International Conference
ENGINEERING MECHANICS 2010
Svratka, Czech Republic, May 10 – 13, 2010

LOGARITHMIC STRAIN IN 1 VERSUS 3(2) DIMENSIONS

Z. Fiala[1]

Summary: *The paper discusses the logarithmic strain in one dimension (1D) from the geometrical point of view to highlight the nature of problems when generalizing it to more dimensions (3D or 2D). Starting from geometry of positive real numbers* $\mathbb{R}^+$*, author advocates the geometrical approach via the Riemannian geometry of the space of symmetric positive-definite* $n \times n$ *matrices (*n *stands for dimension) of real numbers* $Sym^+(n,\mathbb{R}) \cong GL^+(n,\mathbb{R})/SO(n,\mathbb{R})$*, which reduces to* $\mathbb{R}^+$ *in the case of 1D. Based on previous papers, he demonstrates that only such an approach can guarantee consistent and well-defined manipulation with the logarithmic strain in more dimensions. Even though the geometry itself is rather unusual and nonintuitive due to its non-euclidean nature, its profit for the theory of finite deformations is noticeable and has already been demonstrated formerly – the natural and unambiguous linearization for an incremental approach within finite deformations, based on covariant derivative instead of on one of many objective time derivatives.*

1. Introduction

Even though the logarithmic (natural) strain was proposed by Ludwik (1909) for one dimensional extension of a rod, it is also named after Hencky (1928), who extended it to 3D in terms of logarithmic strains appropriate for the three principal directions. However, its use to studies wherein the principal axes of strain rotate in the body is difficult to grasp. Its main advantage in 1D (in 3D only in the previous non-rotating case) is additivity of progressive elongation increments (see (4)). Truesdell et al. (1960), p. 270, critically noted: “Such simplicity for certain problems as may result from a particular strain measure is bought at the cost of complexity for other problems. In a Euclidean space, distances are measured by a quadratic form, and attempt to elude this fact is unlikely to succeed.” Therefore, they advocate using the Cachy-Green or Almansi strain fields instead of the logarithmic one.

An attempt to construct the logarithmic strain from deformation tensors, which are such quadratic forms, was made in (Freed, 1995), though he paid little attention to the emerging problems due to the non-commutativity of matrix multiplication. Based on previous papers (Fiala, 2008, 2009), in which we relate the logarithmic strain to geometry of the space of symmetric positive-definite matrices of real numbers $Sym^+(n,\mathbb{R})$, actually representing deformation tensors, we shall tackle these problems from the mathematical point of view to stress the significance of the non-commutativity of matrix multiplication within the non-euclidean geometry of $Sym^+(n,\mathbb{R})$. Since now $Sym^+(1,\mathbb{R}) \equiv \mathbb{R}^+$, commutativity of multiplication of

[1] RNDr. Zdeněk Fiala, CSc., Institute of Theoretical and Applied Mechanics, Prosecká 76, 190 00 Prague 9, tel. +420 222 363 094, e-mail fiala@itam.cas.cz

real numbers enables to think of deformations in 1D in terms of the (euclidean) vector space $\mathbb{R}$. In $Sym^+(3,\mathbb{R})$, such a situation arises only along geodesic (generalized straight lines). The role of commutivity noticed already Fitzerald (1980), though he did not relate it to the specific geometry of $Sym^+(3,\mathbb{R})$.

2. Deformation process and strain measures

First we remind engineering definitions of strain for rod:

Simple strain

$$\epsilon_s := \frac{l-l_0}{l_0} = \frac{\delta l_0}{l_0} = \frac{\delta l(l_0)}{l_0}; \tag{1}$$

True strain (also natural, logarithmic, or effective specific) defined as the total of each incremental increase in length divided by the current length

$$\epsilon_{tr} := \int_{l_0}^{l} \frac{d\lambda}{\lambda} = \Big[\ln\lambda\Big]_{l_0}^{l} = \ln\frac{l}{l_0}\,, \tag{2}$$

and so

$$\epsilon_{tr} = \ln(1+\epsilon_s). \tag{3}$$

The intrinsic appeal of the true strain is precisely that it can be totalled as compared with the simple strain. That is, for two successive deformations $l_0 \overset{\delta l(l_0)}{\longrightarrow} l_1 \overset{\delta l(l_1)}{\longrightarrow} l_2$ we have

$$\epsilon_{tr}^{02} = \epsilon_{tr}^{01} + \epsilon_{tr}^{12}\,, \tag{4}$$

where

$$\epsilon_{tr}^{ij} = \ln\frac{l_j}{l_i}\,. \tag{5}$$

In generalizing the logarithmic strain to 3D, we shall find that this appealing feature in general no longer applies.

Before proceeding further, we remind definitions of strain within the framework of continuum mechanics: Globally, a deformation

$$x = \Phi(X,t), \qquad F = \frac{\partial\Phi}{\partial X} \tag{6}$$

is represented by a diffeomorphism $\Phi: \mathcal{B} \to \mathcal{E}^3$ (i.e. a one-to-one map, which is differentiable together with its inverse) and a deformation process by a time-dependent diffeomorphism $\Phi: I\times\mathcal{B} \to \mathcal{E}^3$. However, within continuum mechanics one adopts a local point of view to describe a deformation process in terms of a time-dependent deformation field, expressed by means of the deformation gradient F – a linearized diffeomorphism Φ, and its transpose F^{T}. We shall consider here the *right Cauchy-Green deformation field* $C = F^{\mathrm{T}}F$, represented by a symmetric, positive-definite matrix Sym^+, resp $\mathbb{R}^+$ in 1D.

$$C(X) = F^{\mathrm{T}}F \tag{7}$$

and the *rate-of-deformation tensor*

$$d(x) = \frac{1}{2}\,(L + L^{\mathrm{T}}) \tag{8}$$

where

$$L(x) = \dot{F}F^{-1}. \tag{9}$$

Now,

$$\partial C_t(X) := \frac{\partial C}{\partial t} = F^{\mathrm{T}}\dot{F} + \dot{F}^{\mathrm{T}}F = F^{\mathrm{T}}(\dot{F}F^{-1} + F^{-\mathrm{T}}\dot{F}^{\mathrm{T}})F = 2F^{\mathrm{T}}d_tF \equiv 2\Phi_*(d_t) \tag{10}$$

is an image of $2d_t(x)$ in reference configuration, and so $\partial C_t(X)$ represents the deformation rate in terms of a tangent to the ***deformation process described as a curve*** $C_t(X)$ in the space of all deformation tensors from Sym^+, resp $\mathbb{R}^+$.

STARTING POINT: *A deformation process will be represented by a trajectory* $C : I \to Sym^+$ *in the set of all symmetric, positive-definite real matrices. If the initial configuration is unstrained, the initial condition* $C_0 = I$*, where* I *stands for the identity matrix.*

Since both C and I belong to the same tensor space, we can subtract I from C to find a relative deformation, a one more strain – the Green-St. Venant strain tensor $\frac{1}{2}(C - I)$.

For all the strain tensors in 1D we now obtain:

Green - St.Venant strain

$$E_G(X) = \frac{1}{2}(C(X) - I) = \frac{1}{2}\left(F^2(X) - 1\right) \approx F(X) - 1, \text{ for } F \text{ close to } 1. \tag{11}$$

Simple strain

$$E_s(X) = \frac{\delta\lambda}{\lambda} = \frac{F(X)\lambda - \lambda}{\lambda} = F(X) - 1, \tag{12}$$

where λ is a vector emanating from X. For homogeneously deformed rod (that is all the vicinities along the rod deform the same way, independent of X)

$$E_s = \frac{\delta l_0}{l_0} = \frac{l - l_0}{l_0} = \frac{Fl_0 - l_0}{l_0} = F - 1, \tag{13}$$

where l is the length of rod.

True strain

$$E_{tr}(X) = \int_{\lambda_0}^{\lambda_1} \frac{d\lambda}{\lambda} = \ln\frac{\lambda_1}{\lambda_0} = \tag{14}$$

$$= \ln\frac{F(X,t_1)\lambda_0}{\lambda_0} = \ln F(X,t_1) \approx 1 + F(X,t_1), \text{ for } F \text{ close to 1,} \tag{15}$$

where λ is length of a vector emanating from X. Note, that in this case the whole deformation process parameterized by $t \in \langle t_0, t_1\rangle$ is considered when integrating, even though the result mentions only the starting t_0 and terminal t_1 times – at the undeformed state resp. the deformed one. Again for homogeneously deformed rod

$$E_{tr} = \ln\frac{l_1}{l_0} = \ln\frac{F(t_1)l_0}{l_0} = \ln F(t_1) \approx 1 + F(t_1)\,. \tag{16}$$

Comment: Since $ds^2 - ds_0^2 := 2E_G\, dX^2$

$$2E_G = E_s(E_s + 2)\,. \tag{17}$$

In addition,

$$\begin{aligned} E_{tr} &= \ln(1 + E_s) \\ &= \tfrac{1}{2}\ln(1 + 2E_G) = \tfrac{1}{2}\ln C\,. \end{aligned} \tag{18}$$

3. Deformation process as a lagrangian system

In this section we show that properties of deformation tensors endow the configuration space $\mathbb{R}^+$ – the representative of the space of deformation tensors in 1D, with the structure of $\mathbb{H}^1$ – the hyperbolic 1D space to demonstrate its intimate connection with the logarithmic strain in the next section, and compare it with $Sym^+(3, \mathbb{R})$.

First, consider two successive deformations: $0 \xrightarrow{\Phi_{10}} 1 \xrightarrow{\Phi_{21}} 2$ (from the state labeled by 0 to that labeled by 2, through the intermadiate one labeled by 1), where

$$\Phi_{ij} : j\,(\simeq \Phi(t_j, \mathcal{B})) \to i\,(\simeq \Phi(t_i, \mathcal{B}))\,, \tag{19}$$

then

$$\Phi_{20} = \Phi_{21} \circ \Phi_{10} \quad \text{and} \quad F_{20} = F_{21} \cdot F_{10}\,. \tag{20}$$

Denoting by C_{ji} the deformation of body in the configuration j with respect to the configuration i, preceding formulas due to $C = F^{\mathrm{T}}F$ result in

$$C_{20} = F_{21}^{\mathrm{T}} \cdot C_{10} \cdot F_{21} \equiv \Phi_{21}^*(C_{10})\,, \tag{21}$$

introducing thus translation operation Φ^* by F on the space of deformation tensors in terms of*matrix multiplication (or simple multiplication in 1D)*:

$$\Phi^* \colon GL \times Sym^+ \to Sym^+ \tag{22}$$

$$\text{or} \quad \Phi^* \colon \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+ \quad \text{in 1D,} \tag{23}$$

such that

$$\Phi^* \colon (F, C) \mapsto F^{\mathrm{T}}CF, \tag{24}$$

and for which $\Phi_{20}^* = \Phi_{21}^* \circ \Phi_{10}^*$.

Second, one can introduce a Riemannian metric on $\mathbb{R}^+$, resp Sym^+. Let us start with 1D. The stress power

$$\begin{aligned}
\frac{\delta E_i}{\delta t} &:= \int_{\Phi(\mathcal{B})} (\sigma : d)\, dx = \int_0^l \left(F^{-1} J\sigma F^{-\mathrm{T}}\right)\left(F^{\mathrm{T}} dF\right) F\, dx = \\
&= \int_0^L P \tfrac{1}{2}\, \partial C\, dX = \int_0^L \langle P, \tfrac{1}{2}\, \partial C_t \rangle_{T_{C_t(X)}\mathbb{R}^+}\, dX = \langle\!\langle P, \tfrac{1}{2}\, \partial C_t \rangle\!\rangle_{T_{C_t}\mathbb{R}^+} && (25)\\
&\qquad = \int_0^l F^{-1}F^{-\mathrm{T}}\left(F^{\mathrm{T}} J\sigma F\right) F^{-1}F^{-\mathrm{T}}\left(F^{\mathrm{T}} dF\right) F\, dx = \\
&= \int_0^L C^{-1}KC^{-1}\left(\tfrac{1}{2}\, \partial C\right) dX = \int_0^L \Omega_{C_t(X)}\left(K, \tfrac{1}{2}\, \partial C_t\right) dX = \omega_{C_t}\left(K, \tfrac{1}{2}\, \partial C_t\right). && (26)
\end{aligned}$$

Now, $P := F^{-1}J\sigma F^{-\mathrm{T}}$ is the reference configuration's representative of *the Kirchhoff stress* $\tau = J\sigma$, known as *the 1st Piola-Kirchhoff stress*, and $K := F^{\mathrm{T}}\tau F$ – *the convective stress*, also the reference configuration's representative of the Kirchhoff stress. Since a deformation process in 1D can be represented by a *curve* in $\mathbb{R}^+$, we conclude from (25) that the 1st Piola-Kirchhoff stress is a ***covector***, and from (26) that the convective stress is a ***vector***. Thus, the expression in (26)

$$\Omega_C(H, D) = \frac{1}{C^2}\, HD \tag{27}$$

stands for an inner product of any two vectors D, H emanating from a common point C, and so defines the ***Riemannian metric*** on $\mathbb{R}^+ -$ the space of deformation tensors in 1D. This space, endowed with basic operation of simple multiplication and with the Riemannian metric (27) is known as (see Appendix A) the one-dimensional hyperbolic space $\mathbb{H}^1 := (\mathbb{R}^+, \cdot)$.

On the other hand, it has already been proved for more dimensions (see Fiala (2009))

$$\begin{aligned} \frac{\delta E_i}{\delta t} &:= \int_{\mathcal{S}} (\sigma : d)\, dv = \\ &= \int_{\mathcal{S}} \big(\sigma^{ij} d_{ij}\big)\, dv = \int_{\mathcal{B}} \langle \mathbf{P}, \tfrac{1}{2}\, \partial \mathbf{C}_t \rangle_{T_{\mathbf{C}_{t(X)}} Sym^+}\, dV = \big\langle\!\big\langle \mathbf{P}, \tfrac{1}{2}\, \partial \mathbf{C}_t \big\rangle\!\big\rangle_{T_{\mathbf{C}_t} Sym^+} \qquad (28) \\ &= \int_{\mathcal{S}} \big(\sigma^i_j d^j_i\big)\, dv = \int_{\mathcal{B}} \Omega_{\mathbf{C}_{t(X)}} \big(\mathbf{K}, \tfrac{1}{2}\, \partial \mathbf{C}_t\big)\, dV = \omega_{\mathbf{C}_t} \big(\mathbf{K}, \tfrac{1}{2}\, \partial \mathbf{C}_t\big). \qquad (29) \end{aligned}$$

From the viewpoint of $Sym^+(3)$, the deformation process is a *curve*, and so the deformation at a particular time, represented by the deformation tensor $\mathbf{C}_t$, is a *point*. The rate of deformation $\partial \mathbf{C}_t$ at $\mathbf{C}_t$ is thus a ***vector*** attached to this point $\mathbf{C}_t$, as well as the mixed convective stress tensor $\mathbf{K}$, corresponding to the *Kirchhoff stress* $K^\flat = \Phi_t^*(J\sigma^\flat)$. Thus, (29) specifies the ***Riemannian metric*** on $Sym^+(3, \mathbb{R})$

$$\Omega_{\mathbf{C}_t}(\mathbf{H}, \mathbf{D}) := \mathrm{tr}\,(\mathbf{C}_t^{-1} \mathbf{H} \mathbf{C}_t^{-1} \mathbf{D}) \ \big(\equiv B^i_j K^j_k B^k_l \partial C^l_i\big)\,, \qquad (30)$$

which makes it the *Riemannian (globally) symmetric space*. Just as in 1D, the *second Piola-Kirchhoff stress* tensor

$$\mathbf{P} = \mathbf{\Omega}_{\mathbf{C}_t} \mathbf{K} = \mathbf{C}_t^{-1} \mathbf{K} \mathbf{C}_t^{-1} \qquad (31)$$

corresponding to *the Kirchhoff stress* $P^\sharp = \Phi_t^*(J\sigma^\sharp)$ is a ***covector*** (i.e. covariant vector), also attached to the point $\mathbf{C}_t$, and

$$\begin{aligned} \langle \mathbf{P}, \tfrac{1}{2}\, \partial \mathbf{C}_t \rangle_{T_{\mathbf{C}_t} Sym^+} &:= \ \mathrm{tr}\,(\mathbf{P} \partial \mathbf{C}_t) \ \big(\equiv P^i_j \partial C^j_i\big) \qquad (32) \\ &= \ \mathrm{tr}\,(\mathbf{C}_t^{-1} \mathbf{K} \mathbf{C}_t^{-1} \partial \mathbf{C}_t) = \Omega_{\mathbf{C}_t}(\mathbf{K}, \partial \mathbf{C}_t)\,. \qquad (33) \end{aligned}$$

We have thus arrived at formulation of the deformation process in terms of a ***lagrangian system*** (for example: Holm (2008), Marsden et al. (2003), Arnold (1989)), and so we can employ the methods of analytical mechanics:

CONCLUSION: *Deformation process can be described by a curve in the configuration space* $\mathbb{H}^1 = (\mathbb{R}^+, \cdot)$, *resp* $Sym^+(n, \mathbb{R})$, *with a "kinetic energy" (strain energy actually - the terminology here adheres to that of analytical mechanics) given by the Riemannian metric (27), resp (30). Stresses are then represented by vector field (the convected stresses), resp covector field (the 1st Piola-Kirchhoff stress) along this curve, related to each other by this Riemannian metric (31).*

Now, since the rate of change of any vector/covector field along a curve is necessarily expressed in terms of covariant derivative, we thus arrive at the Zaremba-Jaumann time derivative (Fiala, 2008). However, in 1D it reduces to simple time derivative and this fact will be related to the geometry of $\mathbb{H}^1$, which is much simpler in contrast to that of Sym^+.

4. Logarithmic strain and the geometry of $\mathbb{H}^1$ vs $Sym^+(n, \mathbb{R})$

Now supposing $C \in \mathbb{H}^1$, we are ready to analyse the relation (18) : $E_{tr} = \frac{1}{2} \ln C$ from the point of view of the geometry of the one-dimensional hyperbolic space $\mathbb{H}^1$. Without trouble we can start with $\mathrm{Log}_{C_0}(.)$ – see (61)

$$\mathrm{Log}_{C_0} \colon \mathbb{R}^+ \to \mathbb{R}, \qquad (34)$$

generalizing (18). This mapping isometrically transforms geometry of $\mathbb{H}^1$ into euclidean geometry of vector space $(\mathbb{R},+)$ (see Appendix A). Since multiplication of real numbers is *commutative*, it transforms composition of successive deformations (21) from multiplication

$$C_{20} = F_{21}^{\mathrm{T}} \cdot C_{10} \cdot F_{21} = F_{21}^{\mathrm{T}} F_{21} \cdot C_{10} = C_{21} \cdot C_{10} = C_{10} \cdot C_{21}\,, \tag{35}$$

into summation (cf. (4))

$$E_{tr\,20} = E_{tr\,21} + E_{tr\,10}\,. \tag{36}$$

Isometry means that these spaces are geometrically indistinguishable, and since within analytical mechanics the geometry determines the mechanics via the virtual power (i.e. Riemannian metrics), both spaces represent the same mechanics, and we can thus thought of the transformation (34) simply as a change of coordinate system. Moreover, due to its euclidean nature the Zaremba-Jaumann time derivative in $(\mathbb{R},+)$ equals to the simple time derivative, just as in $\mathbb{H}^1$ (see Appendix B), and the theory of finite deformations in this representations looks precisely the same as that of small deformations (simple time derivative, superposition of successive deformations).

In more dimensions the situation is totally different. The transformation

$$\mathrm{Log}_{\mathbf{C}_0}\colon\ Sym^+(n,\mathbb{R}) \to sym(n,\mathbb{R}) \tag{37}$$

between the space of symmetric positive-definite matrices with Riemannian metric (30), and the (euclidean) vector space of symmetric matrices $sym(n,\mathbb{R})$ with euclidean metric is no longer isometry, even though it is one-to-one and differentiable. The Zaremba-Jaumann time derivative then cannot be reduced to the simple time derivative and we have to resort to the logarithmic time derivative when using logarithmic strain (see (Fiala, 2009)). Moreover, since matrix multiplication is *not commutative*, successive deformations cannot be simplified as in (35) and transformed into summation (36). In this respect, Fitzerald (1980) speaks about a non-Abelian group under addition in the space of tensorial logarithmic strain measures, though without being more specific. Next paragraph suggests that $Sym^+(n,\mathbb{R})$ should rather be the (globaly) symmetric space, in particular a left coset space $GL^+(n,\mathbb{R})/SO(n,\mathbb{R})$ with the Riemannian metric (30) (see (Fiala, 2009)).

In fact, for *commuting* matrices, i.e. $\mathbf{AB} = \mathbf{BA}$, and only for them applies

$$\log \mathbf{AB} = \log \mathbf{A} + \log \mathbf{B}, \ \text{ resp} \tag{38}$$

$$\exp \mathbf{A} \cdot \exp \mathbf{B} = \exp(\mathbf{A} + \mathbf{B}). \tag{39}$$

Commuting matrices are characterized by the same principal directions, and so are all of the form $\mathbf{A} = \mathbf{O}^{-1}\mathbf{\Lambda}\mathbf{O}$, $\mathbf{B} = \mathbf{O}^{-1}\mathbf{\Gamma}\mathbf{O}$, where both $\mathbf{\Lambda}$, $\mathbf{\Gamma}$ are diagonal and $\mathbf{O}$ is the same orthogonal matrix. Now, we can select one-parameter subsets $\mathbf{C}_t$ of $Sym^+(n,\mathbb{R})$, for which (35) still holds true:

$$\mathrm{Exp}_{\mathbf{C}_0}\colon\ sym(n,\mathbb{R}) \times \mathbb{R} \to Sym^+(n,\mathbb{R}) \tag{40}$$

$$\mathrm{Exp}_{\mathbf{C}_0}\colon\ (\mathbf{H},t) \ \mapsto\ \mathbf{C}_t \equiv \mathrm{Exp}_{\mathbf{C}_0}(\mathbf{H}t) := \mathbf{C}_0 \exp(\mathbf{C}_0^{-1}\mathbf{H}t). \tag{41}$$

Then for

$$\mathrm{Log}_{\mathbf{C}_0} := \mathbf{C}_0 \log(\mathbf{C}_0^{-1}\mathbf{C})\colon\ Sym^+(n,\mathbb{R}) \to sym(n,\mathbb{R}) \tag{42}$$

on an one-parameter set $\mathbf{C}_t \equiv \mathrm{Exp}_{\mathbf{C}_0}(\mathbf{H}t)$ we have due to (39)

$$\mathrm{Log}_{\mathbf{C}_0}(\mathbf{C}_t) = \mathbf{H}t, \text{ and so} \tag{43}$$

$$\mathrm{Log}_{\mathbf{C}_0}(\mathbf{C}_{t+\tau}) = \mathrm{Log}_{\mathbf{C}_0}(\mathbf{C}_t) + \mathrm{Log}_{\mathbf{C}_0}(\mathbf{C}_\tau) \tag{44}$$

and (cf.relation (35))

$$\mathbf{C}_{t+\tau} = \mathbf{C}_t \cdot \mathbf{C}_\tau = \mathbf{C}_\tau \cdot \mathbf{C}_t \,. \tag{45}$$

These one-parameter subsets indexed by symmetric matrix $\mathbf{H} \in sym$ are in fact geodesics in $Sym^+(n, \mathbb{R})$, i.e. generalized straight lines starting from point $\mathbf{C}_0$ in direction of vector $\mathbf{H}$.

Thus, *generalizing logarithmic strain from one to more dimensions we come across the geometry of $Sym^+(n, \mathbb{R}) \cong GL^+(n, \mathbb{R})/SO(n, \mathbb{R})$ **implicitly** via geodesics. It is quite natural to make use of the geometry **explicitly** when considering general deformation processes*, and not only those made up of geodesics (see (Fiala, 2009)), along which the rate of change of tensor fields again reduces to simple time derivative. In the case of 1D, $Sym^+(1, \mathbb{R})$ reduces to $\mathbb{H}^1$. For 2D $Sym^+(2, \mathbb{R}) = GL^+(2, \mathbb{R})/SO(2, \mathbb{R}) = \mathbb{R}^+ \times SL(2, \mathbb{R})/SO(2, \mathbb{R}) \equiv \mathbb{H}^1 \times \mathbb{H}^2$, where $\mathbb{H}^2$ is the Lobachevsky geometry in its Poincaré representation in terms of the upper half-plane model, and for 3D such equivalence no longer applies and $Sym^+(3, \mathbb{R}) \neq \mathbb{H}^1 \times \mathbb{H}^3$. Though both $\mathbb{H}^n$ and $Sym^+(n, \mathbb{R})$ are the spaces of constant negative curvature, $Sym^+(n, \mathbb{R})$ for $n \geq 3$ is quite different (Bridson et al., 1999).

Second, in this respect, integration of the deformation-rate field along deformation curve deserves also some comment, namely

$$\int_{t_0}^{t} \mathbf{D}_\tau \, d\tau \,, \tag{46}$$

where $\mathbf{D}_\tau := \Phi^*(\mathbf{d}_\tau) = \mathbf{F}^{-1}\mathbf{d}_\tau\mathbf{F} = \frac{1}{2}\mathbf{B}_\tau\partial\mathbf{C}_\tau$. The anticipated result, announced in (Freed, 1995)

$$\int_{t_0}^{t} \mathbf{D}_\tau \, d\tau = \frac{1}{2} \int_{t_0}^{t} \mathbf{B}\partial\mathbf{C} = \frac{1}{2} \log(\mathbf{B}_{t_0}\mathbf{C}_{\mathbf{t}}) \,, \tag{47}$$

is not definitely correct in general as in 1D, but can be made plausible in more dimensions only for geodesics. In fact, along geodesic $2\mathbf{D} = \mathbf{B}_\tau\partial\mathbf{C}_\tau = (\log \mathbf{B}_{t_0}\mathbf{C}_\tau)/(\tau - t_0)$ (cf. (41)) is constant matrix, and so we can naturally set

$$\int_{t_0}^{t} \mathbf{D}_\tau \, d\tau = \mathbf{D} \int_{t_0}^{t} d\tau = \mathbf{D}(t - t_0) = \frac{1}{2} \log(\mathbf{B}_{t_0}\mathbf{C}_t). \tag{48}$$

However, since a general deformation process can be approximated by piecewise connected geodesics, so can be approximated the integral (48) by a sum, and we get

$$\int_{t_0}^{t} 2\mathbf{D}_\tau \, d\tau \approx \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} 2\mathbf{D}_\tau \, d\tau \;=\; \log(\mathbf{B}_{t_0}\mathbf{C}_{t_1}) + \log(\mathbf{B}_{t_1}\mathbf{C}_{t_2}) + \; ... \; + \log(\mathbf{B}_{t_{n-1}}\mathbf{C}_{t_n})$$
$$\neq \; \log(\mathbf{B}_{t_0}\mathbf{C}_t). \tag{49}$$

This inequality is again due to the *non-commutativity* of matrix multiplication.

Third, non-commutativity also accounts for the failure of the return to the starting point of two successive deformations followed by their inversions, i.e. the failure of the closure of the whole deformation process, in the case of finite deformations contrary to small ones. To be more specific, consider two deformations starting from the undeformed state (i.e. at the identity matrix $\mathbf{I}$) with deformation rates $\mathbf{D_I}, \mathbf{H_I}$, from which we shall construct two *constant* vector fields $\mathbf{D}, \mathbf{H}$ by means of translation ρ on $Sym^+(n, \mathbb{R})$ (see Fiala (2009)), so that for any $\mathbf{C} \in Sym^+(3, \mathbb{R})$ a vector value $\mathbf{D_C}$ of the vector field $\mathbf{D}$ at $\mathbf{C}$ reads

$$\begin{aligned} \mathbf{D_C} &= \rho_*(\mathbf{C}^{1/2})\,(\mathbf{D}) \equiv \mathbf{C}^{\frac{1}{2}}\mathbf{D}\mathbf{C}^{\frac{1}{2}}, \text{ resp.} \\ \mathbf{H_C} &= \rho_*(\mathbf{C}^{1/2})\,(\mathbf{H}) \equiv \mathbf{C}^{\frac{1}{2}}\mathbf{H}\mathbf{C}^{\frac{1}{2}} \end{aligned} \tag{50}$$

Now, the whole process takes the form

$$\mathbf{C}(t) = \text{Exp}_{\mathbf{C}_3}(-\sqrt{t}\,\mathbf{H}_{\mathbf{C}_3}) \circ \text{Exp}_{\mathbf{C}_2}(-\sqrt{t}\,\mathbf{D}_{\mathbf{C}_2}) \circ \text{Exp}_{\mathbf{C}_1}(\sqrt{t}\,\mathbf{H}_{\mathbf{C}_1}) \circ \text{Exp}_{\mathbf{I}}(\sqrt{t}\,\mathbf{D_I}), \tag{51}$$

where $\mathbf{C}_1 = \text{Exp}_{\mathbf{I}}(\sqrt{t}\,\mathbf{D_I})$, $\mathbf{C}_2 = \text{Exp}_{\mathbf{C}_1}(\sqrt{t}\,\mathbf{H}_{\mathbf{C}_1})$ etc. and the sign '-' indicates an inverse deformation. Since $\mathbf{C}(t) \neq \mathbf{I}$, the curve $\mathbf{C}(t)$ has its tangent, (and since these geodesics are actually flows of these constant vector fields, i.e. solutions of equation $\dot{\mathbf{C}}_t = \mathbf{D}_{\mathbf{C}_t}$) we obtain (see for example Frankel (1997))

$$\mathbf{C}(t) \approx t\,(\mathbf{D_I}\mathbf{H_I} - \mathbf{H_I}\mathbf{D_I}) \equiv t\,[\mathbf{D_I}, \mathbf{H_I}] \neq \mathbf{0}. \tag{52}$$

For small deformations, the configuration space $sym(3, \mathbb{R})$ is a vector space, in which geodesics are ordinary straight lines, and so the constructed parallelepipeds (51) are closed.

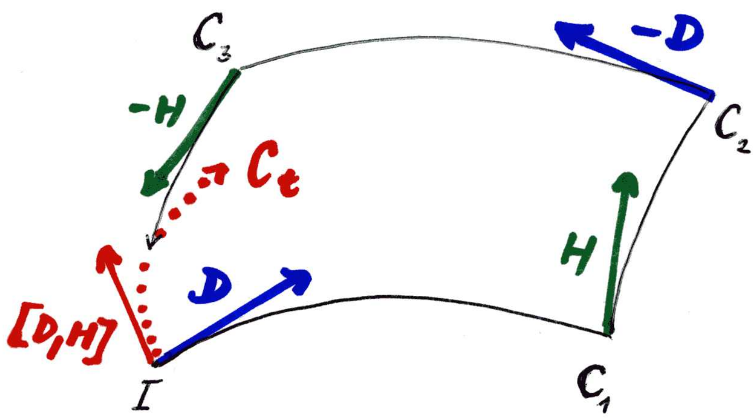

Figure 1: Failure to close two successive deformations followed by their inversions

5. Conclusion

The previous paper (Fiala, 2009) concluded with an observation that the geometrical approach to deformation via geometry of Sym^+ looks remarkably compact and self-consistent, and provides a natural way to linearization of deformation process and to the incremental approach within finite deformations (Fiala, 2008). Now, I would like to stress another aspect of trying to properly master the underlying geometry of the configuration space $Sym^+(3, \mathbb{R})$, namely, the numerical one. Let us conclude with a quotation (Iserles et al., 2000), which pretty well justifies further study in this direction: "An important reason why a manifold, rather than the entire $\mathbb{R}^n$, is a suitable configuration space is that it often expresses crucial geometric attributes of the underlying differential system, e.g. conservation laws, symmetries or symplectic structure. An added bonus of this approach is that it frequently leads to interesting numerical advantages, in particular to slower error accumulation."

Appendix A: Geometry of the one-dimensional hyperbolic space $\mathbb{H}^1$

For more about hyperbolic spaces see (Cannon et al., 1997) - in particular model H, which in 1D matches up precisely with our setting.

Rewriting (27) as $ds^2 = g_{ij}\,dx^i dx^j$ we identify one and only one component of the metric tensor:

$$g_{11} = x^{-2}, \tag{53}$$

and from the relation $g_{ij}\,g^{jk} = \delta_i^k$ its contravariant counterpart

$$g^{11} = x^2. \tag{54}$$

Based on relation $\partial_1 g_{11} = -2x^{-3}$, for the only one component of the Christoffel symbols $\Gamma^i_{jk} = \frac{1}{2}\,g^{il}\,(\partial_k g_{lj} + \partial_j g_{lk} - \partial_l g_{jk})$ we get

$$\Gamma^1_{11} = -x^{-1}, \tag{55}$$

so that the covariant derivative of any vector field $w = w^1\partial_1$ with respect to a vector $u = u^1\partial_1$ reads

$$\nabla_u w = u^i(\partial_i w^k + w^j\Gamma^k_{ji})\partial_k = u^1(\partial_1 w^1 - w^1 x^{-1})\partial_1. \tag{56}$$

Now, a geodesic curve, which is a generalization of a straight line in non-euclidean geometries, is a curve $c(t) \in \mathbb{R}^+$ that parallel-transports its own tangent vector $\partial_t c$, and so the geodesic equation reads

$$\nabla_{\partial_t c}\partial_t c = 0. \tag{57}$$

Considering (56), we obtain

$$\nabla_{\partial_t c}\partial_t c \equiv \partial_t c\left(\partial_t^2 c - \partial_t c\,\frac{1}{c}\right) = \partial_t c \cdot c \cdot \partial_t\left(c^{-1}\partial_t c\right) = 0, \tag{58}$$

which, since $c(t) > 0$, is zero iff $c^{-1}\partial_t c = const$. For initial conditions $c(0) = c_0$, $\partial_t c(0) = \partial_0 c$ we get the solution

$$c(t) = c_0 \exp(t \cdot c_0^{-1}\partial_0 c) \equiv \mathrm{Exp}_{c_0}(t \cdot \partial_0 c)\,. \tag{59}$$

On the other hand, if the geodesic is determined by two points $c(0) = c_0$ and $c(1) = c_1$, then the geodesic reads

$$c(t) = \mathrm{Exp}_{c_0}(t \cdot h(c_1))\,, \tag{60}$$

where $h(c_1)$ stands for vector

$$h(c_1) = c_0 \log\left(c_0^{-1}c_1\right) \equiv \mathrm{Log}_{\,c_0}(c_1)\,. \tag{61}$$

Denoting by $T_{c_0}\mathbb{R}^+(\simeq \mathbb{R})$ a vector space of all vectors emanating from the common point c_0, we interpret the mapping

$$\mathrm{Log}_{\,c_0}\colon\ \mathbb{R}^+ \to \mathbb{R} \tag{62}$$

as that assigning to all points from $\mathbb{R}^+$ the corresponding vectors in $\mathbb{R}$. Moreover, this mapping is an ***isometry*** between $\mathbb{H}^1$ with Riemannian metric (27) and the (euclidean) vector space $(\mathbb{R}, +)$ with euclidean metric $ds^2 := dx^2$. Isometry means that these spaces are geometrically indistinguishable, and since in analytical mechanics the geometry governs the mechanics itself via virtual power (the Riemannian metrics), the corresponding mechanics are also the same.

Even though hyperbolic spaces $\mathbb{H}^n$ for $n > 1$ have constant negative curvature, the case of $n = 1$ is different. The Riemann curvature tensor $R^l_{kij} = \partial_i \Gamma^l_{\ kj} - \partial_j \Gamma^l_{\ ki} + \Gamma^m_{kj} \Gamma^l_{\ mi} - \Gamma^m_{ki} \Gamma^l_{\ mj}$ has now only one component

$$R^1_{111} = 0, \tag{63}$$

and so the curvature of $\mathbb{H}^1$ is zero. This is due to the fact that the operation of multiplication of real numbers in $\mathbb{R}^+$ is commutative, whereas the corresponding matrix multiplication for the case of $n > 1$ not (cf. (24)). This fact also explains why the logarithmic strain in more dimensions behaves more complex than in 1D, and why (and in particular how) the non-euclidean geometry enters the theory of finite deformations.

Appendix B: the Zaremba-Jaumann time derivative in $\mathbb{H}^1$

Note the transformation between vectors (in the sense of $\mathbb{H}^1$) $U := U^1_1$ in reference configuration and $u := u^1_1$ in actual configuration

$$U \longrightarrow u = \Phi^*(U) = F^{-\mathrm{T}} U F^{-1} \tag{64}$$
$$u \longrightarrow U = \Phi_*(u) = F^{\mathrm{T}} u F \tag{65}$$

and covector (again in the sense of $\mathbb{H}^1$) $\Omega := \Omega^1_1$ in reference configuration and $\omega := \omega^1_1$ in actual configuration

$$\Omega \longrightarrow \omega = \Phi^*(\Omega) = F \Omega F^{\mathrm{T}} \tag{66}$$
$$\omega \longrightarrow \Omega = \Phi_*(\omega) = F^{-1} \omega F^{-\mathrm{T}}. \tag{67}$$

Since (commutativity counts in)

$$\Phi^*(\partial U) = \Phi^*[\partial(\Phi_* u)] = F^{-\mathrm{T}}[\partial(F^{\mathrm{T}} u F)] F^{-1} = \dot{u} + L^{\mathrm{T}} u + uL = \dot{u} + 2du \tag{68}$$
$$\Phi^*(\partial \Omega) = \Phi^*[(\partial\, \Phi_* \omega)] = F[\partial(F^{-1} \Omega F^{-\mathrm{T}}] F^{\mathrm{T}} = \dot{\omega} - L\omega - \omega L^{\mathrm{T}} = \dot{\omega} - 2d\omega, \tag{69}$$

for the rates – the Zaremba-Jaumann derivatives (again commutativity counts in) we get

$$\frac{D}{dt} U := \partial U + \nabla_{\partial C_t} U = \partial U - (\partial C_t)\, C_t^{-1} U \tag{70}$$
$$\mathring{u}^{\mathrm{ZJ}} := \Phi^* \left(\frac{D}{dt} U \right) = \dot{u} + 2du - 2du = \dot{u} \tag{71}$$

and

$$\frac{D}{dt} \Omega := \partial \Omega + \nabla_{\partial C_t} \Omega = \partial \Omega + (\partial C_t)\, C_t^{-1} \Omega \tag{72}$$
$$\mathring{\omega}^{\mathrm{ZJ}} := \Phi^* \left(\frac{D}{dt} \Omega \right) = \dot{\omega} - 2d\omega + 2d\omega = \dot{\omega}. \tag{73}$$

We have thus proved that the Zaremba-Jaumann time derivative in 1D reduces to the simple time derivative.

6. Acknowledgment

The support of grant GA ČR 103/09/2101, as well as the project AV 0Z 2071913 is gratefully acknowledged.

7. References

Arnold, V. I. 1989: *Mathematical Methods of Classical Mechanics*, Springer-Verlag.

Bridson, M. R. & Haefliger, A. 1999: *Metric spaces of non-positive curvature*, Springer-Verlag.

Cannon, J. W., Floyd, W. J., Kenyon, R. & Parry, W. R. 1997: Hyperbolic Geometry, in: *Flavors of geometry* (S. Levy ed) MSRI Publications 31, 59–115, Cambridge University Press; http://www.msri.org/communications/books/Book31/contents.html

Fiala, Z. 2008: Geometry of finite deformations, linearization, and incremental deformations under initial stress/strain. Engineering Mechanics 2008, eds: V. Fuis et al., Svratka, 20pp.

Fiala, Z. 2009: Is the logarithmic time derivative simply the Zaremba-Jaumann derivative? Engineering Mechanics 2009, eds: J. Náprstek et al., Svratka, 14pp.

Fitzerald J. E. 1980: A tensorial Hencky measure of strain and strain rate for finite deformations. *J. Appl. Phys.*, vol. 51, 5111–5115.

Frankel T. 1997: *The geometry of physics. An introduction*. Cambridge University Press.

Freed A. D. 1995: Natural strain. *J. Eng. Mater. Technol.*, vol. 117, 379–385.

Hencky H. 1928: Über die Form des Elasticitätsgesetzes bei ideal elastischen Stoffen. *Zeitschrift für technische Physik*, vol. 9, 215–220.

Holm, D. D. 2008: *Geometric mechanics. Part I, Dynamics and symmetry*, Imperial College Press, London.

Iserles, A., Munthe-Kaas, H. Z., Nørsett, S. P. & Zanna, A. (2000): Lie-group methods. *Acta Numerica*, vol. 9, 215–365.

Ludwik, P. 1909: *Elemente der Technologischen Mechanik*, Springer-Verlag, Berlin.

Marsden J. E. & Ratiu T. 2003: *Introduction to Mechanics and Symmetry*, Texts in Applied Mathematics vol. 17, Springer-Verlag.

Truesdell C. & Tupin, R. 1960: *The Classical Fields Theories*, vol.III/1 of Encyclopedia of Physics, eds: S. Flügge, Springer-Verlag, Berlin.