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# VIBRATION AND STABILITY OF ELECTRO-MECHANICAL SYSTEM WITH LIMITED SOURCE OF ENERGY

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**Summary:** *The paper presents an analysis of vibration of electromechanics system consisting of a 1DOF vibrating mechanical subsystem connected by means of spring and crank mechanisms with an electric driving motor with limited power and inertia. After transformation of motion equations and using averaging method, the approximate expressions for stationary and transient vibrations are derived. Owing to the non-ideal excitation, the unstable section of stationary response curve originates and causes jumps at transient processes. Influence of slope of moment characteristics and driving motor inertia on the range and level of instability domain is shown and discussed as well.*

## 1. Introduction

The influence of nonlinear spring and/or dashpot characteristics on the dynamic properties of mechanical system has been investigated in the literature during the last century usually at the assumption of so called hard (ideal) strictly prescribed function of exciting force or exciting motion. Because actually the power of the real exciting energy source has always limited capacity, this idealization is in many cases not acceptable. The non-ideal properties (limited power, limited inertia) of source of exciting force introduce new nonlinear phenomena into dynamic behavior of the whole investigated system.

These properties are studied on a one DOF nonlinear system excited by mechanical cam or crank mechanisms connected by a spring. Mechanical system can contain also a nonlinear element with a characteristic described by function $\varepsilon f(y, \dot{y})$. The other nonlinearity is given by the motor characteristic $M(\dot{\varphi}, \alpha)$ and by excitation mechanisms.

The first kind of non-ideal properties of energy source was mentioned with the connection of the so called Sommerfeld's effect [1] in 1902, but the detailed analysis was given latter by V. O. Kononěnko [2,3] with the main orientation to the centrifugal exciter. The overview of the various systems behavior and selection of main publications on vibrating systems with weak excitation is given in [4 - 9]. In these publications was the main attention oriented to problems of response curve, change of eigen- or resonance-frequencies, variations of forms, arising of new harmonic components, etc.

Theoretical analysis of mentioned nonlinear system in this paper is focused on the investigation of influence of non-ideal energy source on the form of response curve and stability of weakly excited mechanical system.

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For comparison and verification of results gained by analytic-numerical methods with properties of real structures, the laboratory stand of vibrating system excited by electric motor with limited power is described.

## 2. Equations of motion

The combination of both nonlinearities i.e. nonlinear internal element together with the nonlinear characteristic of the source of exciting forces can give rise to some new properties and changes in the motion of such systems.

Let us show these properties on the system with one degree of freedom excited by the mechanical exciter, the main part of which is the motor with the moment characteristic $M(\dot{\varphi},\alpha)$ (or power characteristic $P(\dot{\varphi},\alpha) = \dot{\varphi} M(\dot{\varphi},\alpha)$), where $\alpha$ is the parameter ascertaining the input of constant energy flow into the motor, $\dot{\varphi}$ is the angular velocity. The shaft of the motor is provided by a crank or cam mechanism connected by spring (stiffness $\bar{k}$) with the vibrating system (Fig. 1).

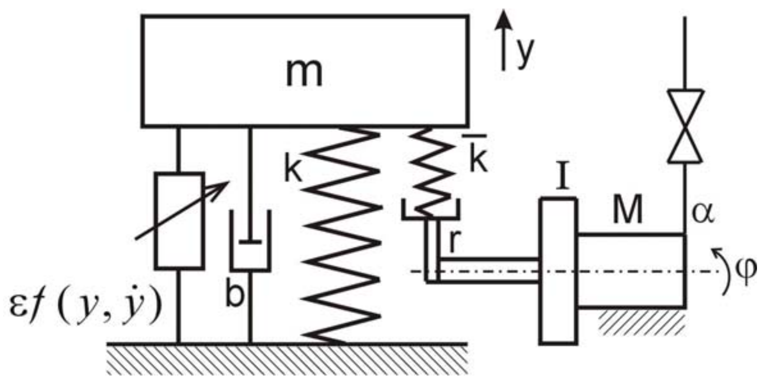


Fig. 1

The motion of this system is described by the differential equations

$$m\ddot{y} + b\dot{y} + ky + \varepsilon f(y,\dot{y}) = \bar{k}(-y + r\cos\varphi) \tag{1}$$

$$I\ddot{\varphi} = M(\dot{\varphi},\alpha) + \bar{k}r\sin\varphi(r\cos\varphi - y)$$

The characteristic of the mechanical nonlinearity is described by the function $\varepsilon f(y,\dot{y})$, where $\varepsilon$ is a small parameter. Second nonlinearity is caused by the conversion of the rotation $\varphi(t)$ into the translation $y$, and is described by the last equation (l).

After introducing the new independent variable $\varphi$ instead of the time *t*, where

$$\vartheta = \frac{d\varphi}{dt} \tag{2}$$

is instantaneous angular velocity and

$$\dot{(\ )} = \frac{d}{dt} = \frac{d}{d\varphi}\vartheta \tag{2a}$$

$$\ddot{(\ )} = \frac{d^2}{dt^2} = \vartheta^2\frac{d^2}{d\varphi^2} + \vartheta\vartheta'\frac{d}{d\varphi}$$

we get

$$m(\vartheta^2 y'' + \vartheta\vartheta' y') + b\vartheta y' + k_1 y + \varepsilon f(y, \vartheta y') = \bar{k} r \cos\varphi$$

$$I\vartheta\vartheta' = M(\vartheta, \alpha) + \bar{k} r \sin\varphi (r\cos\varphi - y), \qquad (3)$$

where $k_1 = k + \bar{k}$.

The motor characteristic we shall suppose in the simple form

$$M(\dot{\varphi}, \alpha) = M_0(\alpha - \dot{\varphi}), \qquad (4)$$

where $M_0$ is the declination of motor moment characteristic and parameter $\alpha$ corresponds to the revolution of the idle run of the motor (Fig. 2).

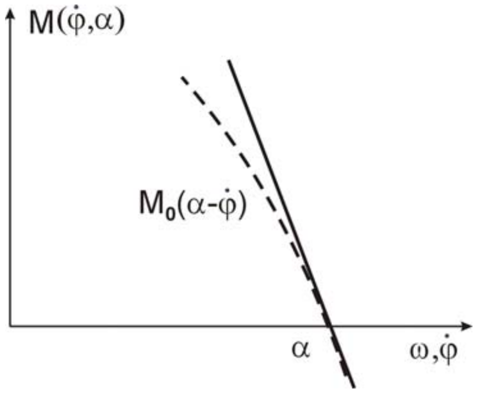


Fig. 2

During study of the forced oscillation of this system, we can apply the direct numerical simulation or use some of the first approximation methods. The direct numerical simulation gives a more exact solution and for given case it expresses the real motion of the nonlinear system, but the first approximation methods give more general results and enable an analytical description of both response curves and domains of instability, even for the wide set of equations (1).

In this paper we shall deal with the solution in the first approximation and we use the averaging method combined with the equivalent linearization method [11, 12].

These methods are based on the application of the transformation of variables $y$, into $C, S$:

$$y = C\cos\varphi + S\sin\varphi, \qquad (5)$$

where $C, S$ are slowly varying variables.

Introducing the auxiliary conditions

$$C'\cos\varphi + S'\sin\varphi = 0 \qquad (5a)$$

we get

$$y' = -C\sin\varphi + S\cos\varphi \qquad (5b)$$

$$y'' = (-C'\sin\varphi + S'\cos\varphi) - (C\cos\varphi + S\sin\varphi).$$

After substituting (5b) into (3) with the consideration of (5a) we get the set of three differential equations of first order in $C', S', \vartheta'$

$$C' = \frac{1}{2m\vartheta^2}\{A + B\sin 2\varphi - A\cos 2\varphi + 2\varepsilon f[(C\cos\varphi - S\sin\varphi), \vartheta(-C\sin\varphi + S\cos\varphi)]\sin\varphi\}$$

$$S' = \frac{-1}{2m\vartheta^2}\{B + A\sin 2\varphi + B\cos 2\varphi + 2\varepsilon f[(C\cos\varphi - S\sin\varphi), \vartheta(-C\sin\varphi + S\cos\varphi)]\cos\varphi\}$$

$$\vartheta' = \frac{1}{I_1\vartheta}\left\{M_0(\alpha - \vartheta) - \frac{\bar{k}r}{2}S + \frac{\bar{k}r}{2}(r - C)\sin 2\varphi + \frac{\bar{k}r}{2}S\cos 2\varphi\right\}, \qquad (6)$$

where $A, B$ stand for

$$A = (k_1 - m\vartheta^2)S - (b + m\vartheta')\vartheta C$$
$$B = (k_1 - m\vartheta^2)C + (b + m\vartheta')\vartheta S - \bar{k}r\,. \tag{7}$$

Let us investigate motion near to the stationary state. Then the right sides and also A, B are small of magnitude $\varepsilon$.

The equations (6) exactly describe the behaviour of the system from Fig. 1 and are fully identical with equations (1). Because an analytical solution is practically impossible, it is advantageous in this case to split the right-hand expressions into the parts containing only members explicitly independent on the angle $\varphi$ and into the parts periodic in $\varphi$.

During this treatment the equivalent linear stiffness $k_e(a_2)$ and damping $b_e(a_2)$ were introduced instead of the nonlinear function $\varepsilon\, f(y, \vartheta\, y')$ according to the relation

$$\varepsilon\, f(y, \vartheta\, y') = k_e(a)y + \vartheta\, b_e(a)y'\,, \tag{8}$$

where

$$k_e(a) = \frac{1}{\pi\, a}\int_0^{2\pi} \varepsilon\, f\big((C\cos\varphi + S\sin\varphi),\, \vartheta(-C\sin\varphi + S\cos\varphi)\big)\cos\varphi\, d\varphi$$

$$b_e(a) = \frac{1}{-\pi\vartheta\, a}\int_0^{2\pi} \varepsilon\, f\big((C\cos\varphi + S\sin\varphi),\, \vartheta(-C\sin\varphi + S\cos\varphi)\big)\sin\varphi d\varphi\,.$$

The amplitude $\alpha$ of the relative motion $y$ is given by

$$a = \sqrt{C^2 + S^2}\,. \tag{8a}$$

The nonlinear function $\varepsilon\, f(y(\varphi), \vartheta\, y'(\varphi))$ of the periodic variables $y, y'$ was also divided into the part of the first harmonic components and the rest $O$ containing the other harmonics

$$\varepsilon\, f\big[(C\cos\varphi + S\sin\varphi),\, \vartheta(-C\sin\varphi + S\cos\varphi)\big] = \tag{9}$$
$$= k_e(a)(C\cos\varphi + S\sin\varphi) + b_e(a)\vartheta(-C\sin\varphi + S\cos\varphi) + O(\cos(n\varphi), \vartheta\sin(n\varphi))_{n\neq 1}\,.$$

For the solution of the equations (6) we shall use the averaging method, based on the assumption that the right-hand terms are proportional to the small parameter $\varepsilon$. Therefore the derivatives $C', S', \vartheta'$ are also small and we can suppose that components $C, S$ of the oscillations change slowly. Then their values during the one period $T = 2\pi/\omega$ can be considered as constant, as well as expressions $A, B$, which are also roughly constant during one period $T$.

By applying the averaging procedure on the eqs. (6), we reduce these equations to a system of three differential equations of the first order in the form

$$C' = \frac{1}{2m_1\vartheta^2}(A + k_e S - b_e\vartheta C)$$
$$S' = \frac{-1}{2m\vartheta^2}(B + k_e C + b_e\vartheta S) \tag{10}$$
$$\vartheta' = \frac{1}{I_1\vartheta}\left\{M_0(\alpha - \vartheta) - \frac{\bar{k}_r}{2}S\right\}.$$

The other terms are rapidly varying terms with the zero average values e.g.

$$\int_0^{2\pi} B\sin 2\varphi d\varphi = 0,\ \int_0^{2\pi} A\cos 2\varphi d\varphi = 0, \ldots\ldots\ldots \text{ etc.}$$

and in the first approximation they can be neglected. The equations (10) describe both the stationary (response curves) and the non-stationary vibrations and can be used also for stability study.

### 3. Stationary forced vibration

The response curves are given by the setting of $C' = S' = \vartheta' = 0$, and $\vartheta = \omega$. For simplicity, we suppose linear system $k_e = 0$, $b_e = 0$.

By means of known procedure we get algebraic equations $A = 0, B = 0$, which can be written in matrix form

$$\begin{bmatrix} k_1 + k_e - m\omega^2 & (b + b_e)\omega \\ -(b + b_e) & k_1 + k_e - m\omega^2 \end{bmatrix} \begin{bmatrix} C \\ S \end{bmatrix} = \begin{bmatrix} \bar{k}_r \\ 0 \end{bmatrix}. \tag{11}$$

The stationary revolution $\omega$ at $a$ given parameter α and $\vartheta' = 0$ can be ascertained from the last equation (10):

$$M_0(\alpha - \vartheta) - \bar{k} r S / 2 = 0. \tag{12}$$

The example of the amplitude response curve $a(\omega)$ for linear ($k_e = 0$, $b_e = 0$) 1DOF system, where $a = \sqrt{C + S}$ is in Fig. 3. By dashed lines are there drawn also responses of components $S(\omega)$ and $C(\omega)$.

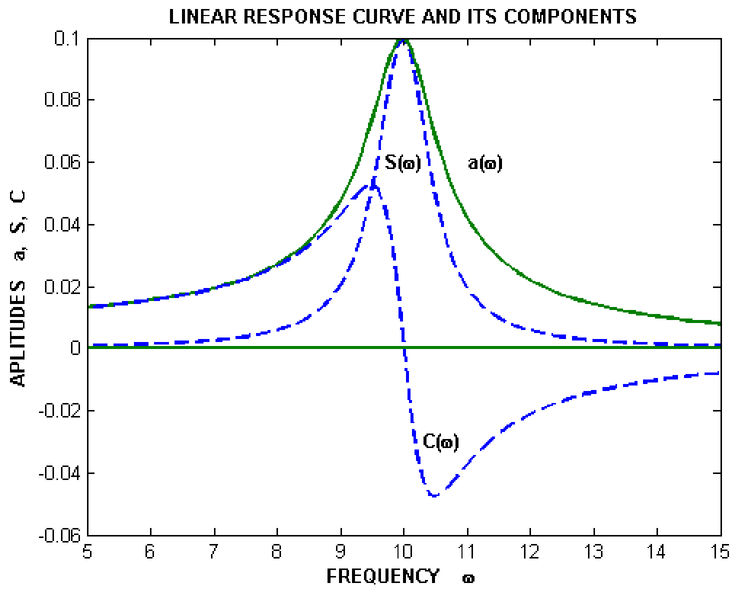


Fig. 3

### 3. Stability

The stationary states of the periodic forced vibrations given by the response curve $a, \omega$ are at non-ideal excitation not stable in all points. The study of stability is based on the perturbation theory. To the equilibrium state at the fixed frequency ω and constant values $C, S$ fulfilling the equations (11) and (12) we add the small perturbations $c, s, \omega_1$ and obtain the perturbed motions:

$$C^* = C + c\,, \quad S^* = S + s\,, \quad \omega^* = \omega + \omega_1\,. \tag{13}$$

After substituting (13) into equations (10), developing it in the series of powers of small quantities $c, s, \omega_1$ and neglecting the powers higher than one, we obtain the linear differential equations in $c, s, \omega_1$

$$2m\omega\dot{x} = \mathbf{L}x \tag{14}$$

$$x = \{c, s, \omega_1\}^T \tag{14a}$$

where the square matrix **L** is of order 3 and contains both the parameters of the system $m, k, k_e(a)$ and the derivatives of the equivalent linear functions $\frac{\partial k_e(a)}{\partial a}, \frac{\partial b_e(a)}{\partial a}$ and also the motor characteristic $M(\vartheta, \alpha)$ and its derivate $\frac{\partial M(\vartheta, \alpha)}{\partial \vartheta}$.

These equations based on the simplificated averaging method where the periodically variable terms are neglected are not quite exact and feasible for all quantitative stability analysis, but are useful for qualitative analysis.

The second and higher approximation must be applied for ascertaining the more exact behaviour in instability domain, but the gained expressions are very complicated and laborious.

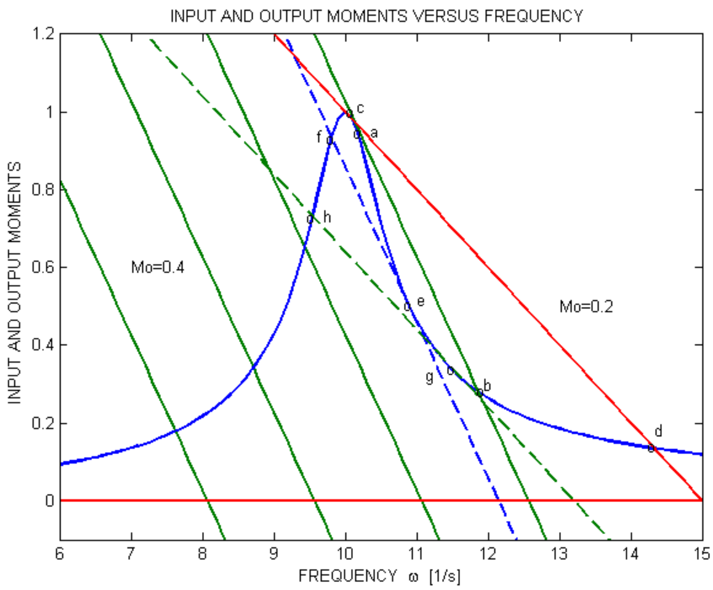


Fig. 4

Another approximate method of determination of stability regions caused by weak excitation is derived from the simple equivalence of input and output moment (or power) of electromechanical exciter described by third equation (10) after substituting values of stationary harmonic oscillations. Graphical representation is in Fig. 4, where the motor moment characteristics $M = M_0(\alpha - \omega)$ are depicted by oblique lines for different α and loading moment from vibration system

$$M = \frac{\bar{k}r}{2} S = \frac{(\bar{k}r)^2}{2} \frac{b\omega}{\sqrt{(k - m\omega^2) + b^2\omega^2}} \tag{15}$$

is expressed by response curves $S(\omega)$. Stability boundaries are given by points of common tangents. At slowly increasing energy input – increasing frequency $\omega$ - with the slope of motor characteristic $M_0$ = 0.4, the jump a→b occur. When the energy input decreases, the back jump occur between points e→f. Section a-e is unstable. Similar, but greater jumps happen at weaker motor $M_0 = 0.2$ between points c→d and back g→h.

This graphical representation of jumps over instability domains does not give any information about the properties of unstable branches (e.g. a‾e, c‾g) of response curve. Character of instability can be ascertained by using equations derived after introducing small perturbations (13).

As examples of application mentioned analysis, the influence of inclination $M_0$ of moment characteristic of driving motor on the exponent $\lambda$ $(c = c_0 e^{\lambda t}, ...)$ is shown in Fig. 5 for values $M_0 = 0.5$; 1; 2 and $I = 1$.

Increasing slope $M_0$ decreases both the unstable range of frequency $\omega$ and the level of instability given by the positive value of characteristic exponent $\lambda$.

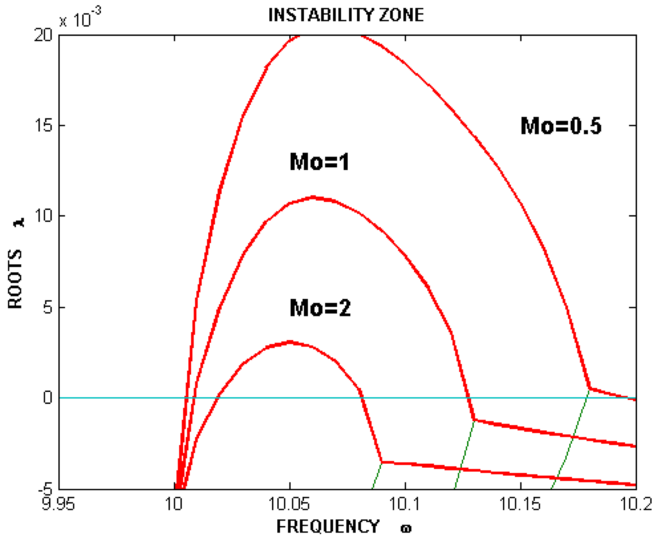


Fig. 5

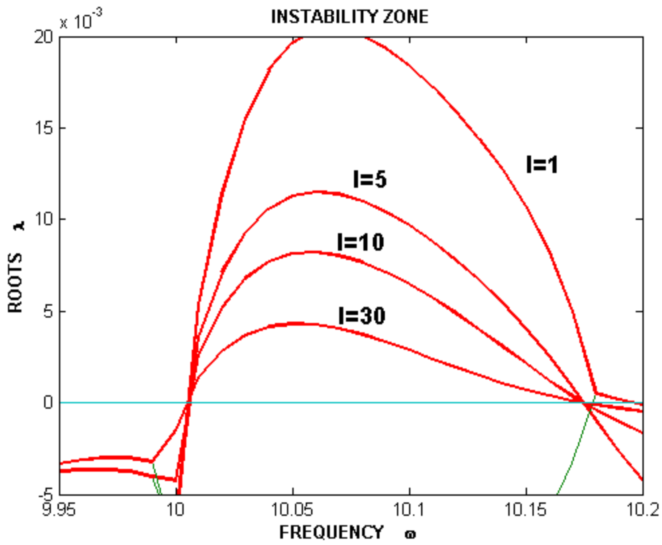


Fig. 6

Influence of moment of inertia $I$ of driving motor is shown in Fig. 6, where at constant value $M_0 = 0.5$ there are plotted characteristic exponents $\lambda$ for four moments of inertia $I = 1$, 5, 10, 30. Magnitude of inertia does not influence the instability range, but has great decreasing effect on the level of instability given by roots $\lambda$.

However these both approximate methods do not enable to find the exact types of unstable motions and of transient processes. Therefore and also for verification of various approximate methods of solution, the experimental research of vibrating system is now prepared in IT ASCR.

**4. Experimental investigation of interaction of energy source with vibrating system**

For completation, verification and improvement of analytical studies of non-ideal excitation, a simple experimental equipment was designed, produced and put into preliminary service in laboratory of IT-ASCR [13].

This experimental set consists in its basic form of one-DOF mechanical subsystem (mass 1,2 kg supported by two steel leaf springs, eigenfrequency 18,4 Hz) connected by a crank mechanism with DC electric motor. Moment characteristic of this motor, fed by a prescribed voltage function, can be set up as weak or hard excitation source in some range.

Driving motor with the crank mechanisms and thin leaf spring for excitation of mechanical vibrating subsystem is shown in Fig. 7.

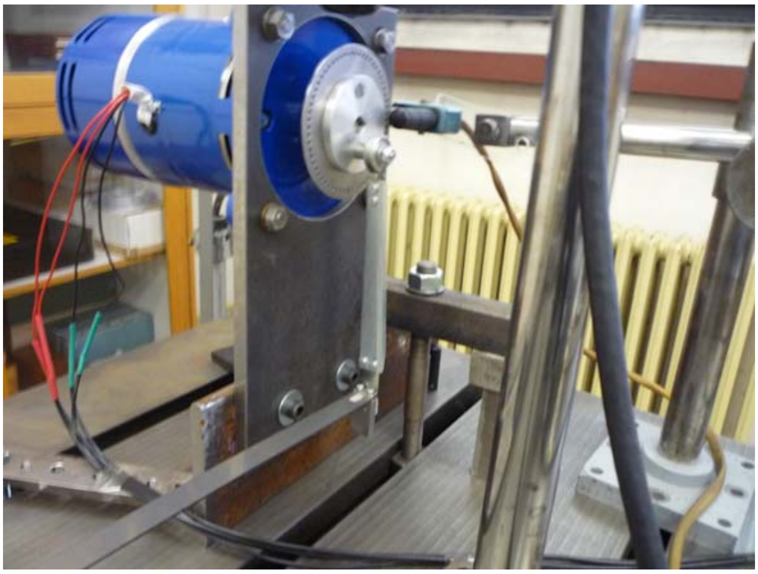

Fig. 7

Examples of selected types of moment characteristics of two electric motors, measured in VUT Brno, will be also mentioned at presentation.

**5. Conclusion**

The object of analytical investigations presented in the paper is a mechanical vibrating system excited through a crank mechanism by an electric motor with limited inertia and limited source of energy.

After transformation independent variables in equations of motion, the expressions for stationary and nonstationary motions were derived.

Unstable sections of response curve were ascertained by means of equivalence of averaged input and output moments of exciter and forced system.

It is shown that the two main parameters of limited source of driving energy, i.e. declination of moment characteristic $M_0$ and moment of inertia $I$ have great effect on the range and level of instability in the unstable section of response curve.

For verification of mentioned approximate methods of solution, the experimental equipment is prepared in laboratory of IT ASCR.

## 6. Acknowledgement

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