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STUDY OF THE CAVITATING VORTEX ROPE IN HYDRAULIC TURBINES

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Summary: *Motion of cavitating vortex rope with influence of its rotation has been studied. The solution is performed in Lagrange coordinates. The nonlinear mathematical model takes into account surface tension on the rope interface. The boundary condition is represented by Laplace equation and state equation is expressed by polytropic law.*

1. Introduction

Flow in the outlet diffuser of the hydraulic turbines - draft tube - is accompanied by severe instability for non-optimal operating points. This phenomenon is result of the vortical flow instability, in turbomachinery is usually termed "vortex rope". Core of the vortical structure is location of the minimum pressure, often the pressure reaches value of saturated vapor pressure and so called "cavitating vortex rope" appears (Illiescu et al., 2008). It is very undesirable effect, which causes vibrations, pressure pulsations and noise. Therefore extensive studies are aimed at its description and understanding. Contrary to many papers recently published, which contain results of computational and experimental research, this contribution is focused on theoretical investigation of the stability mechanism.

Fig. 1 Cavitating vortex rope (experimental visualization, Rudolf(2009))

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2. Trajectory of the fluid particle in Lagrange coordinates

Let us assume that cylindrical vortex rope originated due to rotation of the liquid. Its motion and vibration of its surface will be described using Lagrange coordinates.

$$x_1 = F(a_1, t)\cos[a_2 + \psi(a_1, t)] \quad (1)$$

$$x_2 = F(a_1, t)\sin[a_2 + \psi(a_1, t)] \quad (2)$$

$$x_3 = a_3 + c_0 t$$

We will assume that the rope surface can pulsate. Circumferential and axial velocity distributions can be prescribed in time t = 0. The coordinate system is depicted in Fig. 1.

F and ψ in relations (1), (2) are unknown functions yet, which will be determined from force equillibrium equation.

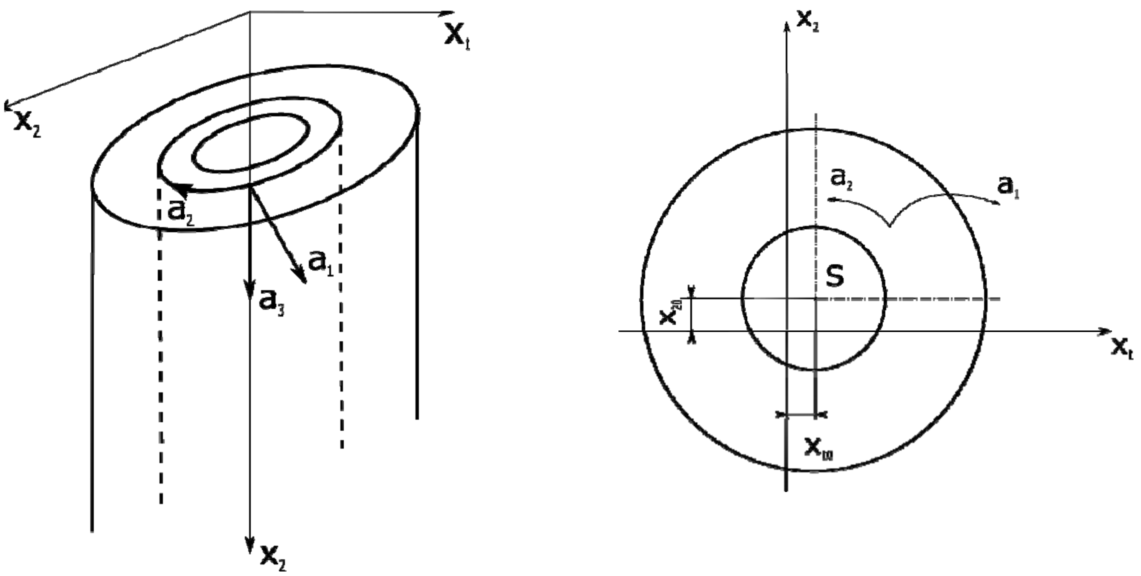

Figure 1 Curvilinear coordinate system

Let us consider following transformation relations (see Fig.2):

$$\Phi = a_2 + \psi(a_1, t) \quad (4)$$

$$\Phi = \varphi + \varepsilon = a_2 + \psi(a_1, t) \quad (5)$$

$$\sin\varepsilon = \frac{x_{10}}{r}\sin\varphi - \frac{x_{20}}{r}\cos\varphi \quad (6)$$

$$\cos\varepsilon = \frac{R}{r} - \frac{x_{20}}{r}\sin\varphi - \frac{x_{10}}{r}\cos\varphi \quad (7)$$

$$R = x_{20} \sin\varphi + x_{10} \cos\varphi + \sqrt{r^2 + \left(x_{20} \sin\varphi + x_{10} \cos\varphi\right)^2} \qquad (8)$$

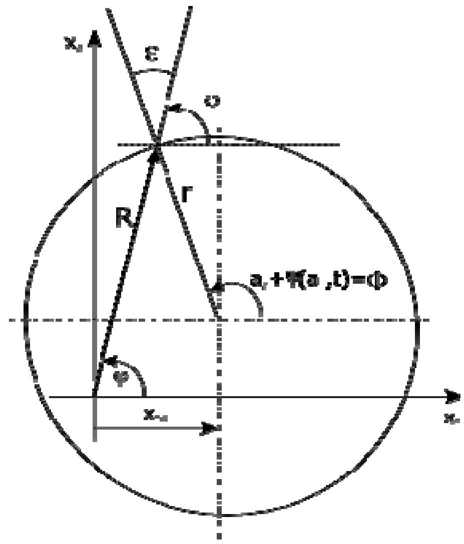

Figure 2 Kinematics of the particle motion

3. Continuity equation

Mass conservation principle, continuity equation, is expressed by jacobian:

$$J(0) = J(t) = \frac{\partial x_1}{\partial a_1}\frac{\partial x_2}{\partial a_2} - \frac{\partial x_2}{\partial a_1}\frac{\partial x_1}{\partial a_2} = a_1 \qquad (9)$$

Partial derivatives with respect to time have following form:

$$\frac{\partial x_1}{\partial t} = F^{\bullet} \cos[a_2 + \psi(a_1, t)] - F\Phi^{\bullet} \sin[a_2 + \psi(a_1, t)]$$

$$\frac{\partial^2 x_1}{\partial t^2} = F^{\bullet\bullet} \cos[a_2 + \psi(a_1, t)] - F^{\bullet}\Phi^{\bullet} \sin[a_2 + \psi(a_1, t)] -$$
$$- F^{\bullet}\Phi^{\bullet} \sin[a_2 + \psi(a_1, t)] - F\Phi^{\bullet\bullet} \sin[a_2 + \psi(a_1, t)] - F\Phi^{\bullet 2} \cos[a_2 + \psi(a_1, t)]$$

$$\frac{\partial^2 x_1}{\partial t^2} = F^{\bullet\bullet} \cos[a_2 + \psi(a_1, t)] - 2F^{\bullet}\Phi^{\bullet} \sin[a_2 + \psi(a_1, t)] -$$
$$- F\Phi^{\bullet 2} \cos[a_2 + \psi(a_1, t)] - F\Phi^{\bullet\bullet} \sin[a_2 + \psi(a_1, t)] \qquad (10)$$

$$\frac{\partial x_2}{\partial t} = F^{\bullet} \sin[a_2 + \psi(a_1, t)] + F\Phi^{\bullet} \cos[a_2 + \psi(a_1, t)]$$

$$\frac{\partial^2 x_2}{\partial t^2} = F^{\bullet\bullet} \sin[a_2 + \psi(a_1, t)] + F^{\bullet}\Phi^{\bullet} \cos[a_2 + \psi(a_1, t)] + F^{\bullet}\Phi^{\bullet} \cos[a_2 + \psi(a_1, t)] -$$
$$- F\Phi^{\bullet 2} \sin[a_2 + \psi(a_1, t)] + F\Phi^{\bullet\bullet} \cos[a_2 + \psi(a_1, t)]$$

$$\frac{\partial^2 x_2}{\partial t^2} = F^{\bullet\bullet}\sin[a_2+\psi(a_1,t)] + 2F^{\bullet}\Phi^{\bullet}\cos[a_2+\psi(a_1,t)] - \\ - F\Phi^{\bullet 2}\sin[a_2+\psi(a_1,t)] + F\Phi^{\bullet\bullet}\cos[a_2+\psi(a_1,t)] \qquad (11)$$

$$\frac{\partial^2 x_1}{\partial t^2} = \left(F^{\bullet\bullet} - F\Phi^{\bullet 2}\right)\cos[a_2+\psi(a_1,t)] - \left(2F^{\bullet}\Phi^{\bullet} + F\Phi^{\bullet\bullet}\right)\sin[a_2+\psi(a_1,t)] \qquad (12)$$

$$\frac{\partial^2 x_2}{\partial t^2} = \left(F^{\bullet\bullet} - F\Phi^{\bullet 2}\right)\sin[a_2+\psi(a_1,t)] + \left(2F^{\bullet}\Phi^{\bullet} + F\Phi^{\bullet\bullet}\right)\cos[a_2+\psi(a_1,t)] \qquad (13)$$

Let us denote to simplify the relations (12) and (13):

$$\alpha = F^{\bullet\bullet} - F\Phi^{\bullet 2} \quad ; \qquad \beta = 2F^{\bullet}\Phi^{\bullet} + F\Phi^{\bullet\bullet} \qquad (13a)$$

$$\frac{\partial^2 x_1}{\partial t^2} = \alpha\cos[a_2+\psi(a_1,t)] - \beta\sin[a_2+\psi(a_1,t)] \qquad (14)$$

$$\frac{\partial^2 x_2}{\partial t^2} = \alpha\sin[a_2+\psi(a_1,t)] - \beta\cos[a_2+\psi(a_1,t)] \qquad (15)$$

Derivatives with respect to spatial coordinates:

$$\frac{\partial x_1}{\partial a_1} = x_1' = F'\cos[a_2+\psi(a_1,t)] - F\Phi'\sin[a_2+\psi(a_1,t)] \qquad (16)$$

$$\frac{\partial x_1}{\partial a_2} = -F\sin[a_2+\psi(a_1,t)] \qquad (17)$$

$$\frac{\partial x_2}{\partial a_1} = F'\sin[a_2+\psi(a_1,t)] + F\Phi'\cos[a_2+\psi(a_1,t)] \qquad (18)$$

$$\frac{\partial x_2}{\partial a_2} = -F\cos[a_2+\psi(a_1,t)] \qquad (19)$$

After inserting (16)-(19) to continuity equation (9):

$$\left(F'\cos - F\Phi'\sin\right)F\cos[a_2+\psi(a_1,t)] + \left(F'\sin + F\Phi'\cos\right)F\sin[a_2+\psi(a_1,t)] = a_1$$

$$\left(F'F\right)\cos^2 + F'F\sin^2 + F^2\Phi'\sin\cos[a_2+\psi(a_1,t)] - F^2\Phi'\sin\cos[a_2+\psi(a_1,t)] = a_1$$

$$F'F = a_1$$

$$\frac{\partial}{\partial a_1}\left(\frac{1}{2}F^2\right) = a_1$$

For circular cross-section of the vortex rope we may write:

$$F = r = \sqrt{A(t) + a_1^2} \tag{20}$$

4. Velocity components for planar motion

Let us determine circumferential and radial velocity components for planar motion of the fluid particle:

$$c_\varphi = -x_1^\bullet \sin\varphi + x_2^\bullet \cos\varphi \tag{21}$$

$$c_r = -x_1^\bullet \cos\varphi + x_2^\bullet \sin\varphi \tag{22}$$

Using relation (5):

$$c_\varphi = -x_1^\bullet \sin(\Phi - \varepsilon) + x_2^\bullet \cos\varphi(\Phi - \varepsilon)$$

$$\begin{aligned} c_\varphi &= \left[-F^\bullet \cos\Phi + F\Phi^\bullet \sin\Phi + x_{10}^\bullet\right]\left[\sin\Phi\cos\varepsilon - \cos\Phi\sin\varepsilon\right] + \\ &+ \left[F^\bullet \sin\Phi + F\Phi^\bullet \cos\Phi + x_{20}^\bullet\right]\left[\cos\Phi\cos\varepsilon + \sin\Phi\sin\varepsilon\right] = \\ &= \left(-F^\bullet \cos\varepsilon - F\Phi^\bullet \sin\varepsilon + F^\bullet \cos\varepsilon + F\Phi^\bullet \sin\varepsilon\right)\sin\Phi\cos\Phi + \\ &+ \left(+F^\bullet \sin\varepsilon - F\Phi^\bullet \cos\varepsilon\right)\cos^2\Phi + \left(F\Phi^\bullet \cos\varepsilon + F^\bullet \sin\varepsilon\right)\sin^2\Phi + \\ &+ x_{10}^\bullet \underbrace{\left(\sin\Phi\cos\varepsilon - \cos\Phi\sin\varepsilon\right)}_{\sin(\Phi-\varepsilon)} + x_{20}^\bullet \underbrace{\left(\cos\Phi\cos\varepsilon + \sin\Phi\sin\varepsilon\right)}_{\cos(\Phi-\varepsilon)} \end{aligned}$$

Final expression for circumferential velocity component:

$$c_\varphi = F^\bullet \sin\varepsilon + F\Phi^\bullet \cos\varepsilon + x_{10}^\bullet \sin\varphi + x_{20}^\bullet \cos\varphi \tag{23a}$$

$$c_r = x_1^\bullet \cos(\Phi - \varepsilon) + x_2^\bullet \sin(\Phi - \varepsilon)$$

$$\begin{aligned} c_r &= \left(F^\bullet \cos\Phi + F\Phi^\bullet \sin\Phi + x_{10}^\bullet\right)\left(\cos\Phi\cos\varepsilon - \sin\Phi\sin\varepsilon\right) + \\ &+ \left(F^\bullet \sin\Phi + F\Phi^\bullet \cos\Phi + x_{20}^\bullet\right)\left(\sin\Phi\cos\varepsilon - \cos\Phi\sin\varepsilon\right) = \\ &= \left(F^\bullet \sin\varepsilon - F\Phi^\bullet \cos\varepsilon + F\Phi^\bullet \cos\varepsilon - F^\bullet \sin\varepsilon\right)cos\Phi sin\Phi + \\ &+ \left(F^\bullet \cos\varepsilon - F\Phi^\bullet \sin\varepsilon\right)\cos^2\Phi + \left(-F\Phi^\bullet \sin\varepsilon + F^\bullet \cos\varepsilon\right)\sin^2\Phi + \\ &+ x_{10}^\bullet \cos\varphi + x_{20}^\bullet \sin\varphi \end{aligned}$$

Velocity component in radial direction

$$c_r = F^\bullet \cos\varepsilon - F\Phi^\bullet \sin\varepsilon + x_{10}^\bullet \cos\varphi + x_{20}^\bullet \sin\varphi \tag{23b}$$

5. Force equillibrium equation

Pressure function will be determined from force equillibrium equation:

$$\frac{\partial^2 x_1}{\partial t^2}\frac{\partial x_1}{\partial a_2}+\frac{\partial^2 x_2}{\partial t^2}\frac{\partial x_2}{\partial a_2}+\frac{1}{\rho}\frac{\partial p}{\rho a_2}=0$$

$$\left[\alpha\cos\Phi-\beta\sin\Phi\ddot{x}_{10}\right]\left(-F\sin\Phi\right)+\left(\alpha\sin\Phi+\beta\cos\Phi+\ddot{x}_{20}\right)F\cos\Phi+\frac{1}{\rho}\frac{\partial p}{\partial a_2}=0$$

$$\left(-\alpha F+\alpha F\right)\sin\Phi\cos\Phi+\beta F\sin^2\Phi+\beta F\cos^2\Phi+F\cos\ddot{x}_{20}-F\ddot{x}_{10}\sin+\frac{1}{\rho}\frac{\partial p}{\partial a_2}=0$$

It follows then:

$$\frac{\partial p}{\partial a_2}=\rho\left(F\ddot{x}_{10}\sin\Phi-F\ddot{x}_{20}\cos\Phi-\beta F\right)$$

Pressure function is obtained after intergration:

$$p=H\left(a_1,t\right)+\rho\left(-F\ddot{x}_{10}\cos\Phi-F\ddot{x}_{20}\sin\Phi-\beta F a_2\right) \tag{24}$$

Because pressure function must be continuous for $x_3 = const.$, it must hold:

$$\beta=0$$

$$2\dot{F}\dot{\Phi}+F\ddot{\Phi}=0 \tag{25}$$

$$\ddot{\Phi}+2\frac{\dot{F}}{F}\dot{\Phi}=0\Rightarrow$$

$$\dot{\Phi}=\frac{B\left(a_1\right)}{F^2} \tag{26}$$

$$p=H\left(a_1,t\right)-\rho\left(F\ddot{x}_{10}\cos\Phi+F\ddot{x}_{20}\sin\Phi\right) \tag{27}$$

It must further hold that:

$$\frac{\partial^2 x_1}{\partial t^2}\frac{\partial x_1}{\partial a_1}+\frac{\partial^2 x_2}{\partial t^2}\frac{\partial x_2}{\partial a_1}+\frac{1}{\rho}\frac{\partial P}{\rho a_1}=0$$

$$\left(\alpha\cos\Phi-\beta\sin\Phi+\ddot{x}_{10}\right)\left(\mathrm{F}'\cos\Phi-F\Phi'\sin\Phi\right)+$$
$$+\left(\alpha\sin\Phi+\ddot{x}_{20}\right)\left(\mathrm{F}'\sin\Phi+F\Phi'\cos\Phi\right)+\frac{1}{\rho}\frac{\partial p}{\partial a_1}=0$$

$$\alpha F'\cos^2\Phi+\alpha F'\sin^2\Phi-\alpha F\Phi'\sin\Phi\cos\Phi+\alpha F\Phi'\sin\Phi\cos\Phi+$$
$$+\ddot{x}_{10}\left(F'\cos\Phi-F\Phi'\sin\Phi\right)+\left(F'\sin\Phi+F\Phi'\cos\Phi\right)\ddot{x}_{20}+\frac{1}{\rho}\frac{\partial p}{\partial a_1}=0$$

$$\frac{1}{\rho}\frac{\partial p}{\partial a_1}+\alpha F'+\ddot{x}_{10}\left(F'\cos\Phi-F\Phi'\sin\Phi\right)+\ddot{x}_{20}\left(F'\sin\Phi+F\Phi'\cos\Phi\right) \tag{28}$$

It follows from equation (27):

$$\frac{\partial p}{\partial a_1} = \frac{\partial H}{\partial a_1} - \rho \ddot{x}_{10}\left(F'\cos\Phi - F\Phi'\sin\Phi\right) - \rho \ddot{x}_{20}\left(F'\sin\Phi + F\Phi'\cos\Phi\right)$$

Let us insert to (28):

$$\frac{1}{\rho}\frac{\partial H}{\partial a_1} + \alpha F' = 0$$

$$\frac{\partial H}{\partial a_1} = -\rho\alpha F' \tag{29}$$

Now let us inset from equation (13a):

$$\frac{\partial H}{\partial a_1} = -\rho F'\left(F^{\bullet\bullet} - F\Phi^{\bullet 2}\right)$$

Rearranging this relation we will finally obtain from expressions (20) and (26):

$$\Phi^{\bullet} = \frac{B}{A(t) + a_1^2} \;; \qquad F = \sqrt{A(t) + a_1^2}$$

$$F^{\bullet} = \frac{1}{2}A^{\bullet}\left(A + a_1^2\right)^{-\frac{1}{2}}$$

$$F^{\bullet\bullet} = \frac{1}{2}A^{\bullet\bullet}\left(A + a_1^2\right)^{-\frac{1}{2}} - \frac{1}{4}A^{\bullet 2}\left(A + a_1^2\right)^{-3/2}$$

$$F' = a_1\left(A + a_1^2\right)^{-\frac{1}{2}}$$

$$F^{\bullet\bullet}F' = \left[\frac{1}{2}A^{\bullet\bullet}\left(A + a_1^2\right)^{-\frac{1}{2}} - \frac{1}{4}A^{\bullet 2}\left(A + a_1^2\right)^{-3/2}\right]a_1\left(A + a_1^2\right)^{-\frac{1}{2}}$$

$$F^{\bullet\bullet}F' = \frac{1}{2}a_1A^{\bullet\bullet}\left(A + a_1^2\right)^{-1} - \frac{1}{4}a_1A^{\bullet 2}\left(A + a_1^2\right)^{-2}$$

$$F^{\bullet\bullet}F'\Phi^{\bullet 2} = \left(A + a_1^2\right)^{//2}a_1\left(A + a_1^2\right)^{-\frac{1}{2}} \cdot\; B^2\left(A + a_1^2\right)^{-2}$$

$$FF'\Phi^{\bullet 2} = a_1B^4\left(A + a_1^2\right)^{-2}$$

$$\frac{\partial H}{\partial a_1} = -\rho\left[\frac{1}{2}a_1A^{\bullet\bullet}\left(A + a_1^2\right)^{-1} - \frac{1}{4}a_1A^{\bullet 2}\left(A + a_1^2\right)^{-2} - a_1B^2\left(A + a_1^2\right)^{-2}\right]$$

$$\frac{\partial H}{\partial a_1} = -\rho\left[\frac{1}{2}A^{\bullet\bullet}\frac{a_1}{A + a_1^2} - \frac{1}{4}A^{\bullet 2}\frac{a_1}{\left(A + a_1^2\right)^2} - \frac{a_1B^2}{A + a_1^2}\right]$$

$$H = U(t) - \rho\left[\frac{A^{\bullet\bullet}}{4}\ln\left(A + a_1^2\right) + \frac{A^{\bullet 2}}{8}\frac{1}{A + a_1^2} - \int\frac{a_1B^2}{A + a_1^2}\,da_1\right] \tag{30}$$

Inserting to (27):

$$p = U(t) - \rho\left\{\frac{A^{\bullet\bullet}}{4}\ln\left(A + a_1^2\right) + \frac{A^{\bullet 2}}{8}\frac{1}{A + a_1^2} - \int\frac{a_1 B^2}{A + a_1^2}da_1 + F\cos\Phi x_{10}^{\bullet\bullet} + F\sin\Phi x_{20}^{\bullet\bullet}\right\} \tag{31}$$

6. Polytropic law and dynamic boundary condition

Equation (31) must be accompanied by state equation of ideal gas. In time instant t = 0:

$$p_0(0)V_0^{\kappa}(0) = p_0(t)V^{\kappa}(t)$$

where p_0 is pressure inside the rope.

For cylindrical vortex rope in plane (2D):

$$p_0(0)r_0^{2\kappa} = p_0(t)r^{2\kappa}(t)$$

$$p_0(t) = p_0(0)\left(\frac{r_0}{r}\right)^{2\kappa}$$

Dynamic boundary condition on the cavitating vortex rope interface is represented by Laplace-Young equation. For $a_1 = a_{10}$:

$$\sigma\frac{1}{r} = p_0(t) - p(t)$$

$$\sigma\frac{1}{r} = p_0(0)\left(\frac{r_0}{r}\right)^{2\kappa} - U(t) + \rho\left\{\frac{A^{\bullet\bullet}}{4}\ln\left(r^2\right) + \frac{A^{\bullet 2}}{8}\frac{1}{r^2} - \int\frac{a_1 B^2}{A + a_1^2}da_1 \Big/ a_1 = a_{10}\right\} \tag{32}$$

Equation (26) must be taken into account :

$$\Phi^{\bullet} = \frac{B(a_1)}{A(t) + a_1^2}$$

together with initial condition for function $\Phi(t = 0)$: $\Phi(t = 0) = 0$.

For $a_1 = a_{10}$:

$$\Phi^{\bullet}\left[A(t) + a_{10}^2\right] = B(a_{10}) \tag{34}$$

$B(a_1)$ must be inserted into relations (32), (34).

7. Solution of for the special case of solid body rotation

Let us assume $a_{10} >> A(t)$. If $B(a_1) = \omega_0 a_1^2$, as it comes from (34), then

$$\Phi^{\bullet} \cong \omega_0 \Rightarrow \Phi = \omega_0 t \tag{34a}$$

Liquid rotates as solid body with constant angular speed ω_0.

Laplace equation has following form for $t = 0$:

$$\sigma\frac{1}{r_0} = P_0(0) - P(t = 0) \tag{35}$$

Let us denote $(t=0)=\sum_0$ and assume:

$$U(t)=\sum_0+\sum(t) \tag{36}$$

$$\frac{\sigma}{r_0}=p_0(0)-\sum_0 \tag{37}$$

$$p_0(0)=-\frac{\sigma}{r_0}-\sum_0 \tag{38}$$

Relations (36), (38) will be inserted to (32):

$$\frac{\sigma}{r}=-\left(\frac{\sigma}{r_0}+\Sigma_0\right)\left(\frac{r_0}{r}\right)^{2\kappa}-\Sigma_0-\Sigma(t)+\rho\left\{\frac{\ddot{A}}{4}\ln(r^2)+\frac{\dot{A}^2}{8}\frac{1}{r^2}-\int\frac{a_1B^2}{\left(A+a_1^2\right)^2}da_1\Big|_{a_1=a_{10}}\right\} \tag{39}$$

$$r^2=A(t)+a_1^2 \tag{40}$$

$$A(t)=r^2-a_1^2 \tag{41}$$

$$\dot{A}=2r\dot{r} \tag{42}$$

$$\dot{A}^2=4(r\dot{r})^2 \tag{43}$$

$$\ddot{A}=2(r\dot{r})^{\bullet} \tag{44}$$

The integral on the right hand side of (39) is computed under assumption of constant angular speed (see (34a)):

$$B(a_1)=\omega_0a_1^2 \tag{45}$$

$$\begin{aligned}\int\frac{\omega_0^2a_1^5}{\left(A+a_1^2\right)^2}da_1&=\omega_0^2\int\frac{a_1^5}{\left(A+a_1^2\right)^2}da_1=\\&=\omega_0^2\int\left\{a_1-\frac{A^2a_1}{\left(A+a_1^2\right)^2}-\frac{2Aa_1^3}{\left(A+a_1^2\right)^2}\right\}da_1=\\&=\omega_0^2\left(\frac{a_1^2}{2}-\frac{A^2}{2}\frac{1}{A+a_1^2}-A\ln\left|A+a_1^2\right|+\frac{A^2}{A+a_1^2}\right)=\\&=\omega_0^2\left(\frac{a_1^2}{2}+\frac{A^2}{2}\frac{1}{A+a_1^2}-A\ln\left|A+a_1^2\right|\right)\end{aligned} \tag{46}$$

After inserting (40)-(44), (46) to (39):

$$\frac{\sigma}{r}=-\left(\frac{\sigma}{r_0}+\Sigma_0\right)\left(\frac{r_0}{r}\right)^{2\kappa}-\Sigma_0-\Sigma(t)+\rho\left\{\frac{(rr^{\bullet})^{\bullet}}{2}\ln(r^2)+\frac{r^{\bullet 2}}{2}-\omega_0^2\left(\frac{r_0^2}{2}+\frac{\left(r^2-r_0^2\right)^2}{2}\frac{1}{r^2}-\left(r^2-r_0^2\right)\ln r^2\right)\right\}$$

$$\frac{\sigma}{r}=-\left(\frac{\sigma}{r_0}+\Sigma_0\right)\left(\frac{r_0}{r}\right)^{2\kappa}-\Sigma_0-\Sigma(t)+\rho\left\{\left(\dot{r}^2+r\ddot{r}\right)\ln(r)+\frac{\dot{r}^2}{2}-\omega_0^2\left(\frac{r_0^2}{2}+\frac{\left(r^2-r_0^2\right)^2}{2}\frac{1}{r^2}-\left(r^2-r_0^2\right)\ln r^2\right)\right\} \tag{47}$$

Equation of cylindrical vortex rope in two dimensions without rotation is presented for comparison (Brennen, 1995):

$$\left(\frac{\sigma}{R_0}+\Sigma_0\right)\left(\frac{R_0}{R}\right)^{2K}2-\frac{\sigma}{R}+\rho\frac{d}{dt}\left(R\frac{dR}{dt}\right)\ln R+\frac{1}{2}\rho\left(\frac{dR}{dt}\right)^2=-\left[\Sigma_0+p_N\left(t\right)\right]. \tag{48}$$

Equation (48) is obtained from (47) for $\omega_0=0$, which is confirmation of validity of derivation of the equation (39).

The final result (39) is a Rayleigh-Plesset like equation for cylindrical vortical structure filled with saturated vapor. From numerical point of view it is highly nonlinear ordinary differential equation, which must be solved numerically. Stability of the numerical solution for different values of angular speed corresponds to stability of the vortex rope motion. Results of the numerical experiments with equation (47) is presented in Figs. 3, 4. Runge-Kutta numerical procedure was applied for integration of the equation (47). Radius change of the cavitating vortex rope is excited by a pressure step function. It is apparent from Fig.3 that for bigger initial radius (R_0 = 0,04 m) vortex rope motion becomes unstable when higher angular speeds of the solid body rotation are reached.

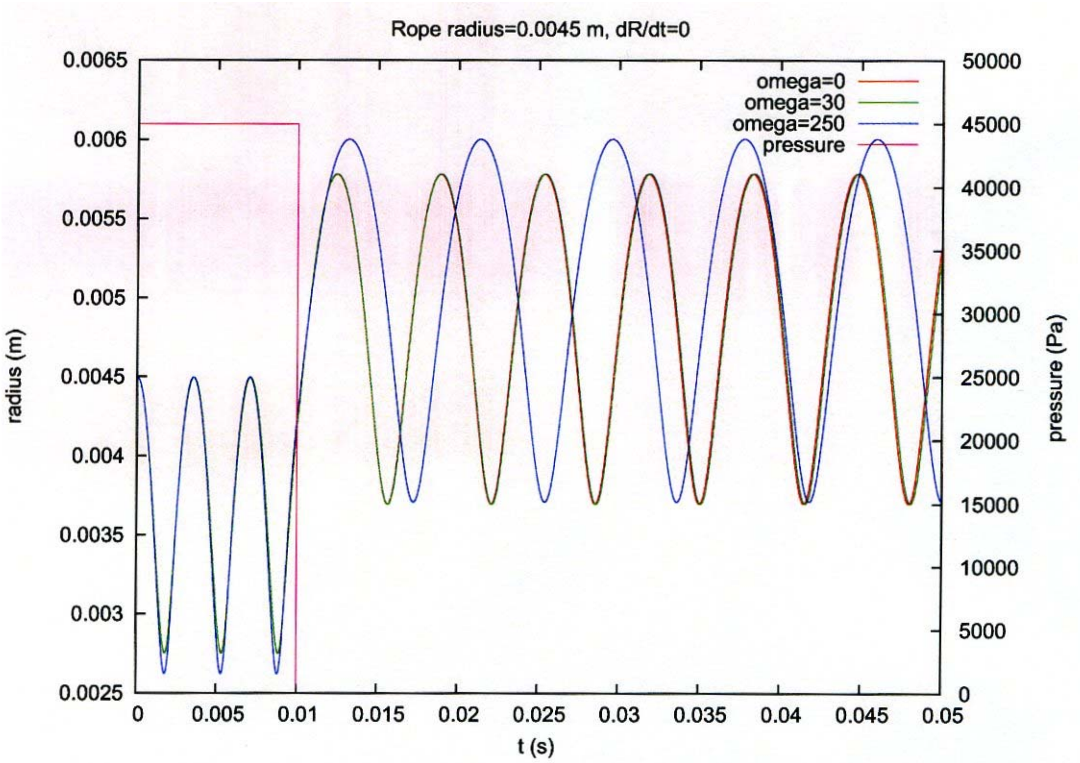

Figure 3 Response of the vortex rope to pressure step function (R_0 = 0,0045 m)

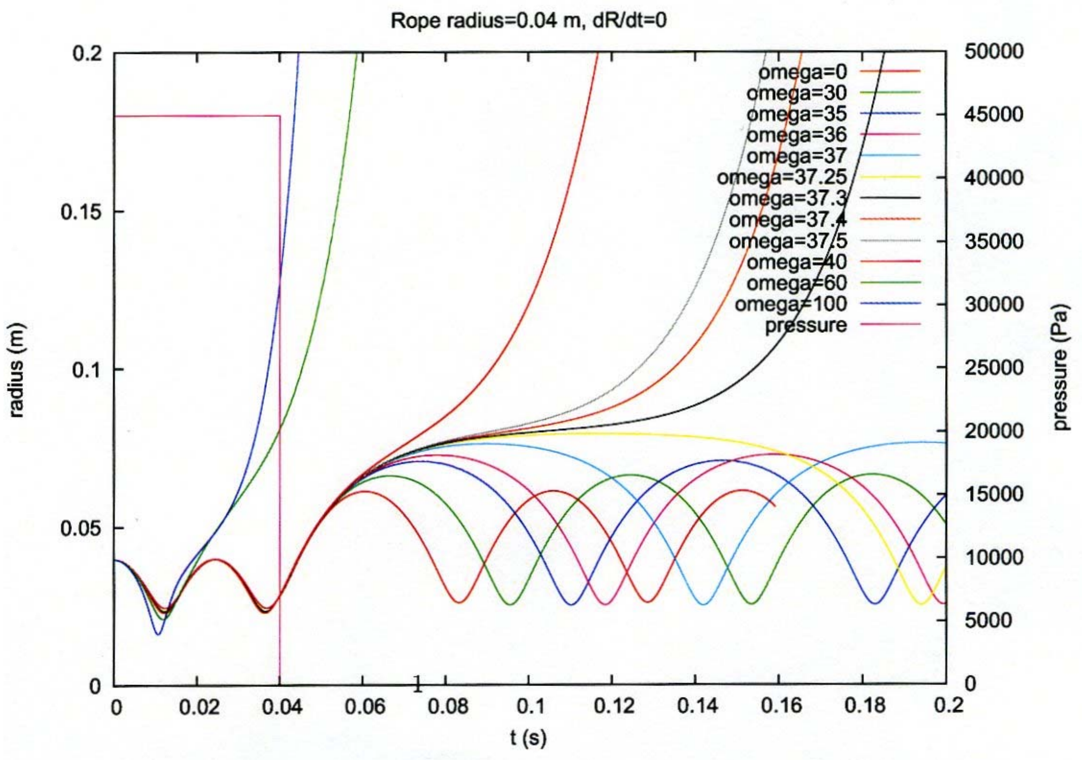

Figure 4 Response of the vortex rope to pressure step function (R_0 = 0,04 m)

8. Conclusion

Derivation of the equation of motion for cylindrical rope cross-section in Lagrange coordinates enabled to include influence of the velocity field. This allows to study stability of the cavitating vortex rope motion for different inlet velocity profiles. However it should be stressed that the final equations are confined to inviscid flow and exclude translational motion of the rope.

Presented analysis forms one of the first steps aimed at understanding of the cavitating rope behavior, especially on the influence of the inlet velocity field on shape of the vortex rope cross-section. Further investigations will comprise experimental and computational (CFD) studies of the problem.

Acknowledgement

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List of symbols:

a_i	curvilinear coordinates
J	jacobian
p	pressure
r	radius
t	time
x_i	cartesian coordinates
ε	angle
φ	angle
Φ	angle
σ	surface tension
ω	angular speed
Σ	pressure in time t = 0

subscripts:

0	respective quantity in time t = 0
1,2,3	axes