International Conference
ENGINEERING MECHANICS 2010
Svratka, Czech Republic, May 10 – 13, 2010

HIGH-QUALITY VECTOR FIELD AND DIRECT VORTICITY ESTIMATION USING THE AFFINE CORRELATION METHOD

P. Gronát [1]

Abstract: *In this paper, a correlation method for direct vorticity computation of the fluid velocity field is introduced. The correlation method allows an affine transformation of a correlation window. It is shown that Lagrangian field of displacements, considering the linear terms only, satisfies the requirements for affine transformation and using the Normalized-Cross-Correlation (NCC) together with affine transformation and Newton-Raphson iterative method, six parameters defining this transformation can be estimated. Subsequently, using these parameters, both displacement and vorticity can be computed, hence, the vorticty is being obtained for every single point directly during the correlation procedure.*

1. Introduction

Typically, vorticty of turbulent flow is computed from a velocity field obtained by standard PIV correlation methods. During the correlation procedure some small enough area around a pixel from reference image is considered and algorithm is searching for similar pixel having similar surroundings in tested image. For every single point in the tested image, the measure of match is evaluated. This measure is typically expressed by a correlation coefficient and the pixel maximizing this coefficient is being searched as shown on Fig. 1.

This method, however, assumes that a pixel and some small area around it did not undergo any deformation but the translation only which is so called fronto-parallel assumption. On practice, however, this assumption is rarely satisfied, especially in fluid dynamics measurements where gradients of velocitiy are relatively high. If too small area around the pixel is considered, the information content is small, hence, the correlation function would have many similar peaks and very probably wrong correspondence pixel would be selected. On the other hand, if area around the pixel is too large, due to this area did undergo not only the translation, peaks of NCC in Fig 1. would become dramatically flat and not significant, therefore, bad detections would occure on practice.

Due to this, always some noise in the velocity vector field is induced using the standard PIV correlation methods. Since typically finite differences are being employed during the vorticity calculation, every noise in the vector field is, consequently, critical for the vorticity estimation and appears critical for the other caluculations where Jacobian of a vector field is needed as well.

[1] Ing. Petr Gronát, MSc.: Institute of Thermomechanics of Czech Academy of Sciences, Department of Fluid Dynamics, Dolejškova 1402/5, 182 00 Prague 8, Czech Republic, tel. +420 286 588 547, e-mail gronat@it.cas.cz

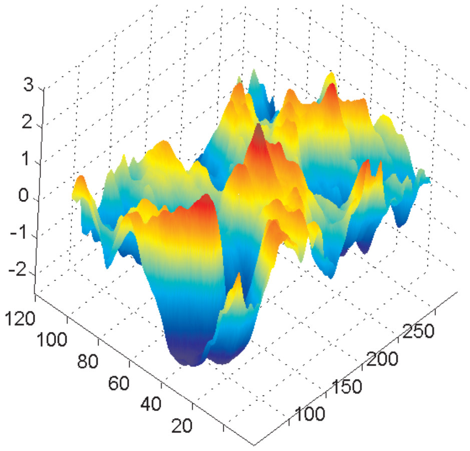


Figure 1: Plot of the correlation coefficient function versus tested pixel position in defined searching area. The best match is identified by the highest peak located in the center.

2. Affine Transformation of a Correlation Window

Affine transformation can be represented as a spatial transformation which preserves collinearity and the ratio of distances of three points lying on a single line. It is the combination of linear transformations translation, scaling, rotation and skew which are not, however, completely independent. In matrix notation, for transformation, it can be written

$$\begin{bmatrix} \vec{y} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \vec{b} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix} \tag{1}$$

where $\mathcal{A}$ is the two by two matrix representing a linear transformation and vector $\vec{b}$ represents shift. Now, consider that some correlation window did undergo not only the translationn but some affine transformation as shown on Fig. 2.

The area aroud the origin of a square correlation window is deformed and the final shape of original correlation window is the parallelogram in the right side of Fig. 2. Using the notation introduced in Fig. 2, this transformation can be written as folows

$$\begin{bmatrix} \vec{x^\star} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \vec{x_0} + \vec{u} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix} \tag{2}$$

where $\Delta\vec{x} = [\Delta x, \Delta y]^T$ and $\mathcal{A}$ is unknown matrix of linear transformation.

3. Lagrangian Field of Displacements and Affine Transformation

To make the correlation method more robust, it is assumed that for every pixel (the reference pixel) and its neighbourhood in the reference image exists some affine transformation such that this pixel moves to the new position and its surroundings is deformed in such a way that it corresponds to the tested image. Let think about a vector field as to be a Lagrangian field of displacements such that

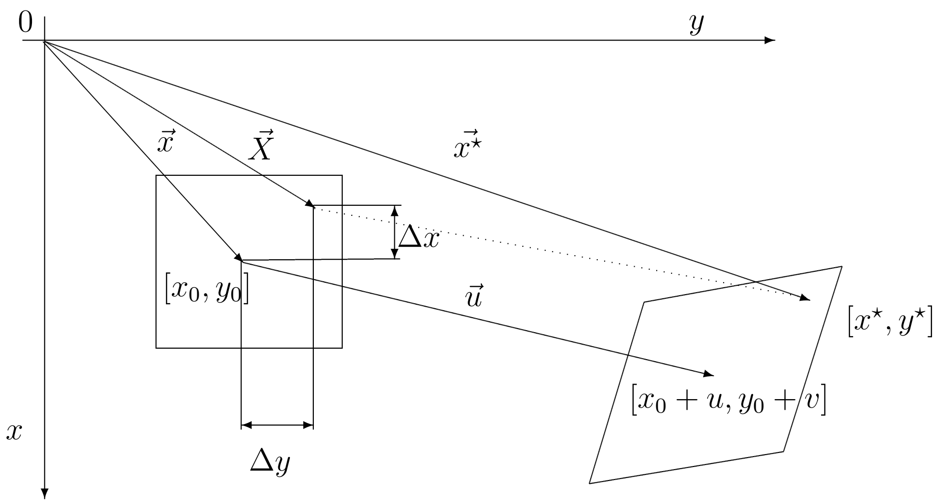


Figure 2: Correlation window before (reference window) and after (tested window) affine transformation. Field of displacements around center point $[x_0, y_0]$ is regarded to be linear. u, v are translations and $\Delta x, \Delta y$ are the distances of a point from the window center in reference window.

$$x^\star = \Gamma(X(t), Y(t), t) \\ y^\star = \Phi(X(t), Y(t), t) \tag{3}$$

where Γ, Φ are unknow functions. Considering the field of displacements is constant in time, Taylor's expansion of (3) around the reference pixel $[x_0, y_0]^T$ considering the linear terms only can be written as

$$x^\star = x_0 + u + \Delta x \frac{\partial u}{\partial x} + \Delta y \frac{\partial u}{\partial y} \\ y^\star = y_0 + v + \Delta x \frac{\partial v}{\partial x} + \Delta y \frac{\partial v}{\partial y} \tag{4}$$

which corresponds to the above–mentioned affine transformation which can be, subsequently, written in matrix form

$$\begin{bmatrix} x^\star \\ y^\star \\ 1 \end{bmatrix} = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y & x_0 + u \\ \partial v/\partial x & \partial v/\partial y & y_0 + v \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ 1 \end{bmatrix} \tag{5}$$

where $u(x, y)$, $v(x, y)$ are displacements, x_0, y_0 are reference pixel coordinates, Δx, Δy are distances of neighbourhood from reference pixel and x', y' are coordinates of the transformed pixel. To describe fully this transformation, six parameters have to be estimated. These are shifts u, v and the partial derivatives in (5). Employing the NCC together with Newton–Raphson iterative method, these parameters can be found for each pixel in tested image.

hence, equations (4) express the affine transformation of a point and its close surrounding. Therefore, to describe this transformation, for each reference pixel $[x_0, y_0]^T$ six parameters has to be found. Indroducing the parameter vector, these unknowns are

$$\vec{p} = \left[\begin{array}{cccccc} u & v & \partial u/\partial x & \partial u/\partial y & \partial v/\partial x & \partial v/\partial y \end{array} \right]^T \tag{6}$$

4. Normalized Cross Correlation (NCC)

To express the measure of match of deformed correlation window on tesdted image for some particular vector $\vec{p}$ the correlation coefficient is introduced as follows

$$S(\vec{p}) = 1 - \frac{\sum_{i,j}^{N} [F(x_i, y_j) \cdot G(x_i^\star(\vec{p}), y_j^\star(\vec{p}))]}{\sqrt{\sum_{i,j}^{N} F(x_i, y_j)^2 \cdot \sum_{i,j}^{N} G(x_i^\star(\vec{p}), y_j^\star(\vec{p}))^2}} \tag{7}$$

In this formulae, function $F(x, y)$ represents the grayscale value of undeformed correlation window at integer coordinates i, j at reference image. The function $F(x^\star, y^\star)$ represents the grayscale value of tested image at transformed coordinates $x^\star, y^\star$. Since these are integers no longer, in general, the grayscale of tested image at non-integer coordinate have to be interpolated. For example, the bicubic or bilinear interpolation respectively can be employed.

5. Newton-Raphson Iterative Method for NCC

To find the minima of the correlation ceofficient function (7) the NewtonRaphson iterative method can be used. It can be shown that correction of parameter $\vec{p}$ at k-th iteration is

$$\Delta\vec{p_k} = -H^{-1}(\vec{p_k}) \cdot J(\vec{p_k}) \tag{8}$$

where H is the Hessian matrix and J is the Jacobian vector. In tensor notation these can be written as follows

$$\begin{aligned} J_i &= \frac{\partial}{\partial p_i} \cdot S(\vec{p}) \\ H_{ij} &= \frac{\partial^2}{\partial p_i \partial p_j} \cdot S(\vec{p}) \end{aligned} \tag{9}$$

Employing the columnwise vectorization of terms inside summation in (7), so that $G(x_i^\star, x_j^\star) = G_{ij} = G_k$, for Hessian matrix and Jacobian in tensor notation it can be obtained:

$$H_{ij} = \frac{-1}{(\sum_k F_k^2)^{1/2}} \left[\frac{\sum_k F_k \frac{\partial^2 G_k}{\partial p_i \partial p_j}}{(\sum_k G_k^2)^{1/2}} - \frac{\sum_k F_k \frac{\partial G_k}{\partial p_i} \sum_k G_k \frac{\partial G_k}{\partial p_j}}{(\sum_k G_k^2)^{3/2}} - \frac{\sum_k F_k \frac{\partial G_k}{\partial p_j} \sum_k G_k \frac{\partial G_k}{\partial p_i}}{(\sum_k G_k^2)^{3/2}} - \right.$$
$$\left. - \frac{\sum_k F_k G_k \sum_k (\frac{\partial G_k}{\partial p_i}\frac{\partial G_k}{\partial p_j} + \frac{\partial^2 G_k}{\partial p_i \partial p_j})}{(\sum_k G_k^2)^{3/2}} + \frac{3 \cdot \sum_k F_k G_k \sum_k G_k \frac{\partial G_k}{\partial p_i} \sum_k G_k \frac{\partial G_k}{\partial p_j}}{(\sum_k G_k^2)^{5/2}} \right] \tag{10}$$

$$J_i = \frac{\partial S}{\partial p_i} = \frac{-1}{(\sum_k F_k^2)^{1/2}} \left[\frac{\sum_k F_k \frac{\partial G_k}{\partial p_i}}{(\sum_k G_k^2)^{1/2}} - \frac{\sum_k F_k G_k \sum_k G_k \frac{\partial G_k}{\partial p_i}}{(\sum_k G_k^2)^{3/2}} \right] \tag{11}$$

Note that terms in front of square brackes remain constant during the iterative process, thus, these can be pre–computed. To express both Jacobian and Hessian, the partial derivatives of G_k with respect to parameter vector $\vec{p}$ have to be calculated. Considering that

$$\begin{aligned} x'_k &= x^\star_k - \lfloor x^\star_k \rfloor \\ y'_k &= y^\star_k - \lfloor y^\star_k \rfloor \end{aligned} \tag{12}$$

where $\lfloor x \rfloor$ denotes floor of x, using the chain rule, first and second partial derivatives are compued as follows:

$$\frac{\partial G_k}{\partial p_i} = \frac{\partial G_k}{\partial x'_k}\frac{\partial x^\star}{\partial p_i} + \frac{\partial G_k}{\partial y'_k}\frac{\partial y^\star}{\partial p_i} \tag{13}$$

$$\begin{aligned} \frac{\partial^2 G_k}{\partial p_i \partial p_j} = &\left(\frac{\partial x^\star_k}{\partial p_i}\frac{\partial x^\star_k}{\partial p_j}\right)\frac{\partial^2 G_k}{\partial x'^2_k} + \left(\frac{\partial y^\star_k}{\partial p_i}\frac{\partial y^\star_k}{\partial p_j}\right)\frac{\partial^2 G_k}{\partial y'^2_k} + \\ &+ \left(\frac{\partial x^\star_k}{\partial p_i}\frac{\partial y^\star_k}{\partial p_j} + \frac{\partial x^\star_k}{\partial p_j}\frac{\partial y^\star_k}{\partial p_i}\right)\frac{\partial^2 G_k}{\partial x'_k \partial y'_k} \end{aligned} \tag{14}$$

Partial derivatives of $x^\star$ and $y^\star$ can be calculated once bicubic interpolation coefficients are estimated. These stay constant during the computation, hence, can be pre-computed in advance.

6. Vorticity and Vortex Detection

Since now parameter vector $\vec{p}$ is estimated using the above-discused method all first partial derivatives of Lagragian field are known. Denote p_i to be the i-th component of $\vec{p}$ vorticity at reference pixel can be computed as

$$\Omega_z = [\nabla \times \vec{u}]_z = (\partial v/\partial x - \partial u/\partial y)$$
$$\Omega_z = (p_5 - p_4) \tag{15}$$

Other parameters such as skew, divergence and scaling can be computed analogically. Since all derivatives are computed directly during correlation process, if experimantal data are pure and noisy the vorticity and other parameters computed from $\vec{p}$ are expected to be less sensitive to noise than those computed numericaly from obtained vector field.

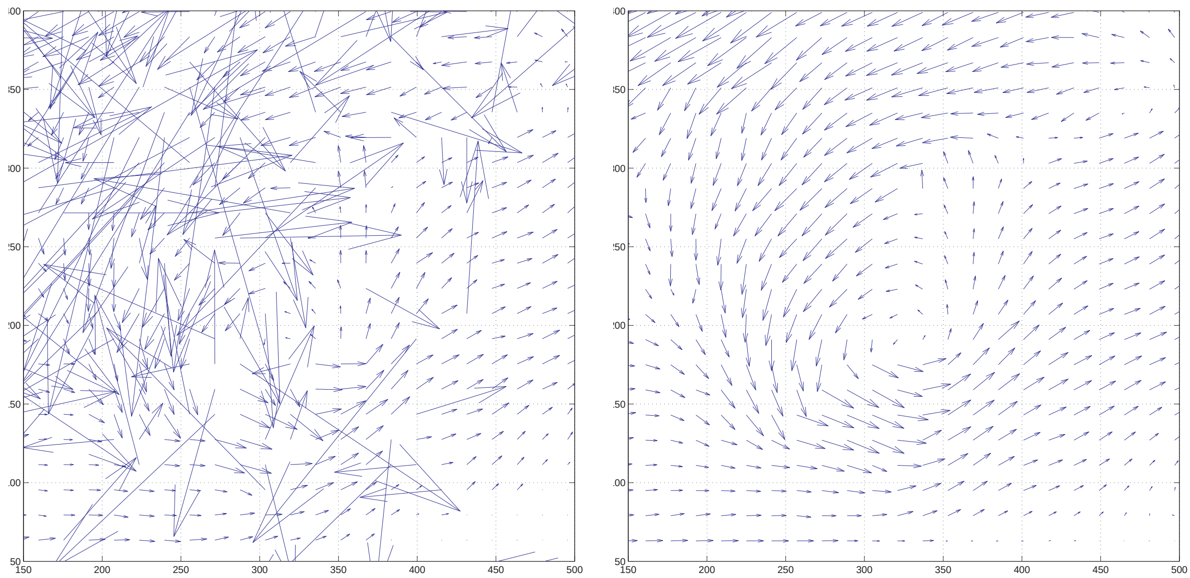
To locate vortex center, discriminant of non-real eigenvalues of Jacobian (DNEJ) of velocity field can be computed [8]. This discriminant separates vortices from other patterns and can be computed as below

$$d_2 = tr(J)^2 - 4 \cdot det(J) \tag{16}$$

where J is the Jacobian matrix of vector field. In terms of parameter vector elements for d_2 it can be written

$$d_2 = (p_3 + p_6)^2 - 4 \cdot (p_3 p_6 - p_4 p_5) \tag{17}$$

Vortices are then indicated in the region with negative alue of d_2.



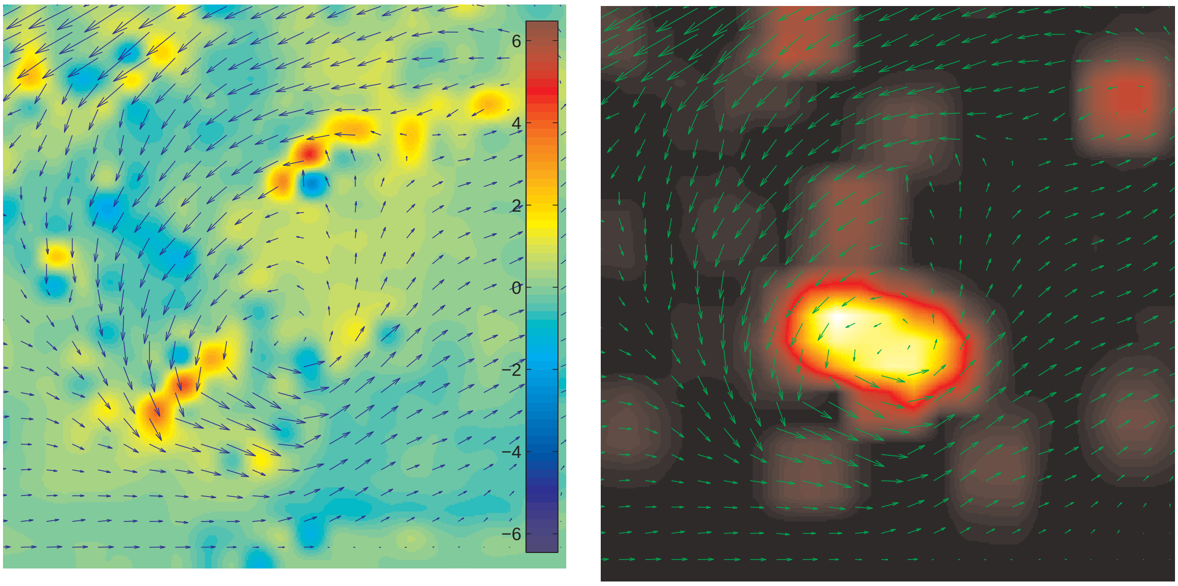
(a) Adaptive correlation - Dynamic Studio

(b) Affine correlation

Figure 3: Vortex structure vector field evaluated by adaptive correlation method in Dantec Dynamic Studio (a) and the same structure interpreted by affine correlation method (b).

7. Practiceal results

The affine correlation method was tested on real data. The airflow behind obstacle was measured and data were processed using the commercial software Dantedc Dynamics Studio and using



(a) Vorticity (b) DNEJ vortex detector

Figure 4: Vorticity estimation (a) and DNEJ vortex detector evaluated using the affine correlation method (b).

proposed method. In Fig. 3 two vector fields near a vortex center are displayed. Data processed by affine correlation method appeares to be more reliable noiseless.

Moreover, since the partial derivatives of the field were obtained directly during correlation important properties such as vorticity or skew can be computed for each reference pixel. In Fig. 4 a vorticity and DNEJ are shown in vortex region using affine correlation.

8. Acknowledgment

The author gratefully acknowledge financial support of the Grant Agency of the Czech Republic, project No. 101/08/1112.

9. References

[1] *Gonzales, R.C.,Woods, R.E.:* **Digital Image Processing.** Prentice-Hall, New Jersey 2002.

[2] *Gronat, P.:* **Plate Deflection on an Elastic Foundation and 3D Computer Vision Software Design.** Thesis, Chung Hua Univeristy 2009, Hsinchu, Taiwan R.O.C.

[3] *Herbst,C.,V.,Splitthof,K.:* **Basics of 3D Digital Image Correlation.** Dantec Dynamics, http://www.dantecdynamics.com, 2008.

[] *O'Neil,P.,V.:* **Advanced Engineering Mathematics.** International Thomson Publishing, Boston 1993.

[4] *Pan, B., Asundi, A., Xie, H., Gao, J.:* **Digital image correlation using iterative least-squares and pointwise least squares for displacement field and strain field measurements.** Elsevier, Optics and Lasers in Engineering 47 (2009) 865-874, 2009.

[5] *Saphiro, L.G., Stockman, G.C.:* **Computer Vision.** Prentice-Hall, New Jersey 2001.

[6] *Sutton, M.A., Oretu, J.J., Schreier, H.W.:* **Image Correlation for Shape, Motion and Deformation Measurements.** Springer, New York, 2009.

[7] *Tiwari, W., Sutton, M.A., McNeill, S.R., Xu, S., Deng, X., Fourney, W.L., Bretall, D.:* **Application of 3D image correlation for full-field transient plate deformation measurements during blast loading.** Elsevier,International Journal of Impact Engineering 36 (2009) 862-874, 2009.

[8] *Vollemrs, H.:* **Detection of vortices and quantitative evaluation of their main parameters from experimental velocity data.** Measurement Science and Technology,p. 1199–1207 Göttingen, 2001.