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MODELING FLOW OF POROUS MEDIUM WITH COULOMB FRICTION ON WALLS OF DUCTS

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Summary: *The paper deals with development of a computational model describing oil expression from oil-containing seeds. The principles of compression in the rotating screw pressing machine are based on interplay between the screw rotor geometry, the related kinematics and the friction forces acting on the surfaces of the rotor and stator of the machine. We consider a poroelastic biphasic material of the Biot type. The deformation is described using a combination of the spatial and material formulations. Within one time increment step, the physical fields involved in the problem are analyzed using the Lagrangian configuration; in order to compute the next time-increment, this configuration is updated using the* upwind *numerical scheme. Such a treatment requires a projection of the data between two non-matching FE meshes. On the rotor/stator surfaces the compressed material is subject to the Coulomb friction conditions, which make each incremental subproblem strongly non-linear. For numerical solution of this problem we employ frictional multipliers and we adopt the semi-smooth Newton method. The model is already implemented in our in-house developed code SfePy.*

1. Introduction

In this paper we introduce a computational time-incremental scheme which is being developed for its application in modeling compression of fluid saturated porous medium (FSPM) in a duct with taking into account the Coulomb friction conditions on walls of the duct. Such modeling is motivated by the technology of oil expression from oil-containing seeds which is based on propelling the material through a compression chamber of the screw compressor, see Fig. 1. This chamber is formed by the space between the rotor (a conical screw) and the cylindrical stator whose surface is perforated to allow for leakage of the interstitial oil expressed from the seeds.

The problem under consideration is featured by the following major difficulties:

- two-phasic material of the solid nature and interstitial fluid flow in the pores;
- plastic deformation and significant change of the permeability of the compressed material;

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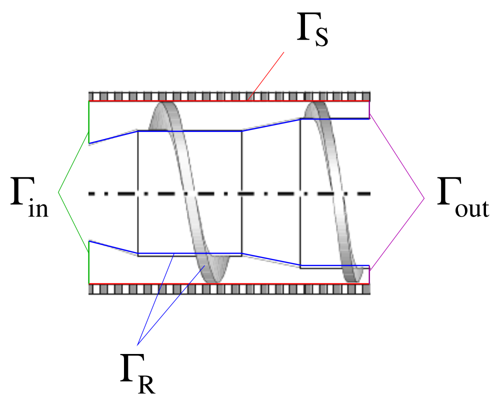

Figure 1: Boundary segments of the duct.

- continuous flow of the material in the duct, i.e. flow of solid deforming skeleton with superimposed diffusion of the interstitial fluid;
- friction between the material and the duct walls – this phenomenon is crucial to describe the driving forces of the material transport in the duct and to simulate genuinely the compression process.

In the present paper we shall not consider the plastic deformation. Here the main focus is on:

- describing the material flow using an incremental formulation which is based on combining the Lagrangian and Eulerian description,
- including the slip/stick conditions on the duct walls.

2. Problem description

In Fig. 1 we display the geometry of the duct bounded by four types of surfaces. The duct is represented by domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$ decomposed, as follows (up to junction lines with zero surface measure):

$$\partial\Omega = \Gamma_S \cup \Gamma_R \cup \Gamma_{\text{in}} \cup \Gamma_{\text{out}} \,, \tag{1}$$

where Γ_S is the stator part of the duct, Γ_R is the "rotor" part of the duct, Γ_{in} and Γ_{out} are the input and output surfaces of the duct, respectively. Since the duct presents the space on the rotor shaft of the compression machine, the reference configuration is rotating with the angular speed of the rotor. The material in the duct rotates with the same revolutions whereby the material particles are translated between the input and output surfaces while being compressed.

2.1. Linear model of the FSPM – Biot model

We assume quasi-static loading of the FSPM occupying domain $\Omega \subset \mathbb{R}^3$. Within a small deformation increment during time step τ the FSPM is assumed to obey the Biot type model (cf. (Rohan et al2008)) with prestressed initial state characterizing the reference configuration. The displacement increment $\boldsymbol{u}$ w.r.t. reference configuration and interstitial fluid pressure increase

$p - p^0$ satisfy at any time level $t + \tau$ the force equilibrium and mass conservation equations imposed in Ω,

$$
\begin{aligned}
-\frac{\partial}{\partial x_j}\left[\left(K - \frac{2}{3}\mu\right) e_{kk}(\boldsymbol{u})\delta_{ij} + 2\mu e_{ij}(\boldsymbol{u}) - b\delta_{ij}(p - p^0) + S^0_{ij}(F^0, p^0)\right] = 0\ , \\
b e_{kk}(\boldsymbol{u}) - \tau\kappa\nabla^2 p + \frac{1}{M} p = 0\ ,
\end{aligned}
\tag{2}
$$

where K, M, μ are elastic constants, b is the Biot poroelastic coefficient, κ the hydraulic permeability, $e_{ij}(\boldsymbol{u})$ the linear strain, $S_{ij}(F^0, p^0)$ is the total stress at the reference configuration Ω (further characterized "point-wise" by total deformation F^0 and pressure p^0) which is being updated after computing $(\boldsymbol{u}, p)$ at each τ-time increment.

2.2. Application of the linear model for the continuous process modelling

We are dealing with compression flow of the FSPM in duct Ω. In contrast with Newtonian fluids, where the stress depends on the actual pressure and velocity fields, in our case the state of FSPM at a given material point depends on the history of deformation. Therefore we propose to use a combination of the Lagrangian and Eulerian formulations. The idea of modeling the continuous flow is based on constructing a sequence (in time) of updated configurations $\{\mathcal{C}(t)\}_t$ such that $\mathcal{C}(t + \tau) = \Pi^\tau \circ \mathcal{C}(t)$. The map Π^τ is constructed as follows.

1. given reference configuration at t, $\mathcal{C}(t) = \{\Omega^0, S^0(t), F^0(t)\}$, where Ω^0 is the reference (to be updated afterwards) domain, S^0 is the initial second Piola-Kirchhoff (2PK) stress and F^0 is the deformation.

2. In order to compute S^τ, the 2PK stress (related still to Ω^0) and the deformation "increment" $f^\tau_{ij} = \partial u_i / \partial X_j$, where $X \in \Omega^0$ and $u = u(\tau, X)$ the relative displacement w.r.t. Ω^0 during time increment τ, one uses the Biot linearized model (2), where the effective stress related to $\mathcal{C}(t)$ is

$$
S^\tau_{ij} = S^0_{ij} + C_{ijkl} e_{kl}(u(\tau)) - b_{ij}(p(t + \tau) - p(t))\ , \tag{3}
$$

   where C_{ijkl} is the tangent stiffness tensor depending on K and μ, b_{ij} is the Biot poroelastic tensor (both C_{ijkl} and b_{ij} possibly dependent on deformation $F_{ij}(t)$) and S^0_{ij} is given by the previous time level computation, see (8) below.

3. The boundary conditions are prescribed on the input boundary Γ_{in}, on the output part Γ_{out} and on the walls in the duct $\Gamma_S \cup \Gamma_R$. We shall list the particular types of boundary conditions prescribed on these boundary segments.

   - **input**: Dirichlet conditions on Γ_{in} w.r.t. $\boldsymbol{u}$ (given velocity $\bar{\boldsymbol{u}}/\tau$), zero-Neumann conditions w.r.t. p (no flow of the interstitial fluid),

$$
\begin{aligned}
\boldsymbol{u} = \bar{\boldsymbol{u}}/\tau \quad \text{on } \Gamma_{\text{in}}\ , \\
\boldsymbol{n} \cdot \boldsymbol{w} = -n_i \kappa_{ij} \partial_j p = 0 \quad \text{on } \Gamma_{\text{in}}\ ,
\end{aligned}
\tag{4}
$$

- **output**: Neumann conditions w.r.t. displacement $\boldsymbol{u}$ (zero stress σ_{ij} projected to the normal), Dirichlet conditions w.r.t p (zero pressure)

$$\begin{aligned} n_j \sigma_{ij} = n_i \sigma_1 = 0 \quad \text{on } \Gamma_{\text{out}} \ , \\ p = p_1 = 0 \quad \text{on } \Gamma_{\text{out}} \ , \end{aligned} \tag{5}$$

- **walls**: the Coulomb friction condition is considered there, which governs the tangential material velocity and the tangential traction forces using the "stick/slip", as described in detail in Section 3.; for pressure the Neumann or Newton conditions are considered: non-penetration of the solid skeleton, fluid can penetrate on Γ_S only:

$$\begin{aligned} \boldsymbol{n} \cdot \boldsymbol{w} = -n_i \kappa_{ij} \partial_j p = 0 \quad \text{on } \Gamma_R \ , \\ \boldsymbol{n} \cdot \boldsymbol{u} = 0 \ , \quad \text{on } \Gamma_R \ , \end{aligned} \tag{6}$$

  drainage through sieve of the stator part

$$\begin{aligned} \boldsymbol{n} \cdot \boldsymbol{w} = -n_i \kappa_{ij} \partial_j p = \varkappa p \quad \text{on } \Gamma_S \ , \\ \boldsymbol{n} \cdot \boldsymbol{u} = 0 \ , \quad \text{on } \Gamma_S \ , \end{aligned} \tag{7}$$

4. **Stress and deformation updating procedure.** Given $\mathcal{C}(t)$, the reference configuration, and relative displacements $u_i(\tau, X)$ and total interstitial pressure $p(t+\tau)$ provided by the linearized model, we compute:

$$\begin{aligned} \text{deformation} \quad & F_{ij}(t+\tau, X) = f_{ik}(\tau) F_{kj}(t) \ , \\ \text{updated Cauchy stress:} \quad & \sigma_{ij}(t+\tau, x) = \det(f)^{-1} f_{ik} S_{kl} f_{jl} \ , \\ & \text{where } x_i = X_i + u_i(\tau) \ , \end{aligned} \tag{8}$$

   and $S_{kl}(t+\tau, X)$ is given by (3).

5. Updated configuration $\mathcal{C}(t+\tau)$:

$$\begin{aligned} \Omega(t+\tau) &:= \Omega(t) + \{u(\tau, X)\} \ , \quad X \in \Omega(t) \\ S^0(x) &:= \sigma(t+\tau, x) \ , \quad x \in \Omega(t+\tau) \cap \Omega^0 \ , \\ F^0(x) &:= F_{ij}(t+\tau, X) \ . \end{aligned} \tag{9}$$

   In the input sector $\Omega_{\text{in}} = \{y \in \Omega^0 | \ y = X + \tau' \dot{u}(X) \ , X \in \Gamma_{\text{in}}, 0 < \tau' < \tau\}$ the stress and deformation must be defined. There are several possible ways how to proceed, we use one of the simplest ones: we consider zero initial stress and deformation in Ω_{in}

   - on Γ_{in} – the same conditions are considered, as for the "τ-increment" problem in whole Ω^0.
   - on $\Gamma_{\text{in}} + \tau\{\dot{u}\}_{\Gamma_{\text{in}}}$ (shifted input) – trace of the total Cauchy stress and pressure (fluid) from $\Omega(t+\tau)$, as computed when solving the "τ-increment" problem.
   - on walls – the same conditions as for the "τ-increment" problem in whole Ω^0.

   This method is motivated by the *steady state solution*.

3. Coulomb friction conditions

Let us consider a general situation, when the sliding/sticking conditions are prescribed on $\Gamma_c \subset \partial\Omega \subset \mathbb{R}^3$; so in the problem treated in this paper $\Gamma_c = \Gamma_S \cup \Gamma_R$. We consider the contact pressure (negative stress, scalar) $\sigma_n \geq 0$ which depends on the state variables of the continuum (it can be approximated using the "given stress" computed in configuration $\mathcal{C}(t)$). The following friction conditions must be satisfied a.e. on Γ_c:

$$\begin{cases} |\boldsymbol{\tau}| - f_c\sigma_n & \leq 0 \quad \text{friction cone} \\ -\lambda & \leq 0 \quad \text{"sliding switch"} \\ \lambda(|\boldsymbol{\tau}| - f_c\sigma_n) & = 0 \quad \text{complementarity} \\ \boldsymbol{u}_t - \boldsymbol{u}_c + \lambda\boldsymbol{\tau} & = 0 \quad \text{slip-drag co-linearity} \\ \boldsymbol{u}_t \cdot \boldsymbol{n} & = 0 \quad \text{contact / no penetration} \end{cases} , \tag{10}$$

where $\boldsymbol{\tau}$ is the friction traction, λ is the friction multiplier, f_c is the friction coefficient which may depend on the state variables (locally at Γ_c), $\boldsymbol{u}_c/\tau$ is the convection velocity on Γ_c and $\boldsymbol{u}_t$ is the displacement vector tangent to Γ_c ($\boldsymbol{n}$ is the normal vector in the last condition). The convection velocity is the velocity of Γ_c w.r.t. the reference configuration. In our application characterized by the boundary decomposition (1), $\boldsymbol{u}_c = 0$ on $\partial\Omega \setminus \Gamma_S$, while on Γ_S the rotation of the reference configuration (associated with the rotor of the compression machine) gives $\boldsymbol{u}_c = \tau R\omega \times \boldsymbol{n}$, where R is the radius of the rotor, ω is the angular velocity vector and $\boldsymbol{n}$ is the unit normal vector.

$\mathbf{u}$	displacements column vector
$\mathbf{p}$	fluid pressure column vector
$\mathbf{A}$	stiffness matrix (sym. pos. def.)
$\mathbf{K}$	permeability matrix (sym. pos. def.)
$\mathbf{Q}$	matrix associated with the poroelasticity coefficients
$\mathbf{M}$	fluid compressibility matrix
$\mathbf{B}$	slip-stress projection matrix
$\mathbf{C}$	stress projection matrix

Table 1: Matrix notation employed in (11).

Matrix formulation of the discretized time increment problem. The FE discretized problem involving equilibrium and mass conservation equations (2) and friction conditions (10) attains the form of a non-linear system of algebraic equations. At each time step we find $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{p} \in \mathbb{R}^p$ and $\boldsymbol{\lambda}, \mathbf{g} \in \mathbb{R}^m$ such that (see Tab. 1 for notation)

$$\begin{aligned} \mathbf{Au} - \mathbf{Q}^T\mathbf{p} - \bar{\mathbf{B}}^T\mathbf{g} &= \mathbf{f} \,, \\ \mathbf{Qu} + (\tau\mathbf{K} + \mathbf{M})\mathbf{p} &= 0 \,, \\ \mathbf{B}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_c) + \boldsymbol{\Lambda}\mathbf{C}\mathbf{g} &= 0 \,, \\ \max\{\hat{\mathbf{D}}\hat{\mathbf{g}} + f_c\hat{\mathbf{q}}\,| \; - \boldsymbol{\lambda}\} &= 0 \,, \end{aligned} \tag{11}$$

where $\hat{\mathbf{q}} := (\hat{\mathbf{A}}\mathbf{u} - \hat{\mathbf{Q}}^T\mathbf{p} - \hat{\mathbf{f}})$ represents the normal-projected traction forces on Γ_c (computed as the equilibrium residuum in the FE-discretized form) and $\bar{\mathbf{u}}_c$ are the convection displacements involved in $(10)_4$. It is worth noting that $\hat{\mathbf{q}} \leq 0$ is assumed for the physical relevance (otherwise

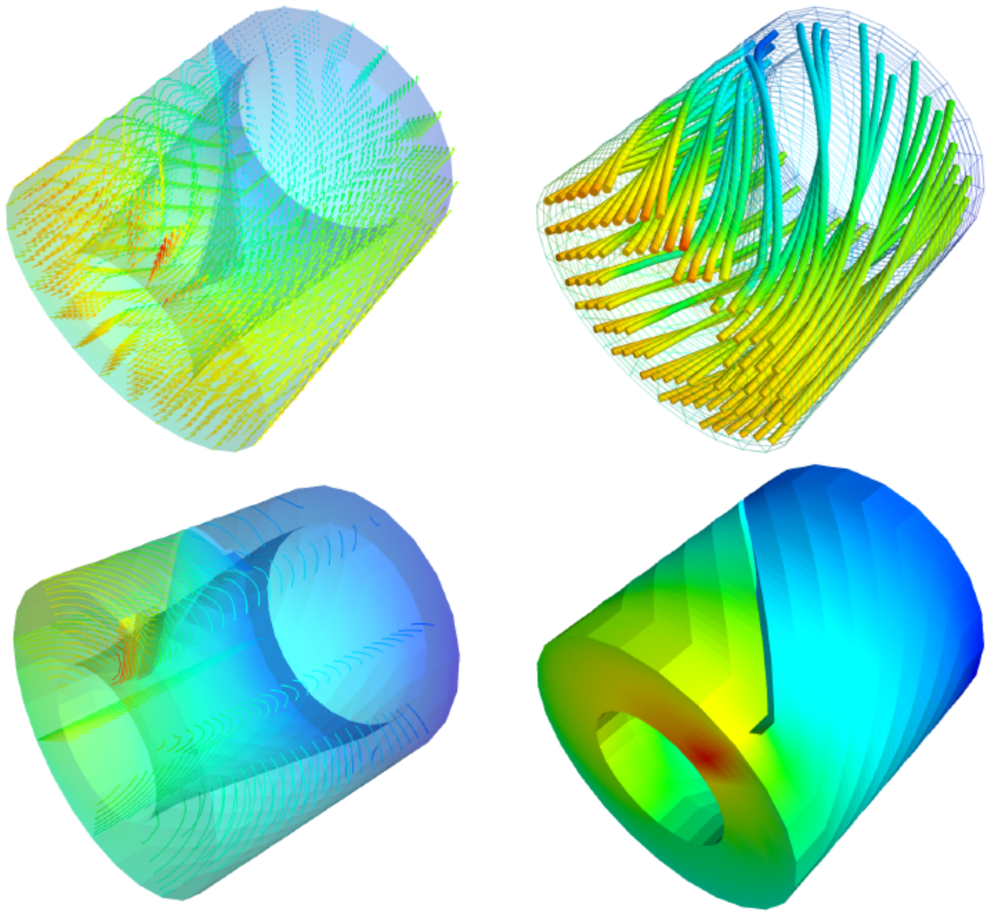

Figure 2: Illustration of the oil expression simulation. The computational domain Ω presents a segment of the screw machine. Above: relative displacements $\boldsymbol{u}$ (left) and reconstructed streamlines (right). Bottom: oil pressure distribution.

unilateral contact condition would have to be treated). To obtain numerical solutions of (11) we use the semismooth Newton method, cf. (Rohan and Whiteman 2000), the model reported in this paper was implemented in our in-house developed FEM code SfePy, (Cimrman et al2010).

In this paper we proposed a numerical scheme for simulation of flow of the fluid saturated porous material in a duct with slip/stick conditions on the walls. Numerical results and details related to the model application in simulating the oil expression in the screw compression machine will be discussed in a forthcoming publication. In figure 2 we introduce just an illustration of the oil expression process simulated with simplified friction slip conditions.

4. Acknowledgment

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5. References

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