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SOME PROPERTIES OF NON-LINEAR RESONANCE OF THE PENDULUM DAMPER

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Summary: *The pendulum damper modelled as a two degree of freedom strongly non-linear auto-parametric system is investigated using an approximate differential system. Uni-directional harmonic external excitation at the suspension point is considered. The resonance phenomenon cannot be described using the semi-trivial solution. In this contribution, nature of the numerical solution in the state of resonance is thoroughly investigated as a preliminary work for the detailed analytical study. Stationary and non-stationary (quasiperiodic) character of the resonance solution is determined and characterised. General form of a solution is proposed.*

1. Introduction

Many structures encountered in the civil and mechanical engineering are equipped with various devices for reducing dynamic response component due to external excitations. Among other low cost passive systems the pendulum dampers are still very popular for their reliability and simple maintenance, see e.g. (Haxton & Barr, 1972). However the dynamic behaviour of such a pendulum is significantly more complex than it is supposed by a widely used simple linear SDOF model working in the (xz) vertical plane only, see Figure 1. The conventional linear model is satisfactory only if the kinematic excitation $a(t)$ introduced at the suspension point is very small in amplitude and if its frequency remains outside a resonance frequency domain.

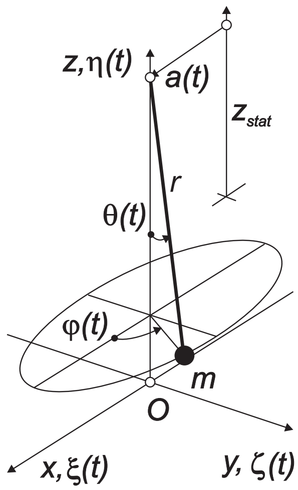

Figure 1: The pendulum and coordinate systems.

The authors have presented several papers on the topic, see Náprstek & Fischer (2007-2009) and Fischer & Náprstek (2009). However, the detailed description of the behaviour in the state of the strong resonance is not available yet. In this state the system exhibits quasi-periodic or other non-stationary behaviour. Amplitudes of the longitudinal and transversal motions are comparable. The both amplitudes do not converge to any constants and remain quasi-periodical with fluctuating internal structure within individual periods. Behaviour of the amplitudes of both components of the motion is studied in the presented paper.

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2. System setup and its basic properties

Using the mechanical energy balance conditions and the Lagrangian principle one can derive the differential system describing the model in the Figure 1:

$$\begin{aligned} \ddot{\xi} + \frac{1}{2r^2}\xi\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\xi^2+\zeta^2) + 2\omega_b\dot{\xi} + \omega_0^2(\xi + \frac{1}{2r^2}\xi(\xi^2+\zeta^2)) &= -\ddot{a} \\ \ddot{\zeta} + \frac{1}{2r^2}\zeta\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\xi^2+\zeta^2) + 2\omega_b\dot{\zeta} + \omega_0^2(\zeta + \frac{1}{2r^2}\zeta(\xi^2+\zeta^2)) &= 0 \end{aligned} \quad (1)$$

where r - suspension length of the pendulum
ω_b - relative scale of the approximate linear damping equivalent in both components ξ, ζ
$\omega_0^2 = g/r$
$a = a(t)$ - kinematic excitation at the suspension point.

The semi-trivial solution is searched in the form of

$$\xi_0 = a_c \cos\omega t + a_s \sin\omega t \; ; \;\; \zeta_0 = 0 \quad (2)$$

and the excitation is assumed to be harmonic $a(t) = a_0 \sin\omega t$ (see Tondl (1991) for details). The coefficients a_c, a_s in general should be considered as functions of time: $a_c = a_c(t)$, $a_s = a_s(t)$. If a stationary solution exists for a given excitation frequency ω, then a_c, a_s should converge to constants for increasing $t \to \infty$. For this reason coefficients a_c, a_s can be considered constant only under special conditions when stable stationary response can be expected.

Substituting (2) into Eq. (1), multiplying it by $\sin(\omega t)$ or $\cos(\omega t)$ and integrating the resulting expressions over the interval $t \in (0, 2\pi/\omega)$ (so called *harmonic balance* operation) results in an algebraic system consisting of two equations. From them one can obtain the equation for the amplitude of the response ($R_0^2 = a_c^2 + a_s^2$):

$$R_0^2\left[4\omega^2\omega_b^2 + \left((\omega^2-\omega_0^2) + \frac{R_0^2}{2r^2}\left(\omega^2 - \frac{3}{4}\omega_0^2\right)\right)^2\right] - 4\omega^4 a_0^2 \;=\; 0 \quad (3)$$

The semi-trivial solution (2) can be endowed with small (in the meaning of a norm) perturbations u, v in both coordinates:

$$\begin{aligned} \xi &= \xi_0 + u \qquad & u = u(t) = u_c\cos\omega t + u_s \sin\omega t \\ \zeta &= 0 + v \qquad & v = v(t) = v_c\cos\omega t + v_s \sin\omega t \end{aligned} \quad (4)$$

As the perturbations are expected to be small, only the first powers of u, v and their derivatives are kept after inserting expressions (4) into Eq. (1). After the harmonic balance operation and some algebra one obtains two homogeneous linear algebraic systems for u_c, u_s and v_c, v_s. Their non-trivial solution exists only if it holds:

$$\frac{1}{2r^2}\Omega_1 R_0^2\left(\Omega_2 + \frac{3}{32r^2}\Omega_1 R_0^2\right) + \Omega_2^2 + 4\omega^2\omega_b^2 = 0 \quad (5)$$

$$\frac{1}{2r^2}R_0^2\left(\omega_0\Omega_2 + \frac{1}{32r^2}\Omega_1\Omega_3 R_0^2\right) + \Omega_2^2 + 4\omega^2\omega_b^2 = 0 \quad (6)$$

where

$$\Omega_1 = 3\omega_0^2 - 4\omega^2 \; ; \; \Omega_2 = \omega_0^2 - \omega^2 \; ; \; \Omega_3 = \omega_0^2 + 4\omega^2 \; ; \; \Omega_4 = \omega_0^2 - 4\omega^2$$

The Eqs (5-6) can be interpreted as limits dividing the plane (R_0^2, ω) into the stable and unstable domains. For given parameters r, ω_b, a_0 the unstable interval of excitation frequency is defined by the position of the intersections of the resonance curve (3) with the corresponding stability limits (5-6).

3. Response in the resonance domain

Let us try to assume a more general expressions as the basic solution:

$$\xi(t) = a_c(t)\cos\omega t + a_s(t)\sin\omega t\ ; \qquad \zeta(t) = b_c(t)\cos\omega t + b_s(t)\sin\omega t \tag{7}$$

Increasing the number of unknown functions to four, one can exploit a possibility to formulate two arbitrarily selectable additional conditions. Then the following expressions for the first derivatives of the general solution (7) can be stated:

$$\dot{\xi}(t) = -a_c\omega\sin\omega t + a_s\omega\cos\omega t\ ; \qquad \dot{\zeta}(t) = -b_c\omega\sin\omega t + b_s\omega\cos\omega t \tag{8}$$

where $a_c = a_c(t), a_s = a_s(t), b_c = b_c(t), b_s = b_s(t)$.

Using the expressions (7) and (8) in the differential system (1) and applying the operation of the harmonic balance, the differential system for amplitudes a_c, a_s, b_c, b_s arises:

$$\mathbf{A}\begin{pmatrix}\dot{a}_c\\ \dot{a}_s\\ \dot{b}_c\\ \dot{b}_s\end{pmatrix} = -\frac{1}{2}\begin{pmatrix} a_c(8\Omega_2 r^2 + R_A^2\Omega_1) + 2b_sS_A^2\Omega_4 + 4a_s\omega_b\omega r^2 \\ a_s(8\Omega_2 r^2 + R_A^2\Omega_1) + 2b_cS_A^2\Omega_4 + 4a_c\omega_b\omega r^2\ \ +8\omega^2 a_0 r^2 \\ b_c(8\Omega_2 r^2 + R_A^2\Omega_1) + 2a_sS_A^2\Omega_4 + 4b_s\omega_b\omega r^2 \\ b_s(8\Omega_2 r^2 + R_A^2\Omega_1) + 2a_cS_A^2\Omega_4 + 4b_c\omega_b\omega r^2 \end{pmatrix} \tag{9}$$

where it has been denoted

$$R_A^2 = a_c^2 + a_s^2 + b_c^2 + b_s^2\ ; \qquad S_A^2 = a_sb_c - a_cb_s \tag{10}$$

The system matrix $\mathbf{A}$ depends only on $a_c, a_s, b_c, b_s, \omega$ and has the form of

$$\mathbf{A} = \omega\begin{pmatrix} -2a_ca_s, & 4r^2 + 3a_c^2 + a_s^2, & -a_sb_c - a_cb_s, & 3a_cb_c + a_sb_s \\ -4r^2 - a_c^2 - 3a_s^2, & 2a_ca_s, & -a_cb_c - 3a_sb_s, & a_sb_c + a_cb_s \\ -a_sb_c - a_cb_s, & 3a_cb_c + a_sb_s, & -2b_cb_s, & 4r^2 + 3b_c^2 + b_s^2 \\ -a_cb_c - 3a_sb_s, & a_sb_c + a_cb_s, & -4r^2 - b_c^2 - 3b_s^2, & 2b_cb_s \end{pmatrix} \tag{11}$$

The explicit solution of Eqs (9) is generally not possible in the resonance interval. However, it can be seen from the numerical analysis, that at least part of the resonance interval (interval of the non-trivial transversal response) can be described by a steady state (stationary) solution.

For the case, where the stationary solution occurs, one can neglect the left hand side of the Eq. (9). Indeed, if the response is stationary, the amplitudes a_c, a_s, b_c, b_s are constant and thus their derivatives vanish. The Eq. (9) reduces itself into the algebraic system, which gives the relation between R_A^2, S_A^2 and ω

$$\begin{aligned} &R_A^2\left((8\Omega_2 r^2 + R_A^2\Omega_1)^2 + 4\left(4\omega^2\omega_b^2 r^4 + S_A^4\Omega_4^2\right)\right) - 8S_A^4\left(8\Omega_2 r^2 + R_A^2\Omega_1\right)\Omega_4 = 64r^4a_0^2\omega^4 \\ &S_A^2\left(2R_A^2\left(8\Omega_2 r^2 + R_A^2\Omega_1\right)\Omega_4 - (8\Omega_2 r^2 + R_A^2\Omega_1)^2 - 16\omega^2\omega_b^2 r^4 - 4S_A^4\Omega_4^2\right) = 0 \end{aligned} \tag{12}$$

The case, where the solution is non-stationary, is much more interesting. However, currently only the numerical analysis is available.

Figure 2 shows the results of direct simulation of the Eq. (1) for several excitation frequencies ω from the resonance interval. Four cases in the Figure 2 correspond to the non-stationary response: $\omega = 3.0$ to $\omega = 3.15$. In this cases a periodic exchange of energy between components occurs. The length of a such period depends on the excitation frequency. This phenomenon will be studied in detail later. The case where $\omega = 3.2$ corresponds to the stationary

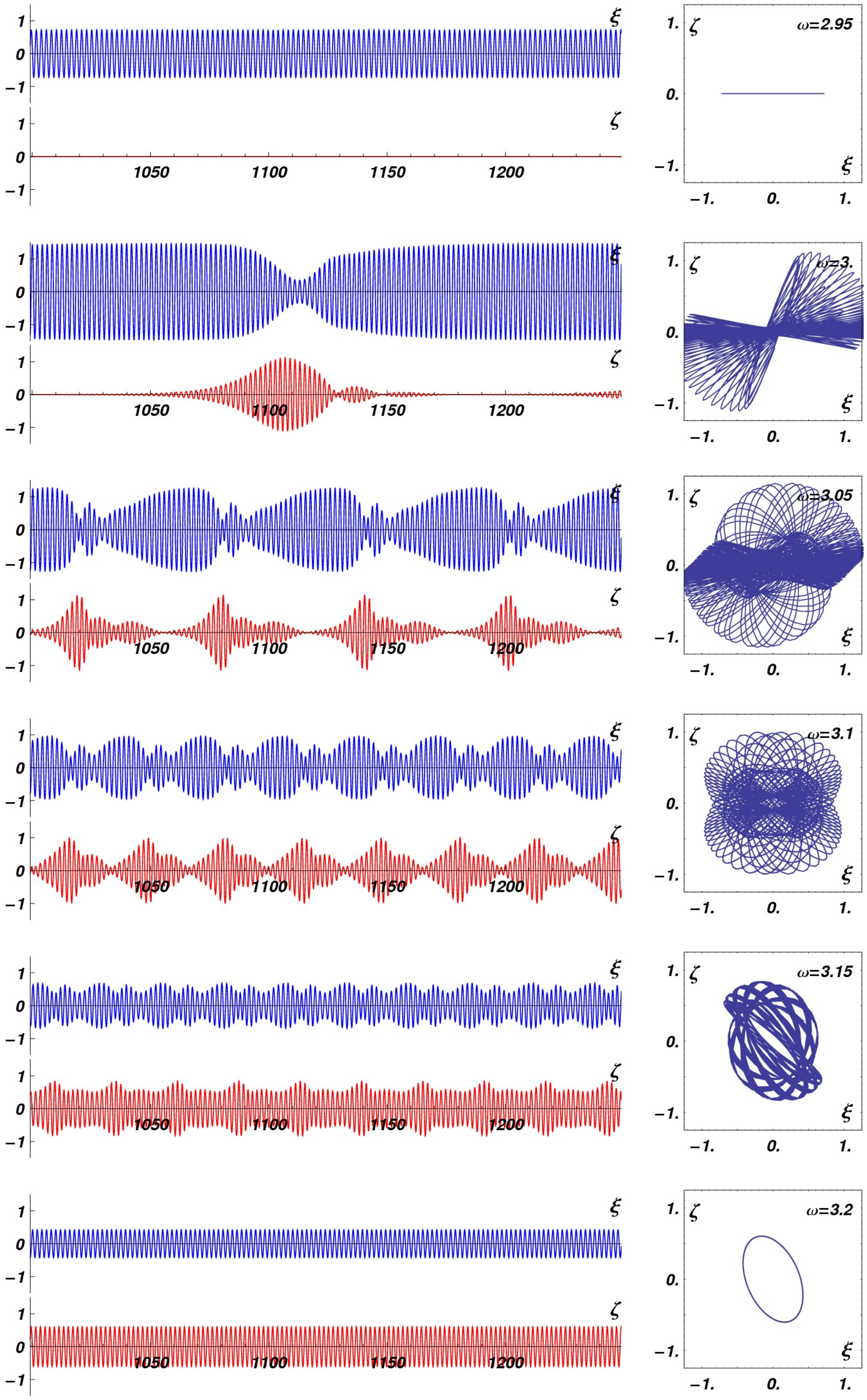

Figure 2: The time histories of pendulum's response, obtained using the direct simulation for various excitation frequencies.

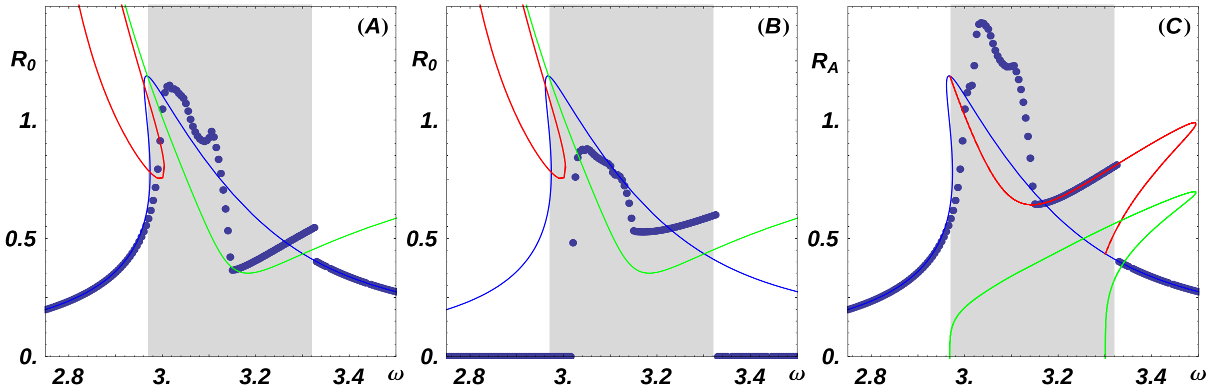

Figure 3: A) Numerical (dotted) and analytical resonance curves (blue) in ξ component, stability limits Eqs (5-6) (red and green curves respectively)
B) Numerical resonance curve in ζ component
C) Total amplitude $\sqrt{\xi^2+\zeta^2}$ (dotted - numerical computation, solid red - R_A from Eq. (12), solid green - S_A from Eq. (12)
The gray area indicates the complete resonance interval

but spatial response, the transversal component ζ is significant, but the overall movement is governed by the single frequency ω.

Figure 3A shows the resonance curve, obtained by means of the numerical solution of the differential system (1) (ξ component - dotted curve), the resonance curve of the semi-trivial solution (3) (blue line) and stability limits (5-6) (red and green curves respectively). Figure 3B shows the numerically obtained resonance curve of the transversal motion (ζ component - dotted curve). Figure 3C shows the course of the total amplitude $\sqrt{\xi^2+\zeta^2}$ (dotted curve - numerical computation), together with the theoretically obtained one by means of the Eq. (12). Let us emphasise the good agreement for theoretical and numerical values of the total amplitude S_A in the second part of the resonance interval. In this part, the assumption of stationary character of the system response is fulfilled ($\dot{a}_c = \dot{a}_s = \dot{b}_c = \dot{b}_s = 0$) and the movement of the pendulum forms the limit cycle in the horizontal ξ, ζ plane.

Numerical solution of the full system (9) for the unknown amplitudes gives very interesting results. Firstly, the numerical solution allows to clearly distinguish between stationary and non-stationary part of resonance interval. The stationary part is characterised by the rapid convergence of the individual amplitudes a_c, a_s, b_c, b_s to constants. This corresponds to the existence of the simple limit cycle, see the case $\omega = 3.2$ in the Figure 4.

In the non-stationary part of the resonance interval the amplitudes a_c, a_s, b_c, b_s seem to have form of biased periodic functions. Their lowest frequency depends on the excitation frequency. However, for the individual values of the excitation frequency ω the amplitudes contain also different number of super-harmonic components. Time histories of the computed amplitudes for several excitation frequencies are depicted in the Figure 4. The excitation frequencies were selected from the first part of the resonance interval, where the response is non-stationary and the Eq. (12) cannot be used. It can be seen from the Figure 4, that the periodic character of the individual components is rather complicated.

The Figure 5 shows results of the Fourier analysis of the individual amplitudes a_c, a_s, b_c, b_s for different excitation frequencies. It comes to light, that the general form of any amplitude

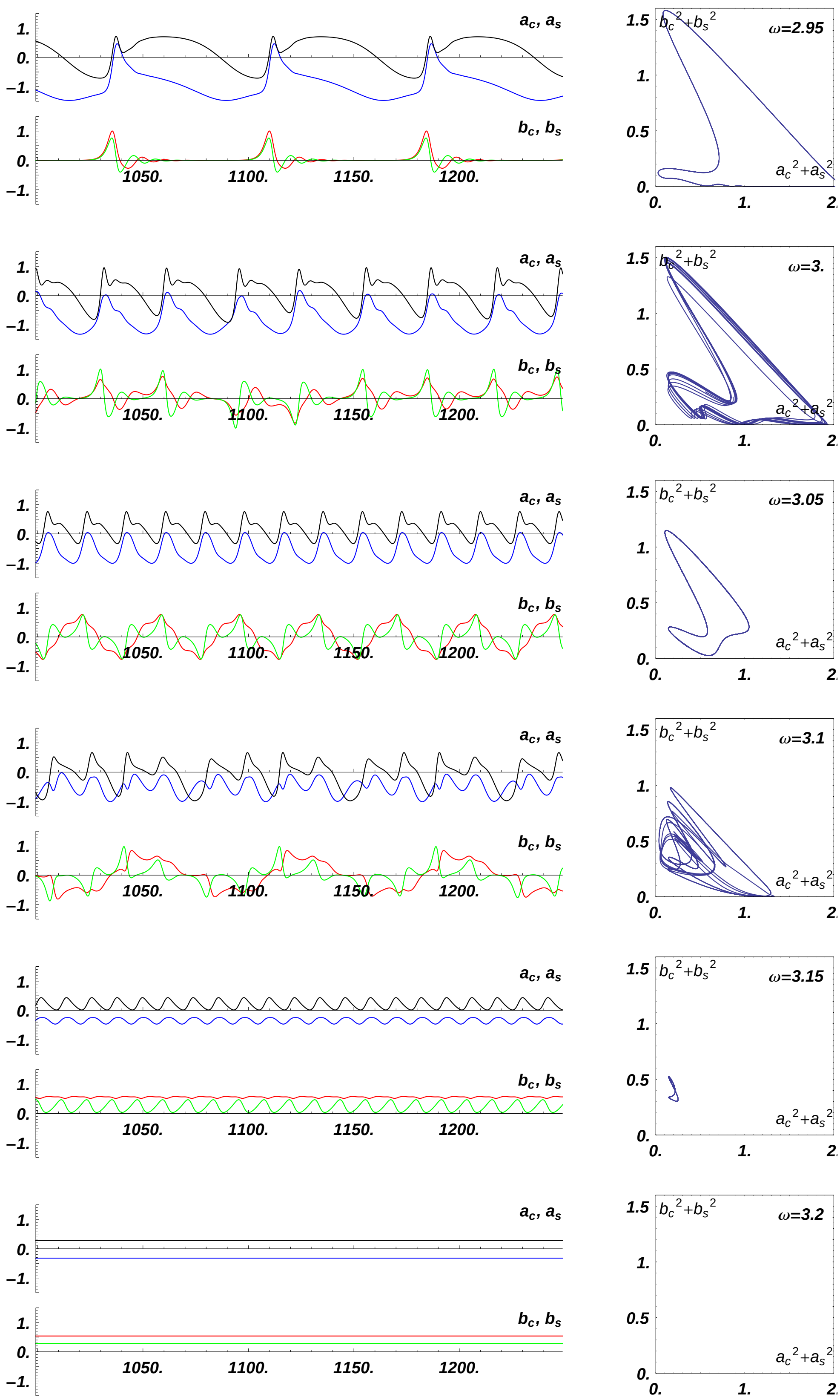

Figure 4: The time histories of non-stationary amplitudes a_c, a_s, b_c, b_s computed according to Eq. (9). Mutual plot of amplitudes of the both components $a_c^2 + a_s^2$ vs. $b_c^2 + b_s^2$ is on the right hand side.

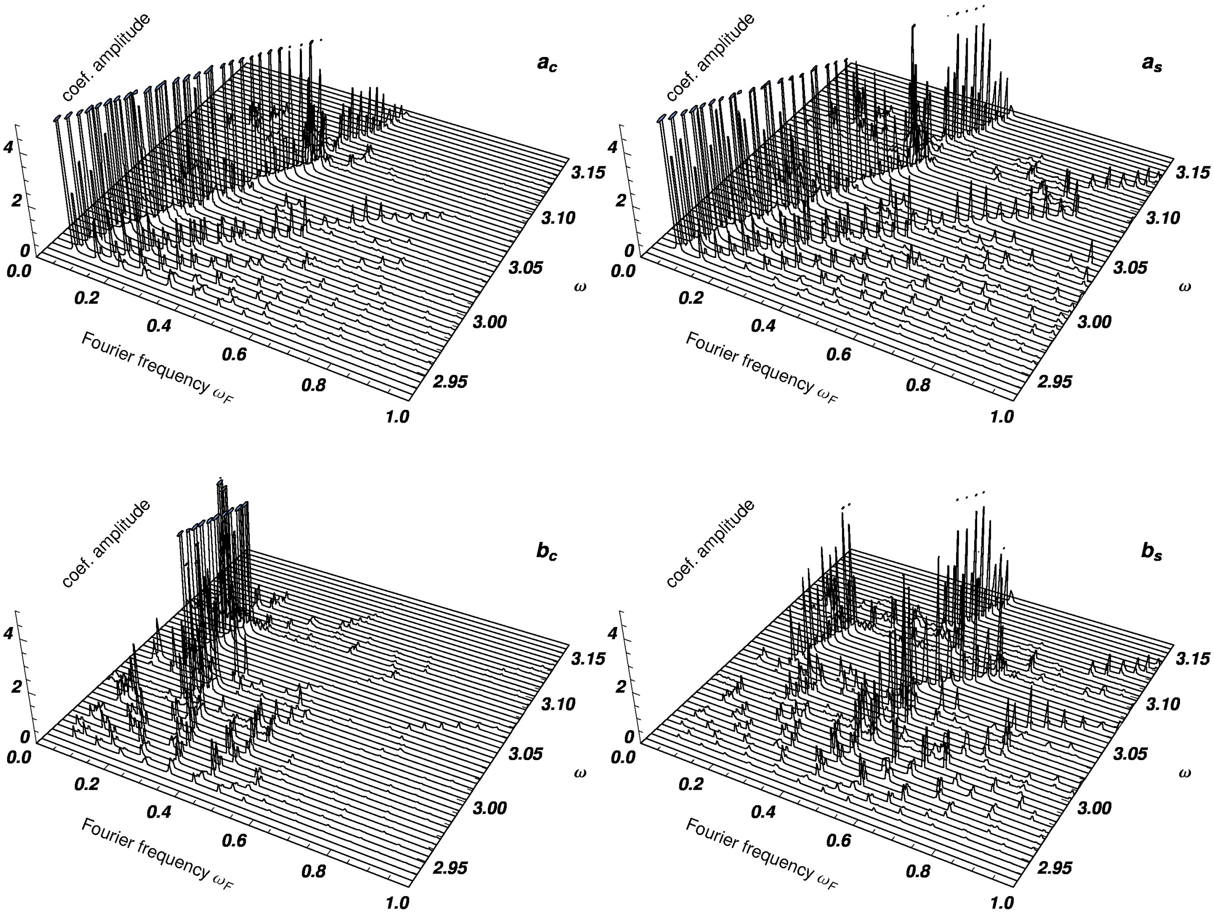

Figure 5: Plot of the significant parts of the periodograms of the individual components a_c, a_s, b_c, b_s for increasing excitation frequency ω. Actual values of the Fourier coefficients were clipped at value 5.

should be sought in the form of finite Fourier series:

$$x(t) = x_0 \sum_{i=n}^{N} \alpha_{xi} \cos\left(n\omega_F t\right) + \beta_{xi} \sin\left(n\omega_F t\right) \tag{13}$$

where x stands for a_c, a_s, b_c, b_s.

Figures 6 and 7 depict the pattern of non-zero Fourier coefficients, computed for increasing excitation frequency. Numerical solution of the Eq. (9) has been used, followed by the numerical Fourier transform. While the Figure 6 includes coefficients exceeding threshold of 1.5 in the absolute value, in the Figure 7 is the threshold taken 0.5. Unfortunately, the dependence of ω_F on the excitation frequency ω is not linear, although a roughly linear trend can be seen from the Figures 6 and 7. On the other hand, only a limited number of Fourier coefficients is necessary for satisfactory description of the general form solution.

Assuming, e.g. $N = 2$ in Eq. (13) and introducing (13) into (9) as the substitution for unknowns a_c, a_s, b_c, b_s, a system of 20 non-linear algebraic equations can be obtained. (4 equations by setting $t = 0$ and 16 using the harmonic ballance approach). However, number of equations is 21, (4 times $x_0, \alpha_{x1}, \alpha_{x2}, \beta_{x1}, \beta_{x2}$ and ω_F), the system is under-determined and one additional condition should be introduced in the future.

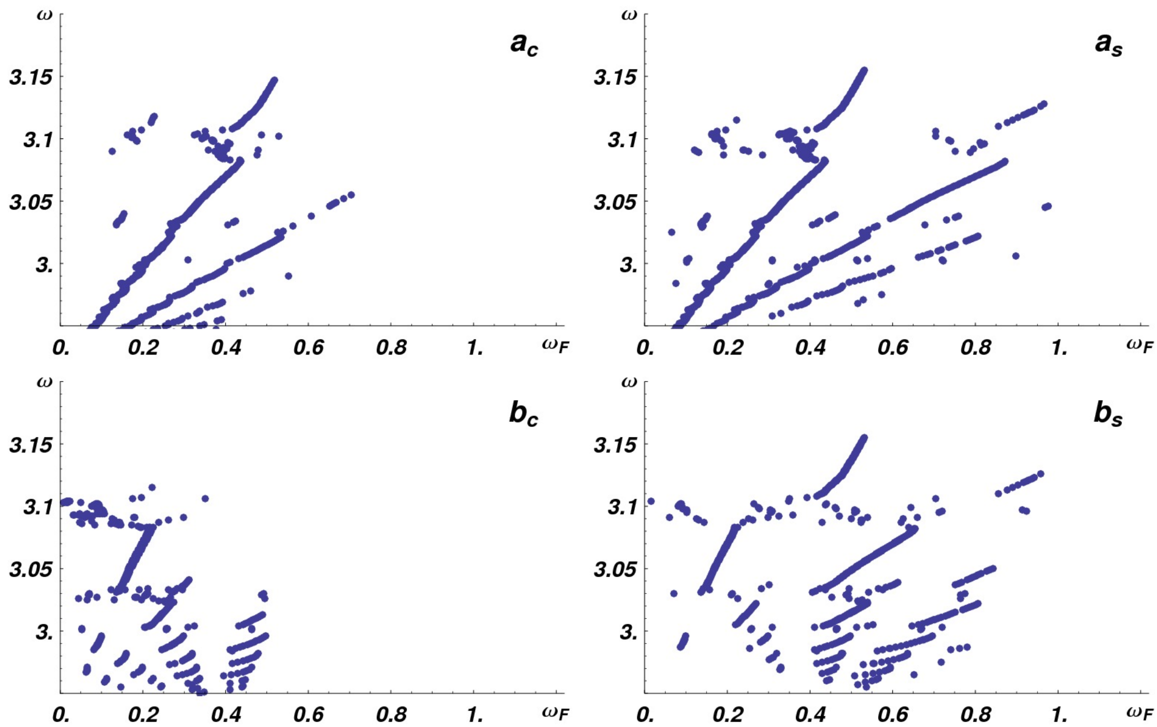

Figure 6: Pattern of the non-zero Fourier coefficients with respect to the excitation frequency of the individual components a_c, a_s, b_c, b_s. Each point corresponds to a Fourier coefficient greater than 1.5.

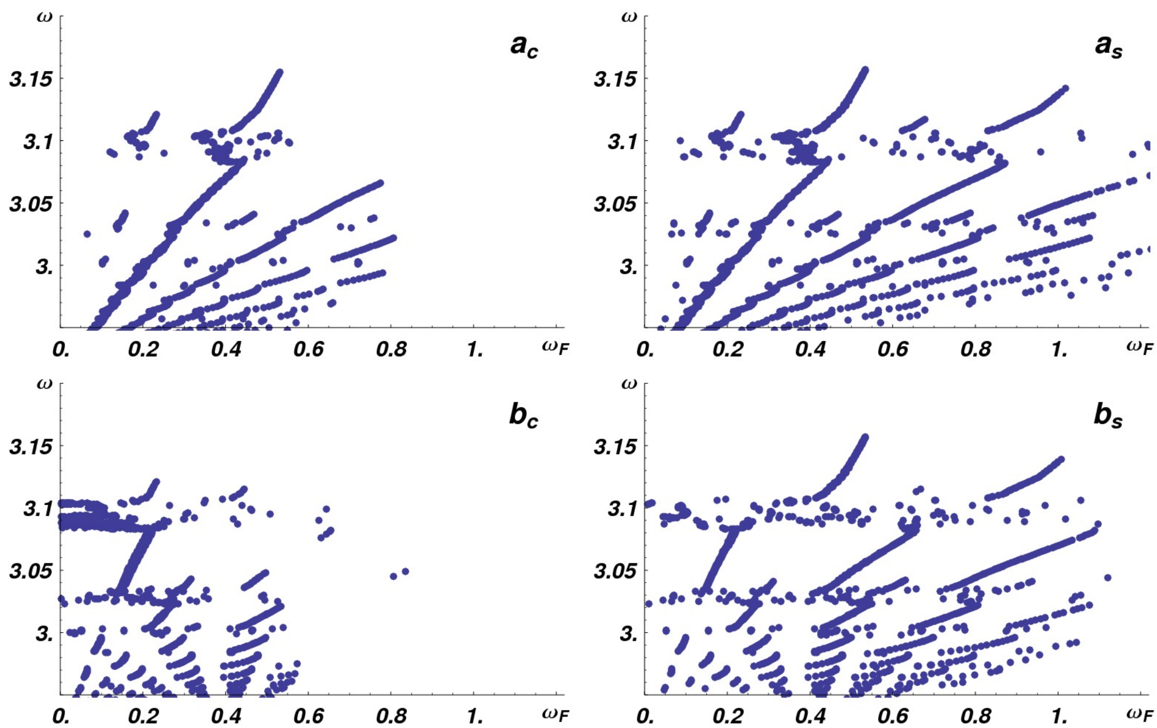

Figure 7: Pattern of the non-zero Fourier coefficients with respect to the excitation frequency of the individual components a_c, a_s, b_c, b_s. Each point corresponds to a Fourier coefficient greater than 0.5.

4. Conclusions

Analytical and numerical investigations have shown that the widely used linear model of the damping pendulum is acceptable only to a very limited extent in parameters of pendulum characteristics and excitation properties. Two degrees of freedom non-linear model either in spherical or Cartesian coordinates must be introduced. The harmonic kinematic external excitation in the suspension point was applied. Using the harmonic balance method, the resonance curves of a planar stationary response as well as stability limits of the semi-trivial solution in both response components were determined. These results are quantitatively acceptable in the non-resonance interval only where a stationary response exists.

The detailed numerical analysis in the resonance domain was completed. Three types of the resonance were identified with respect to the relation of a preliminary resonance curve and stability limits: magnified in-plane response (in ξ coordinate), stationary spatial response (limit cycle exists, amplitudes of oscillations in ξ and ζ directions are non-zero but constant) and non-stationary spatial response (amplitudes in ξ and ζ directions are non-zero and variable). It reveals, that the variability of the individual amplitudes of the non-stationary solution exhibits periodic character with one base frequency ω_F and several its super-harmonics. This frequency ω_F depends on all the parameters of the system (excitation frequency and amplitude, damping etc.) However, for several small sub-intervals of the resonance region, the response behaviour seems to be more complicated.

The general form of the solution in the resonance region was proposed, based on several terms of the Fourier series. Detailed analytical study should follow.

From the practical point of view, it is highly recommended to design the damping pendulum in such a way that any intersections of the resonance curve with the stability limits are avoided. Especially the intersection with the ξ stability limit should be avoided otherwise a negative influence of the pendulum in the resonance domain is to be expected in both along-wind as well as in cross-wind directions. Taking into account that the excitation in the open air has rather a broad band character, details of such a device should be thoroughly thought out.

5. Acknowledgment

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6. References

Haxton, R.S. & Barr, A.D.S. (1974) The auto-parametric vibration absorber, *ASME Jour. Applied Mechanics*, **94**, 119-125.

Fischer, C. & Náprstek, J. (2009) Two equations for description of the pendulum damper. In: *Engineering Mechanics 2009*. Prague: Institute of Theoretical and Applied Mechanics. pp. 253-259.

Náprstek, J. & Fischer, C. (2007) Auto-parametric post-critical behaviour of a spherical pendulum damper. In: B. Topping (Ed.) *Proc. 11th conference on civil, struct. and env. eng. computing.* Malta : Civil-comp Press.

Náprstek, J. & Fischer, C. (2008) Stochastic stability and post-critical random behaviour of auto-parametric systems with cyclic symmetry. In: *EURODYN 2008* Southampton : University of Southampton, pp. 81-92.

Náprstek, J. & Fischer, C. (2009) Auto-parametric semi-trivial and post-critical response of a spherical pendulum damper. *Computers and Structures*, Vol. **87**, No. 19-20, pp. 1204-1215.

Tondl, A. (1991) *Quenching of Self-Excited Vibrations*, Academia, Prague.

Tondl, A., Ruijgrok, T., Verhulst, F. & Nabergoj, R. (2000) *Auto-parametric Resonance in Mechanical Systems*, Cambridge University Press, Cambridge.