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COMPARISON OF THE DEFORMATION AND INCREMENTAL THEORY OF PLASTICITY USED FOR EXPERIMENTAL MECHANICS PROBLEMS

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Summary: *Deformation Theory of Plasticity is often used for evaluation of the experimental data. Main reason of this theory using is its simple implementation into experimental-numeric software. It is supposed that difference between power low and appropriate incremental theory applied is negligible if monotonic loading is presented and isotropic material is used. It will be shown in this paper that this presupposition can be no longer valid if high ductile material is used although external loading is strictly monotonic and proportional.*

1. Introduction

It is well known by the solid and fracture mechanics that incremental theory of plasticity is more appropriate for general loading than deformation theory often used. Incremental theories are nowadays commonly employed for numerical simulations however these have higher computation demands and these simulations can be numerical instable. Related experiments are standardly used for confirmation of boundary conditions of such simulations. Evaluation of the plastic strain evolution based on full strain field measurement on the loaded body is quite ordinarily based on deformation theory of the plasticity on other hand.

Three main reasons can be mentioned for deformation theory using. Firstly it is easy implementation; secondly it is enough to measure strain field in one required loading level only. Finally standard loading tests are commonly done using some simple external proportional loading (i.e. usually pure tensile, bending or torsion test) when validity of the deformation theory is supposed without any troubles. It will be shown in this paper that this presupposition can be no longer valid if high ductile material is used for the specimen with the crack although external loading is strictly monotonic and proportional.

2. Theory

Deformation (also named as total strain or power low) theory can be mathematically taken as extension of the linear elasticity theory. It implies that history of the loading does not play role in this case. Consequently unloading and/or non proportional loading is not allowed when yield criterion is reached if this theory is strictly applied. We will concentrate on the isotropic metal materials when external unloading is not supposed in the text below. External proportional and monotonic loading under plane stress state will be supposed in addition.

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Assuming deformation theory validity proportional loading should result in proportional ratio between main stress components in given place of the specimen. This condition is crucial when we are looking for the places where yield stress was reached. It is well known that yielding is driven by the stress deviator tensor defined as follows:

$$s_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij}, \qquad \textit{for general stress strain state.}$$
$$s_{ij} = \sigma_{ij} - \frac{\sigma_{11}+\sigma_{22}}{3}\delta_{ij}, \ \textit{for plane stress state.} \tag{1}$$

The stress deviator s_{ij} is obtained by subtracting the hydrostatic stress tensor from the stress tensor σ_{ij}. It is clear from this equation, that stress deviator can differ for two different loading states even if their stress tensor is equal. The two stress deviators with the same stress tensor differ in dependence on relation:

$$k=\sigma_{22}/\sigma_{11} \approx \varepsilon_{22}/\varepsilon_{11} \tag{2}$$

This relation has meaning of loading proportion. This proportion has to be constant if proportional loading is presented as for tensile smooth bar test for instance. The different loading proportion results in different stress deviator, i.e. different yielding condition. The limit possibility exists that actual strain state indicates zero plastic strain intensity although plastic strain was developed during loading before. Generally say, deformation theory leads into underestimation of plastic strain intensity when proportionality is lost. This behavior can be observed locally as well as globally for the specimen analyzed. It is reason why proportionality is required not only for boundary loading condition but also for specimen stress-strain field development.

Let's suppose that first yielding appears when von Mieses stress σ_y is reached, which is defined for plane stress as follows:

$$\sigma_y = (\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2)^{1/2} = \frac{E}{1-\nu^2}\left[(\varepsilon_1 + \nu\cdot\varepsilon_2)^2 + (\varepsilon_2 + \nu\cdot\varepsilon_1)^2 - (\varepsilon_1 + \nu\cdot\varepsilon_2)(\varepsilon_2 + \nu\cdot\varepsilon_1)\right]^{1/2}, \tag{3}$$

where Yielding stress (3) is invariant of the stress deviator (1). The Von Mieses stress differs 1.8 times for steel just by switching strain sign if absolute values of both strains are identical; $k=\pm 1$ in this case.

As all three strain tensor components are measured using X-ray digital image correlation (see Vavrik et. all, 2008) used by the author, the strain intensity can be directly calculated using well known equation:

$$\varepsilon_i = \frac{\sqrt{2}}{3}\sqrt{(\varepsilon_{11}-\varepsilon_{22})^2 + (\varepsilon_{22}-\varepsilon_{33})^2 + (\varepsilon_{33}-\varepsilon_{11})^2} \tag{4}$$

Relation between strain intensity and the plastic strain intensity defines deformation theory by the function of plasticity φ, which is calculated from the stress-strain record. Plastic and elastic strain components can be consequently calculated using next equations:

$$\varepsilon_i^{el} = \frac{\varepsilon_i}{(1+\varphi)}, \quad \varepsilon_i^{pl} = \varphi\varepsilon_i^{el} \tag{5}$$

The third strain component ε_{33} can be calculated iteratively if it is not measured, see Kulis&Reznicek 1995 for instance.

Implementation of the incremental Prager-Ziegler theory of plasticity used in this work is little bit more complicated. The increments of principal stresses $d\sigma_1, d\sigma_2$ can be calculated using measured strain component increments as was shown by Vavrik (2004):

$$\begin{aligned} d\sigma_1 &= \frac{E\left(1+\alpha E \cdot s_{22}^2\right)}{\left(1+\alpha E \cdot s_{11}^2\right)\left(1+\alpha E \cdot s_{22}^2\right)-\left(\nu-\alpha E \cdot s_{11}s_{22}\right)^2}\left[d\varepsilon_1+\left(\frac{\nu-\alpha E \cdot s_{11}s_{22}}{1+\alpha E \cdot s_{22}^2}\right)d\varepsilon_2\right] \\ d\sigma_2 &= \frac{E\left(1+\alpha E \cdot s_{11}^2\right)}{\left(1+\alpha E \cdot s_{22}^2\right)\left(1+\alpha E \cdot s_{11}^2\right)-\left(\nu-\alpha E \cdot s_{11}s_{22}\right)^2}\left[d\varepsilon_2+\left(\frac{\nu-\alpha E \cdot s_{11}s_{22}}{1+\alpha E \cdot s_{11}^2}\right)d\varepsilon_1\right] \end{aligned}, \quad (5)$$

While increments of the plastic strain can be expressed as:

$$\begin{aligned} d\varepsilon_{1p} &= \alpha\left(s_{11}^2 \cdot d\sigma_{11} + s_{11}s_{22} \cdot d\sigma_2\right) \\ d\varepsilon_{2p} &= \alpha\left(s_{22}^2 \cdot d\sigma_{22} + s_{11}s_{22} \cdot d\sigma_1\right), \quad \textit{for plastic strain components and} \\ d\varepsilon_{3p} &= -\left(d\varepsilon_{1p} + d\varepsilon_{2p}\right) \end{aligned} \quad (6)$$

$$d\varepsilon_{ip} = \sqrt{2/3} \cdot \sqrt{d\varepsilon_{1p}^2 + d\varepsilon_{2p}^2 + d\varepsilon_{3p}^2} \quad \textit{for plastic strain intensity.} \quad (7)$$

The α is function of hardening. Total plastic strain intensity is obtained by the summing of the plastic strain increments over whole loading history analyzed.

It will be shown in the next chapter that loading proportion k is significantly changing for intensive deformations surrounding crack tip especially if crack advances. Generally say, plastic strain intensity is similar for both theories in the crack plane, contrary outside of the crack plane this plastic strain intensity (and plastic strain zone area) is underestimated using deformation theory because local changing of the loading proportion. Applying of deformation theory for analyzing of the specimen during growing stable crack is one of the quite common mistakes. Using of the deformation theory missing plasticity around the crack faces during crack advance.

It is necessary to note, that incremental theory is much more sensitive for the experimental noise in comparison of the deformation theory. It is because all errors of the plastic strain measurement are transferred from the lower into successive loading levels using the incremental theory.

3. Experimental

The specimen with dimensions 5x50x170 mm was prepared from a high-ductile aluminum alloy. The initial 3 mm long precrack was prepared by fatigue loading on both sides of the pre-machined 10 mm long slit.

The specimen was loaded in uni-axial tension by grips displacement with velocity 0.4 µm/sec until initial cracks prolonged to several millimeters. Full strain field was evaluated using X-ray Digital Image Correlation (XDIC) technique, where natural material structure served as measuring grid. Strain-stress and consequent plastic intensity field were calculated in 113 loading levels until crack advance occurred, see Vavrik et all. 2008. Both plasticity theories were tested for this paper.

Loading proportionality depicted in the Fig. 1 was calculated in the form of $k=\varepsilon_1/\varepsilon_2$, because strains are measured variable while stresses are dependent variables. It is visible that strain-stress state is close to the pure tensile mode in vicinity of the crack tip vicinity at loading level 85. Zero or positive *k* value visible top right has meaning of the pure compressive loading state. Area with the pure tensile loading state is decreasing in the loading level 95 at middle. Crack was growing in the loading level 105 depicted right. Loading proportion is very different from the loading levels before.

Ductile fracture mechanics works with the concept of the strain constraint. It means that amount of the hydrostatic stress (see eqv. 3) influences plastic strain intensity in vicinity o the stress tip and consequently also actual fracture toughness. It is because energy consumed by the plastic strain evolution prior new crack development. This mentioned constrain can be characterized by the loading proportion at the crack vicinity.

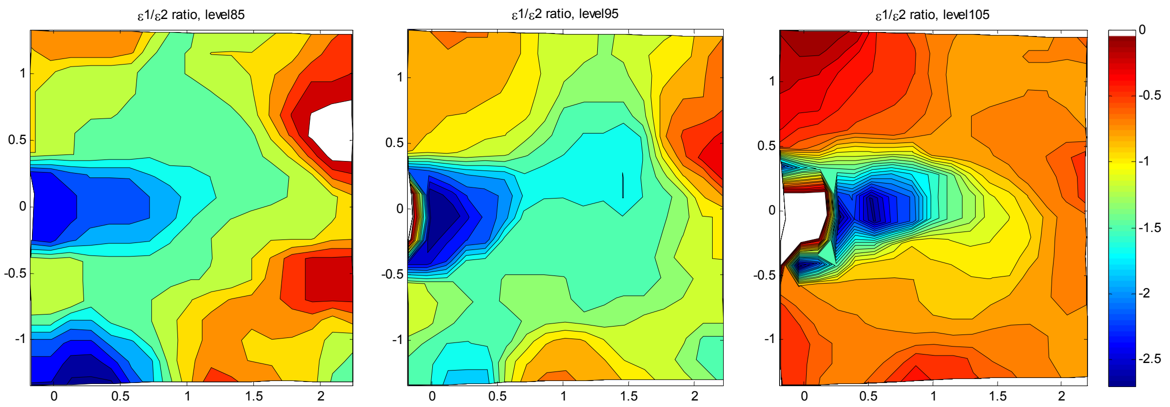

Fig.1: ratio $\varepsilon_1/\varepsilon_2$ characterizing level of the "hydrostatic stress" in the plane stress case. Zero or plus value means pure compressive state (first sub image right).

Plastic strain intensity was calculated using both theory of the plasticity. Results obtained using deformation theory for the same loading levels as in the Fig. 1 are depicted in the Fig. 2. Maximal value is at the crack tip as expected and it exceeds 20 %. Note that crack is not such blunted as it looks like at the last loading level imaged. It is because measuring grid is not so dense to precisely contour crack faces. Measuring grid points are plotted left.

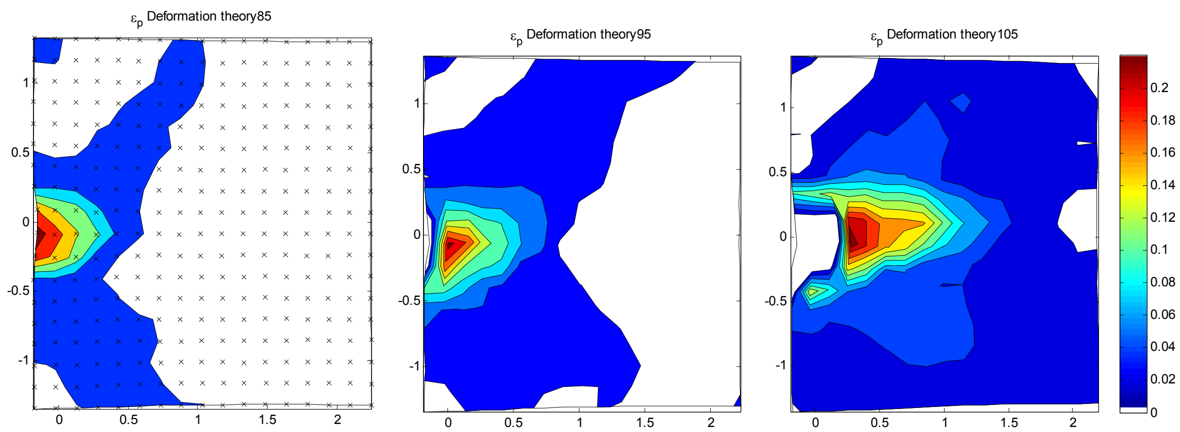

Fig. 2: Plastic strain intensity calculated using deformation theory at three successive loading levels.

Plasticity calculated using incremental theory is in the Fig. 3. Compare corresponding levels for both plastic strain calculations. It is clearly visible, that plastic strain zone range is underestimated using deformation theory as expected while maximal values at the crack tip are similar. Differences of the plastic strain intensity are most pronounced when crack advancing. Significant plasticity at the top left corner probably came from the noisy data.

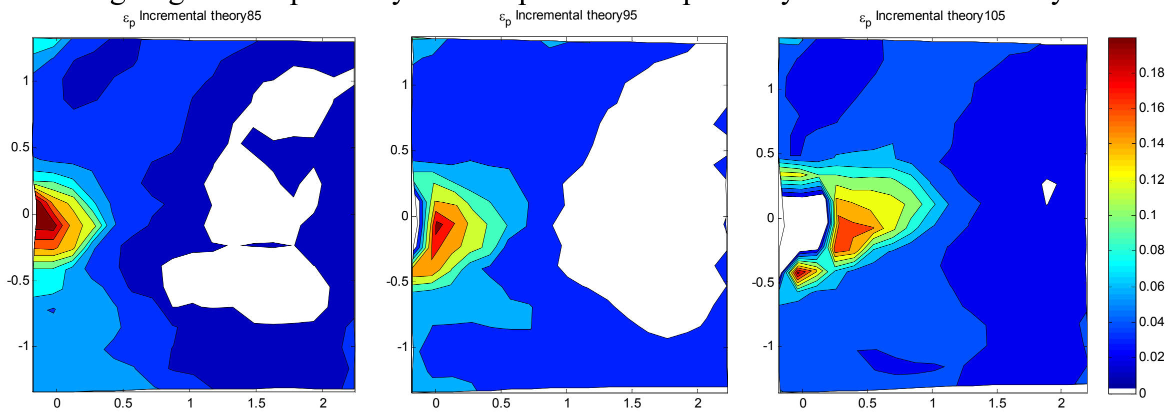

Fig. 3: Plastic strain intensity calculated using Prager-Ziegler incremental at three successive loading levels. Data scale as the same as for Fig. 2.

4. Conclusions

It was shown, that non proportional loading which is observed at the crack tip vicinity strongly influences plastic strain evolution.

This loading non proportionality is significant although external loading was strictly proportional (tensile loading with constant displacement velocity).

Deformation theory is applicable for ductile specimens with the crack only for validation of the plastic strain intensity at the crack tip.

5. Acknowledgement

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6. References

Vavřík, D.; Zemánková, J. (2004) Crack Instability in Ductile Materials Analyzed by the Method of Interpolated Ellipses, *Experimental Mechanics*, Vol. 44, pp. 327-335

Kuliš Z., Řezníček J. (1995) *Experimental Stress Analysis in the Range of Elastic-plastic Strains*. Sb.33, EAN '95, Třešť, s. 153

Vavřík, D. ; Jandejsek, I. ; Jakůbek, J. ; Jakůbek, M. ; Holý, T. (2008) Microradiographic Observation of the Strain Field in Vicinity of the Crack Tip. In 17th European Conference on Fracture. ISBN 978-80-214-3692-3, CD-ROM