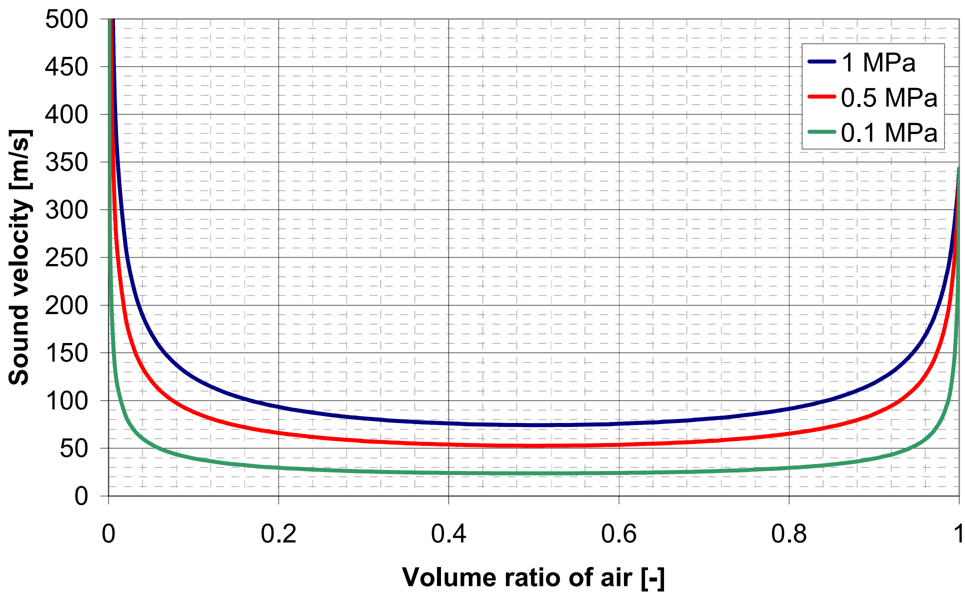

Figure 2 Dependence on the volume ratio of air for different pressures

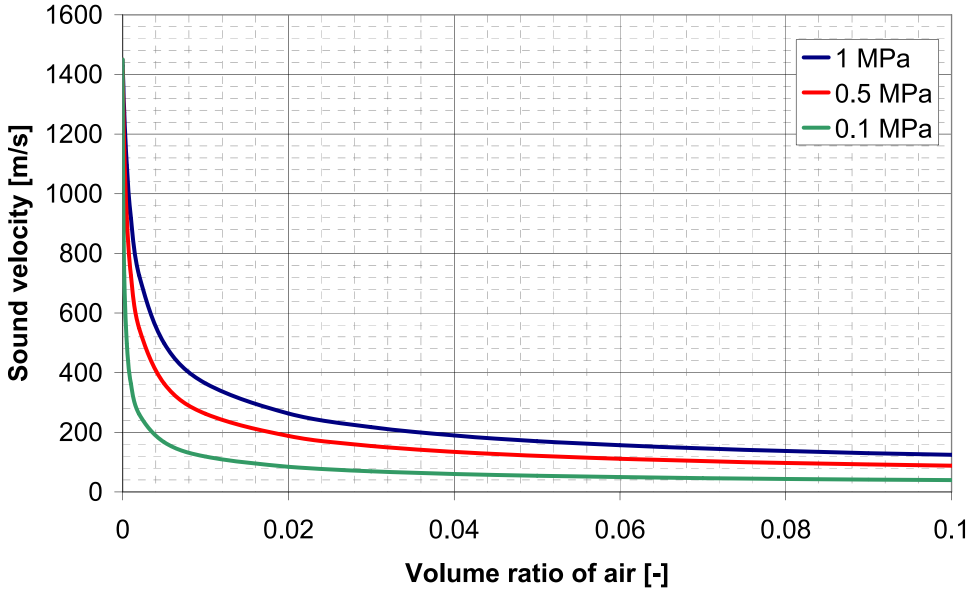

Figure 3 Dependence on the volume ratio of air – detail

6. Isothermal behaviour – constant air mass

We assume isothermal behaviour of the air now. It means that $K_{vz} = p$. It has no impact on the density of mixture, but bulk modulus is different.

With same procedure like above we obtain expression for bulk modulus of mixture. It is similar with equation (8). The only difference is that adiabatic exponent disappeared.

$$K_s = \frac{\left[(1 - M_{vz}) \cdot p + M_{vz} \cdot r \cdot T \cdot \rho_v\right] \cdot v_v^2 \cdot p \cdot \rho_v}{v_v^2 \cdot \rho_v^2 \cdot M_{vz} \cdot r \cdot T + p^2 \cdot (1 - M_{vz})} \tag{17}$$

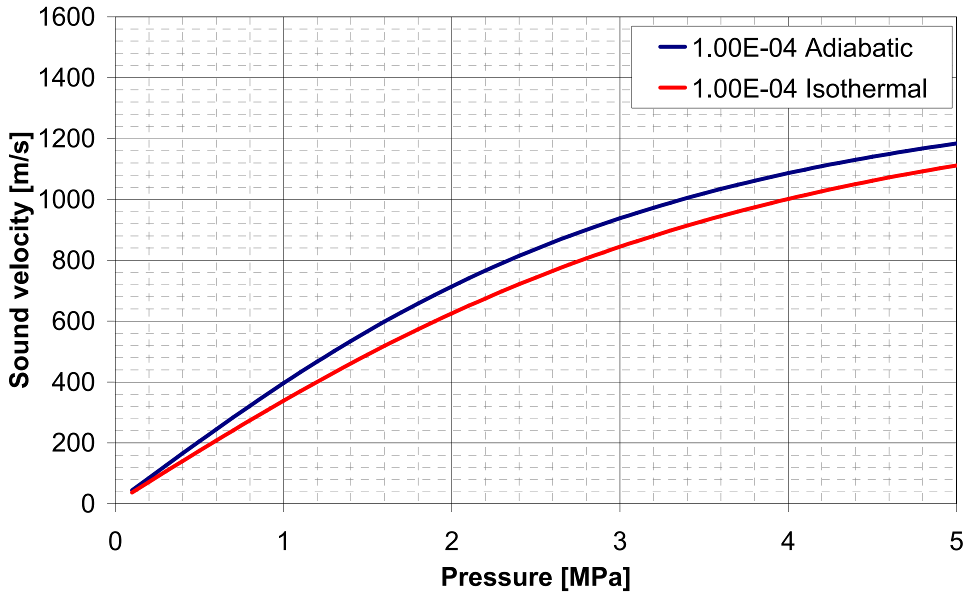

Figure 4 Comparison of adiabatic and isothermal hypothesis ($M_{vz} = 10^{-4}$)

7. Isothermal behaviour – constant pressure

Adiabatic exponent disappeared again.

$$K_s = \frac{v_v^2 \cdot \rho_v \cdot p}{v_v^2 \cdot \rho_v \cdot O_{vz} + p \cdot O_v} \tag{18}$$

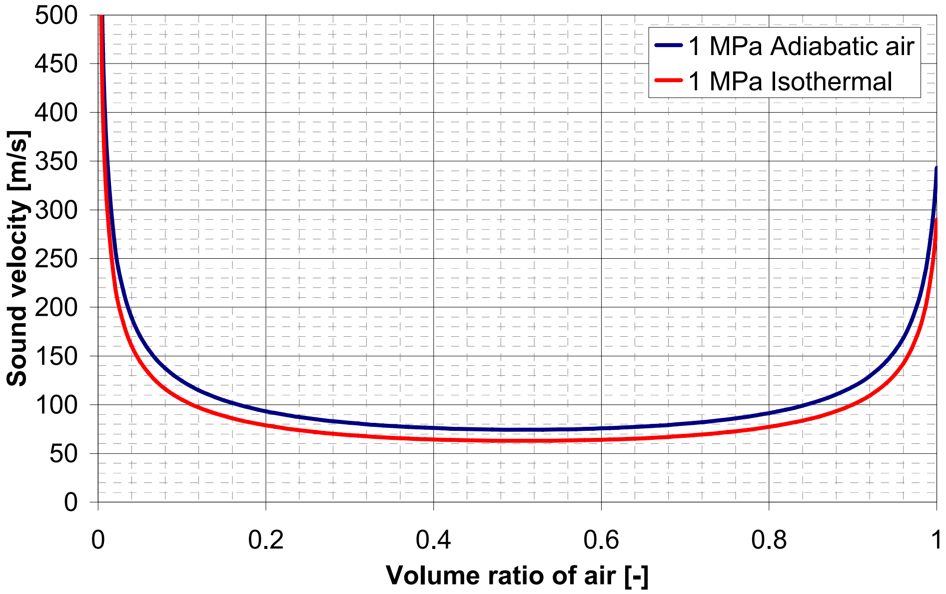

Figure 5 Adiabatic and isothermal hypothesis (p = 1 MPa)

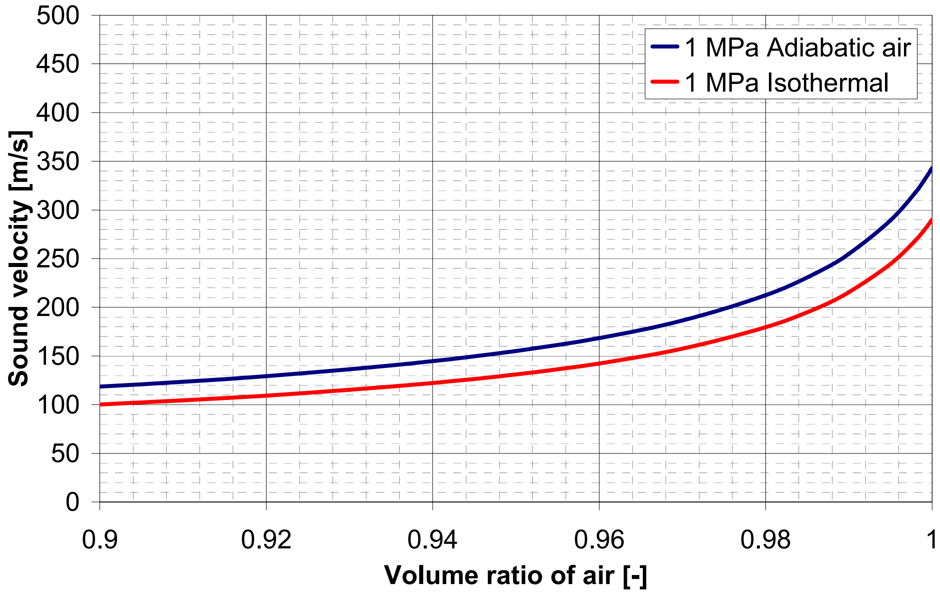

Figure 6 Adiabatic and isothermal hypothesis (p = 1 MPa) – detail

8. Adiabatic behaviour – whole system

Lastly, we can assume that whole system is adiabatic. It means that heat is exchanged between water and air but overall heat is constant.

Following equation is valid for air:

$$dQ = dI - dA_t \tag{19}$$

and water heat flow is described by:

$$-dQ = m_v \cdot c_w \cdot dT \tag{20}$$

We obtain relationship for air in the mixture by putting together equations (19) and (20). We specify also terms dI and dA_t.

$$-m_v \cdot c_w \cdot dT = m_{vz} \cdot c_p \cdot dT - V_{vz} \cdot dp \tag{21}$$

We express volume of the air from state equation and then divide whole relationship with total mass ($m_{vz} + m_v$).

$$-M_v \cdot c_w \cdot dT = M_{vz} \cdot c_p \cdot dT - \frac{M_{vz} \cdot r \cdot T}{p} \cdot dp \tag{22}$$

After rearrangement:

$$M_{vz} \cdot r \cdot \frac{1}{p} \cdot dp = \left(M_{vz} \cdot c_p + M_v \cdot c_w\right) \cdot \frac{1}{T} \cdot dT \tag{23}$$

In order to simplify:

$$M_{vz} \cdot c_p + M_v \cdot c_w = E \tag{24}$$

Now, we can integrate equation (23). Expression $\mathbf{ln(K_i)}$ is integration constant.

$$M_{vz} \cdot r \cdot \ln(p) = E \cdot \ln(T) + \ln(K_i) \tag{25}$$

We remove logarithms and use state equation again.

$$p^{M_{vz} \cdot r} = \left(\frac{p}{\rho_{vz} \cdot r}\right)^E \cdot K_i \tag{26}$$

After rearrangement:

$$p^{M_{vz} \cdot r - E} \cdot \rho_{vz}^E = \frac{K_i}{r^E} \tag{27}$$

finally

$$p \cdot \rho_{vz}^{-n} = \text{invariable} \tag{28}$$

where

$$n = \frac{M_{vz} \cdot c_p + M_v \cdot c_w}{M_{vz} \cdot c_v + M_v \cdot c_w} \tag{29}$$

Variable **n** depends on the ratio water/air!

Bulk modulus of the air is:

$$K_{vz} = \rho_{vz} \cdot \frac{\partial p}{\partial \rho_{vz}} = n \cdot p \qquad (30)$$

Next derivation of the sound speed in the mixture is same like in chapter 3 and 4. Only variable **n** replaces **κ**.

$$K_s = \frac{\left[(1-M_{vz}) \cdot p + M_{vz} \cdot r \cdot T \cdot \rho_v\right] \cdot v_v^2 \cdot n \cdot p \cdot \rho_v}{v_v^2 \cdot \rho_v^2 \cdot M_{vz} \cdot r \cdot T + n \cdot p^2 \cdot (1-M_{vz})} \qquad (31)$$

Dependence of sound speed is almost same as for isothermal assumption, because water has higher heat capacity then air but with $M_{vz} = 1$ is speed same as adiabatic.

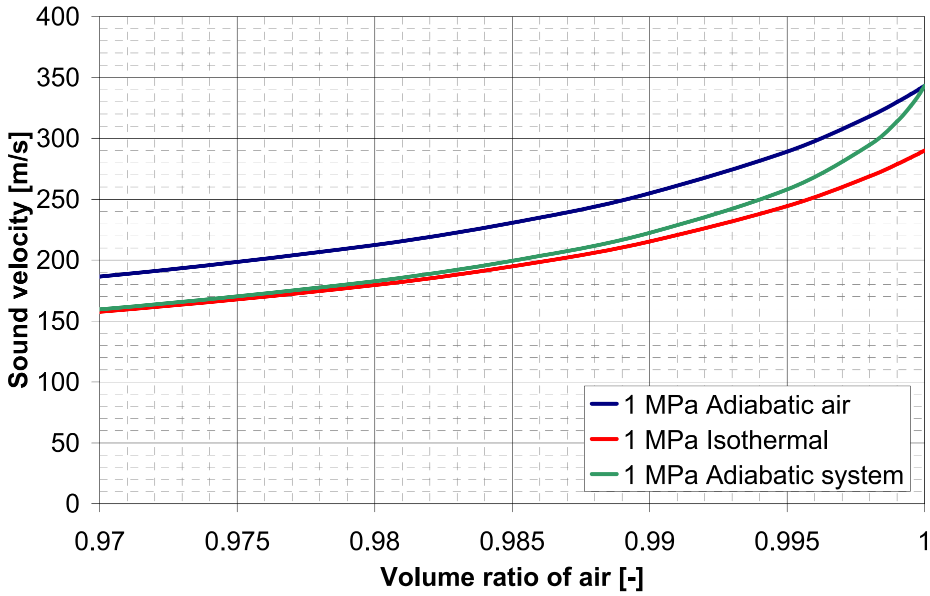

Figure 7 Comparison – adiabatic system (p = 1 MPa)

9. Conclusion

Sound speed in the water is markedly influenced by the included air. Computation gives similar results for the both adiabatic and isothermal assumptions. It is interesting that the sound speed in the mixture is in the major part of the area of the graph O – v lower than sound speed in the air.

10. Acknowledgement

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11. References

International Association for the Properties of Water and Steam (IAPWS) [online], last revision September 26, 2008 [cit. 2009-3-1]. *Revised Release on the IAPWS Industrial Formulation 1997 for the Thermodynamic Properties of Water and Steam.* <http://www.iapws.org/>

International Association for the Properties of Water and Steam (IAPWS) [online], last revision September 26, 2008 [cit. 2009-3-1]. *Supplementary Release on Properties of Liquid Water at 0.1Mpa.* <http://www.iapws.org/>

International Association for the Properties of Water and Steam (IAPWS) [online], last revision September 26, 2008 [cit. 2009-3-1]. *Revised Advisory Note No.3. Thermodynamic Derivatives from IAPWS Formulations.* <http://www.iapws.org/>

Pavelek, M. a kol. (2003) *Termomechanika.* Student text, VUT v Brně, Brno.

Varchola, M. & Knížat, B. & Tóth, P. (2007) *Hydraulic Solution of Pipeline Systems*, ISBN, Bratislava.