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DAMAGE CUMULATION FUNCTIONS OF CONCRETE

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Summary: *The reliability of concrete structure is immediately connected with the damage development in the material. The expression of the ascending and descending branches of the stress-strain diagram with the damage cumulation function offers an insight into the progress of damage. It is favourable to approximate the damage cumulation function with the probability distribution functions. The Weibull, gamma and log-normal distributions seem to be convenient. While in the graphical form the damage density is very illustrative, in the numerical form the characteristics of the position and variability of the probabilistic distribution could be used with advantage.*

1. Introduction

The process of damage is immanent to our existence. We are exposed to it and likewise our environment. We expect its occurrence and calculate with its presence. We are searching for its extend and are interested in the resistance against it. From this point of view it is logical, if the analytical model of behaviour of the subject of our interest will be expressed by the damage characteristic. Mechanical material properties are such a subject, which attract attention of engineers and stimulate investigation and progress of knowledge. Within the set of materials, concrete is the one, which is still the matter of interrogation, due to its heterogeneity and the dependence on various influences during the production and service.

Load level belongs among the parameters significantly influencing the progress of damage. Miscellaneous time dependent force redistributions and cracking of different scale inside and between the concrete components are growing during loading. The stress-strain diagram is an illustrative expression of what could be expected from the material. The first task is to try to express it by the characteristic of damage, the second one is to find a simple parameter for the sake of comparison. A set of straightforward coefficients could help sometimes to identify materials with unsuitable properties for the intended purpose. The use of damage cumulation function for the expression of the stress-strain diagram is discussed below. The characteristics of the position and variability of the probabilistic distribution are tested for the comparison of the state of damage.

2. Stress-strain diagram expressed by damage cumulation function

There are various ways to construct the stress-strain diagram. Apart from the different force action the stress-strain diagram could be obtained from tests at the stress rate controlled (soft)

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testing procedure and strain rate controlled (stiff) testing procedure. The first mode of test is characteristic e.g. for common loading by hydraulic jacks – only the ascending branch up to the failure could be registered. However the descending branch of the stress-strain diagram is exceptionally important, hence the necessity of stiff testing (constant velocity of deformation increase – in post-peak region achieved by force reduction). An example of the stress-strain diagram for tension element with both the ascending and descending branch is in Fig. 1. The onset of damage at the increasing loading is signalized by the deviation of the curve from its tangent AC crossing its origin. The determination of arising of microcracks is important for the evaluation of the action of material at sustained or repeated loading. It couldn't be omitted also the importance of the rate of load or strain increase on the shape of the diagram.

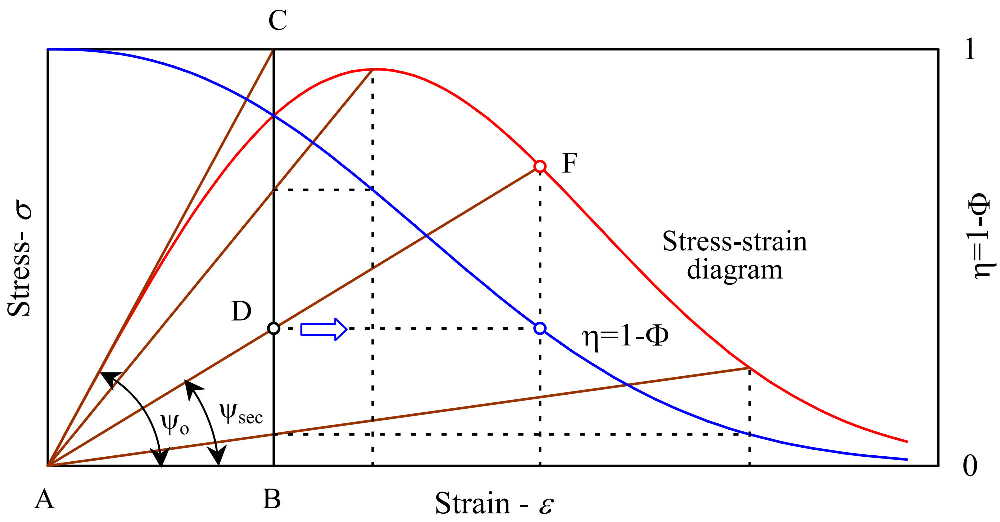

Fig. 1 Stress-strain diagram and damage cumulation function according to (Hájek & Komloš, 1984)

Each point (F) on the stress-strain diagram (Fig. 1) for the monotonously growing deformation (Hájek and Komloš, 1984) given by strain and stress coordinates defines the secant modulus of elasticity E_{sec} which is equal to tg ψ_{sec}. The initial tangent modulus of elasticity E_0 at the strain $\varepsilon = 0$ is given by tg ψ_0. It is obvious, that the concrete state deterioration is indicated by the decrease of E_{sec}. Helpful could be the comparison of triangles ABC and ABD (D is the intersection point of the abscissa AF with BC) in Fig. 1. AC is the tangent in the initial point A of the stress-strain diagram and BC is the parallel with the stress axis crossing the strain axis in the arbitrary point B. Let the distance BC is equal to 1. Important here is the distance BD, which could be derived from both triangles having the same leg AB

$$\frac{BD}{BC} = \frac{BD}{1} = \frac{tg\psi_{\text{sec}}}{tg\psi_0} = \frac{E_{\text{sec}}}{E_0} = \eta . \tag{1}$$

For given assumptions BD is equal to the ratio of secant and initial moduli – it is practical to denote it as η. The point on the stress-strain diagram (by introducing E_{sec} from Eq. 1) is given by the equation

$$\sigma = E_{\text{sec}}\varepsilon = E_0\eta\varepsilon . \tag{2}$$

The axial force N acting on the area of the cross section A could be expressed from (Eq. 2) as

$$N = \sigma A = E_0 \varepsilon \eta A. \tag{3}$$

To explain (Eq. 3) the material model according to (Heilmann, 1976) could be used. The cross-section is composed of parallel fibres with constant modulus of elasticity E_0, different strength and with linear stress-strain dependence up to the failure. It means that at the force N part of the fibres is already damaged (we can denote it as Φ), only the area ηA is still acting. From the definition it follows for the sum of the acting part of the cross-section (η) and of the damaged part (Φ)

$$\eta + \Phi = 1 \tag{4}$$

We can plot the parameter η (= BD) in Fig. 1 in the position ε. By this way we receive the whole curve starting with the value equal to 1 (= BC) for $\varepsilon = 0$ and approaching 0 for higher strains. The complement to 1 is creating the curve of the parameter Φ, which is considered in (Hájek & Komloš, 1984) as the *damage cumulation function* of the cross section. At intact material $\Phi = 0$ and at fully damaged material $\Phi = 1$. Introducing (Eq. 4) into (Eq. 2) we receive the expression of the stress-strain diagram by the use of the damage characteristic.

3. The expressions of damage cumulation function

It is the endeavour to express the damage cumulation function as simple as possible at contemporary sufficient accuracy. At first sight it is apparent its similarity in shape as well as in the nature with the cumulative (probability) distribution functions known from the theory of probability. Hájek & Komloš (1984) chose for the expression of damage cumulation of concrete in axial tension the three-parameter Weibull distribution (Wikipedia) in the form

$$\Phi(\varepsilon, \varepsilon_0, \alpha, \beta) = 1 - e^{-\left(\frac{\varepsilon - \varepsilon_0}{\beta}\right)^{\alpha}}, \tag{5}$$

where ε_0 is the threshold of damage, α the shape parameter, β the scale parameter and ε the relative deformation as the independent variable. Its complementary cumulative distribution function (in our case of η) is a stretched exponential.

As a first step of the damage cumulation function determination it is preferable to separate ε_0. One of the best ways to observe the progress of damage is the exploration of the volume deformations. The origin of microcracks is identified by the deviation of the curve from the straight line. The turn from the decrease to increase of the volume at higher load levels is a signal of advanced stage of destruction. The disadvantage is the need of strain measurements in three mutually perpendicular axes, what is not made usually. Another procedure was used by (Hájek & Komloš, 1984). Best-fit regression lines through two, three or four experimental points of the stress-strain diagram were calculated successively (in Fig. 2a for four points) and their intersections with the vertical (σ) axis were determined. The position of the damage starting could be located in Fig. 2b from the course of magnitudes (a_0) plotted for the average strain in the specific regression.

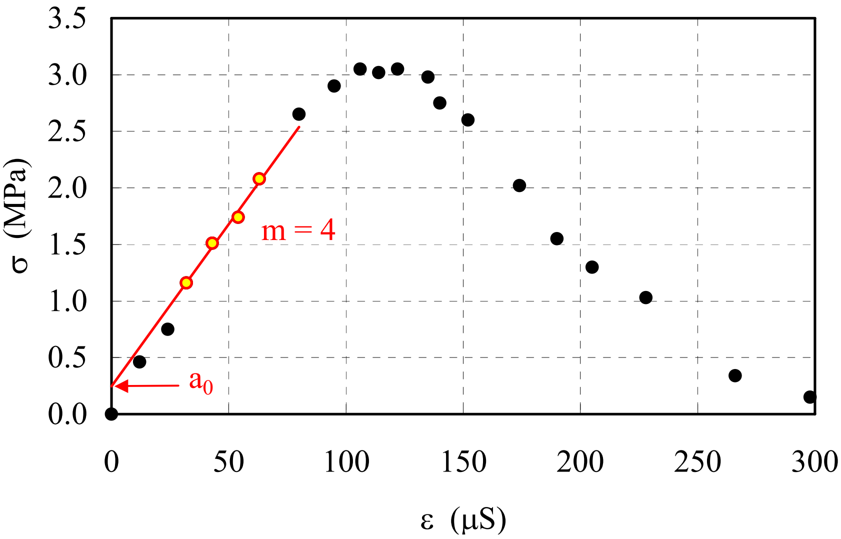

Fig. 2a Best-fit regression lines through four experimental points of the stress-strain diagram – a_0 determination (Hájek & Komloš, 1984).

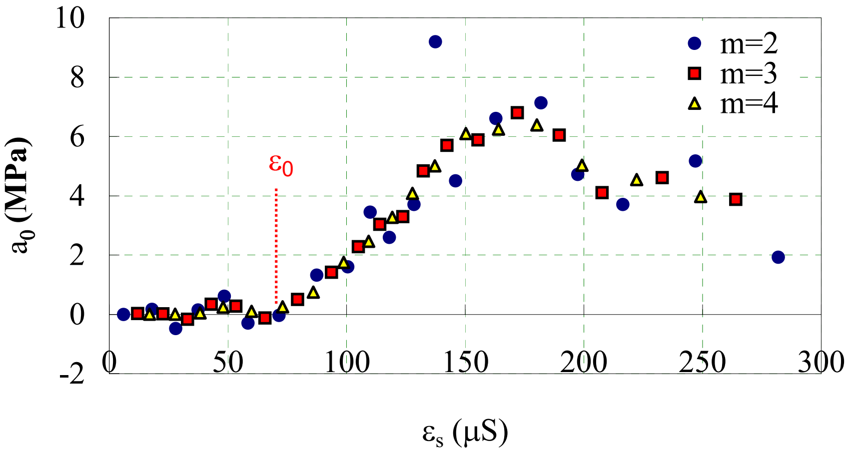

Fig. 2b Threshold of damage determination (Hájek & Komloš, 1984).

At the use of Weibull distribution it is supposed, that no damage occurs up to ε_0. Experimental values of Φ are derived from experimental values of the stress-strain diagram according to Fig. 1 (Eq. 1 and Eq. 4) – the initial modulus of elasticity is calculated from the slope (linear approximation) of the stress-strain diagram up to ε_0. The shape (α) and scale (β) parameters are derived then from the least squares of the difference of experimental and calculated (Eq. 5 or Excel function) values of Φ (only points above the damage threshold are included). The plot of experimental (Hájek & Komloš, 1984) and calculated damage cumulation functions for concrete in tension is in Fig. 3. Very illustrative in the theory of probability is the derivation of the cumulative distribution function known as probability density function (for continuous random variables). If it is used for damage cumulation it is called as *damage density* (Hájek & Komloš, 1984). As can be seen in Fig. 4, its course identifies the regions of strain where the damage activities are concentrated. However if we are interested in the amount of damage related to the acting part of the cross section – η (remaining fibres), we can use the *damage intensity* (λ), defined in (Hájek & Komloš, 1984) as the ratio of damage density (ϕ) and η

$$\lambda = \frac{\phi}{1-\Phi} = \frac{\phi}{\eta}. \tag{6}$$

Though the damage density decreases at higher strains, the damage intensity is still increasing (Fig. 4).

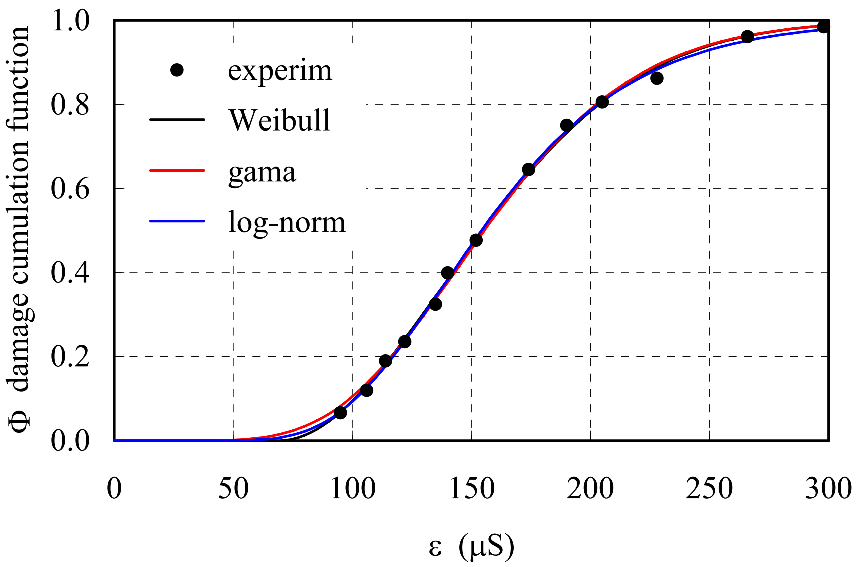

Fig. 3 Damage cumulation function

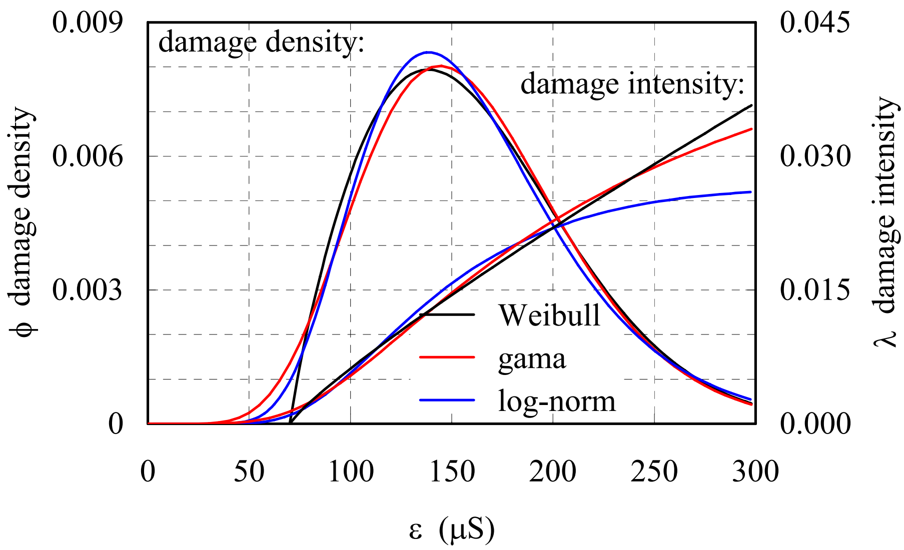

Fig. 4 Damage density and damage intensity

Having the α and β parameters from the approximation we can introduce Eq. 5 into Eq. 4 and Eq. 2 receiving the expression for the theoretical course of the stress-strain diagram, as depicted in Fig. 5. By use of the Weibull distribution the part up to ε_0 (= 0.07 ‰) is created by

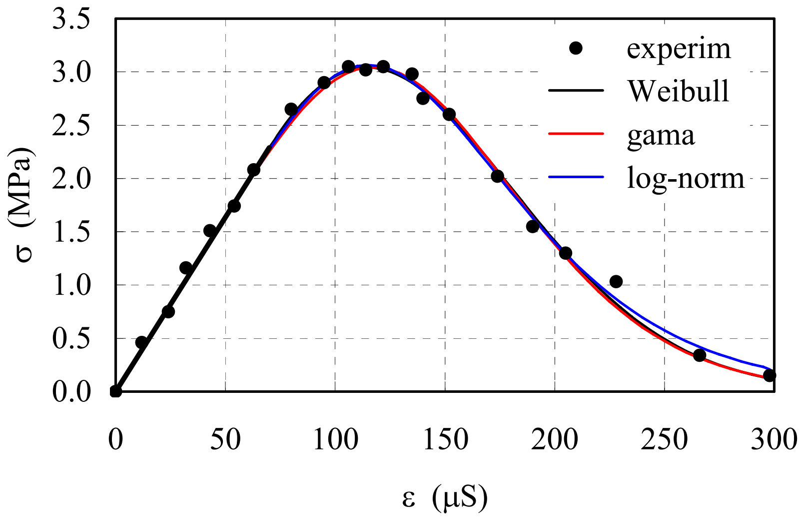

Fig. 5 Stress-strain diagram for measurements according to (Hájek & Komloš, 1984).

the straight line with the initial modulus of elasticity (derived from the slope of experimental values up ε_0). Satisfactory fitting of experimental measurements could be observed.

Two other probability distributions seem to be appropriate for the expression of the damage cumulation function – the gamma distribution and the log-normal distribution. They are defined by the probability density functions

$$f(\varepsilon,\varepsilon_0,\alpha,\beta)=\frac{1}{|\beta|^{\alpha}\Gamma(\alpha)}|\varepsilon-\varepsilon_0|^{(\alpha-1)}e^{-\frac{(\varepsilon-\varepsilon_0)}{\beta}} \tag{7}$$

$$f(\varepsilon,\varepsilon_0,\mu_u,\sigma_u)=\frac{1}{\sqrt{2\pi}|\sigma_u|}\frac{1}{|\varepsilon-\varepsilon_0|}e^{-\frac{1}{2}\left(\frac{\ln|\varepsilon-\varepsilon_0|-\mu_u}{\sigma_u}\right)^2}. \tag{8}$$

Both relationships are three-parameter. The parameters are ε_0, α, β, μ_u (mean value) and σ_u (standard deviation of the function u = ln|ε-ε_0|). $\Gamma(\alpha)$ is the gamma function. The cumulative distribution functions of both distributions are defined as an integral of density functions. With regard to the fact, that the modern programs have already built-in distribution functions, this does not present a problem at the approximation. Another way of approximation is to vary all three parameters, the condition is that only experimental values with $\varepsilon > \varepsilon_0$ are taken into consideration at the partial step. As can be seen in Figs. 3 and 4 the course of the damage cumulation function, so as of the damage density and damage intensity is starting from values of ε_0 different from that calculated for the Weibull distribution. The experimental measurements of the stress-strain diagram (Fig. 5) are satisfactorily fitted by all three curves.

The graphical presentation of the damage density function is very illustrative. For the sake of comparison it will be desirable to characterize each distribution by numerical parameters of the position and variability. The task is therefore to find the relationship between the parameters of the probability equations and the mean value, the standard deviation and the skewness. In the following the expressions for the gamma distribution (Mrázik, 1987) are presented.

If $f(\varepsilon)$ is the probability density function according to Eq. (7), the first moment about the threshold value ε_0 could be written as

$$m_{\varepsilon_0,1} = \int_{\varepsilon_0}^{\infty} f(\varepsilon)(\varepsilon-\varepsilon_0)d\varepsilon = \int_{\varepsilon_0}^{\infty} \frac{1}{\beta^{\alpha}\Gamma(\alpha)}(\varepsilon-\varepsilon_0)^{\alpha} e^{-\frac{(\varepsilon-\varepsilon_0)}{\beta}} d\varepsilon. \tag{9}$$

Taking into consideration the definition of the gamma function and the relationship between $\Gamma(\alpha+1)$ and $\Gamma(\alpha)$ we receive that

$$m_{\varepsilon_0,1} = \alpha\,\beta, \tag{10}$$

what is the distance of the mean from the threshold value ε_0. The distance from zero is then

$$\mu_{\varepsilon} = \alpha\,\beta + \varepsilon_0. \tag{11}$$

Similarly could be written for the second and the third moment about the threshold value ε_0

$$m_{\varepsilon_0,2} = \alpha(\alpha+1)\beta^2, \tag{12}$$

$$m_{\varepsilon_0,3} = \alpha(\alpha+1)(\alpha+2)\beta^3. \tag{13}$$

The second and the third central moments can be expressed by moments about ε_0. Using Eq. (10) and (12) we receive for the standard deviation σ_{ε}

$$\sigma_{\varepsilon} = \sqrt{\alpha}\,|\beta|. \tag{14}$$

The skewness, defined as the ratio of the third central moment and the third power of the standard deviation, could be written by use of Eq. (10), (12) and (13) in the form

$$A = \frac{2}{\sqrt{\alpha}}. \tag{15}$$

If we apply Eqs. (11), (14) and (15) to the approximation of results presented in (Hájek & Komloš, 1984) with the gamma cumulative distribution function we receive for the mean value, the standard deviation and the skewness the values 161.59, 52.35 and 0.757 respectively. Using the analogous expressions for log-normal distribution (Wikipedia) the corresponding values will be 163.65, 56.26 and 1.254. Though the differences, especially at the first two parameters, are not significant, the use of one distribution at the given comparison will undoubtedly lead to the enhancement of the accuracy.

More intricated models were developed (Li & Li, 2001), taking into account the contribution of fibres and their debonding and slip. The question of damage cumulation is highly actual also when dealing with masonry structures of inferior quality (Bažant & Strnad, 2008) or subjected to the action of fire (Bellová, 2008), so as by the repair of structures with prestressing (Klusáček, 2002; Klusáček & Bažant, 2003; Klusáček et al, 2008). The application of artificial neural networks to the determination of damage and strengths parameters (Hoła & Schabowicz, 2005) could be promising. The strain analysis of aerated concrete members (Hroncová & Piták, 2005), so as the probability based solutions at the analysis of the cross-section and special structures (Hroncová, 1996; Králik, 2008; Králik & Králik, 2008; Hudoba & Grešlík, 2005; Melcer, 2007) are close connected with the damage evaluation. It is the important question of the corrosion influence (Janotka & Krajči, 2008; Krajči, 2004; Krajči, 2006; Krajči & Janotka, 2005) on the damage distribution detected e.g. by use of the course of the damage density.

4. Conclusion

For the damage state of concrete determination it is preferable to express the stress-strain diagram of concrete by the damage cumulation function.

The damage cumulation function approximated with the cumulative distribution functions is together with the damage density and damage intensity a sensitive and reliable indicator of the concrete quality.

The Weibull, gamma and log-normal cumulative distribution functions were found to be suitable for damage cumulation approximation.

The state of damage could be transparently expressed by the characteristics of the position and variability of the probabilistic distribution.

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