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# UNCERTAIN MODELING OF MECHATRONIC SYSTEMS

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**Abstract:** *The paper proposes an approach to modeling of systems with a parametric uncertainty. The approach is based on use of upper linear fractional transformation which is commonly used for the modeling of the uncertain systems. The main difference between the classical methods and the proposed one is that the proposed method describes the uncertainty as a difference between values of corresponding matrices elements of nominal and perturbed state-space model of the described system. This result into a general analytic description of the uncertain model which may be easily used for building of an uncertain model based on a nominal and perturbed state-space model of any system. The obtained uncertain model respects the form for a robust controller design. The approach is presented on simple mass-damper-spring system.*

**Keywords:** ***Parametric uncertainty, uncertain model, mass-damper-spring.***

## 1. Introduction

The present tendencies in the area of modeling of the mechatronic systems are mostly concentrated on building of a model as precise as possible. However building of a precise model is not always possible. This might be caused by change of some of system parameters during the action of the system or just simply by inaccurate or incomplete information about the system, neglected nonlinearities, etc. These inaccuracies or deviations from the reality may be described by introducing an uncertainty to the model. Such a model is then established by its nominal parameters which may vary within a certain range given by the uncertainty.

The standard methods of the parametric uncertainty modeling (Petkov et al., 2002, 2008) are typically based on **M** - Δ configuration (Fig. 1). The **M** matrix represents the augmented model obtained by the upper linear fractional transformation (ULFT) (Safonov, 1982) and Δ matrix represents a diagonal perturbation matrix.

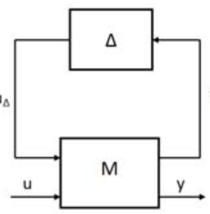


*Fig. 1: The standard* ***M*** *- Δ configuration for a model with a parametric uncertainty.*

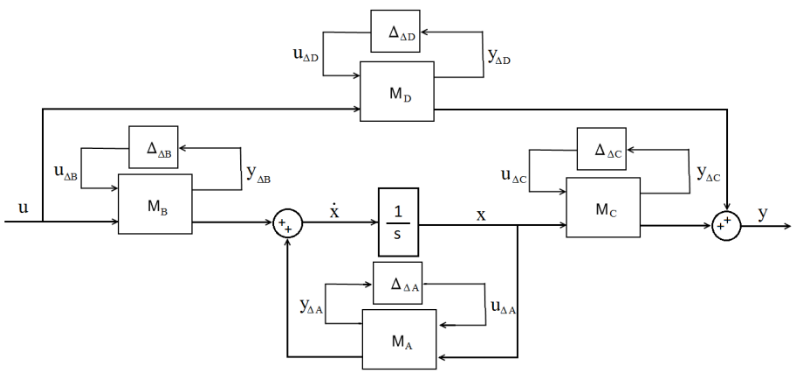


*Fig. 2: Scheme of particular transfer function matrices.*

The matrix **M** is typically constructed individually for every uncertain parameter in the model (Gu, 2005). Let's note that the ULFT must be then also used individually. The obtained matrices are then

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combined into a compact uncertain model. This is very uncomfortable in cases where is needed a high number of uncertain parameters in the model. The advantage of the proposed approach is that the uncertain model is constructed analytically in the most general form. Such a general description may be consequently easily used for building of an uncertain model of any system without need for precedent design steps typical for constructing of uncertain models (ULFT, etc.).

The possibility of the uncertain modeling is also offered by commercially available tool Robust Control Toolbox in Matlab with functions ‘ureal’, ‘umat’ and ‘uss’. The crucial disadvantage of these functions is that even the application of one of them transforms the standard ‘ss’ (state-space) Matlab model into an ‘uss’ (uncertain state-space) form. Such a transformed model is suitable for simulations but it is impossible to use it for a robust controller design (e.g. ‘hinfsyn’ function in Matlab).

It was already mentioned that proposed method leads to a general form of the uncertain model in the analytic form. The modeling of the system as uncertain then requires the knowledge of the nominal and perturbed model in state-space form only. The advantage of the method is the possibility of its application to models of high orders where modeling of the uncertainty individually for desired parameters might be demanding or for models where the inner structure of the model is not exactly known (for example models obtained by identification). Obtained model also respects the form for a robust controller design in Matlab.

## 2. Modeling of a system with parametric uncertainty

The approach to the modeling of the Stewart platform with a parametric uncertainty based on state-space models of the system was described in (Březina, L. & Březina, T., 2010). Now the more general approach will be presented. The basic idea is to determine the difference between particular values of state matrices elements of a nominal system and a maximally perturbed system. Then the uncertain model is obtained by upper linear fractional transformation.

The nominal system is described as

$$\begin{aligned}\dot{\mathbf{x}} &= \bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{B}}\mathbf{u} \\ \mathbf{y} &= \bar{\mathbf{C}}\mathbf{x} + \bar{\mathbf{D}}\mathbf{u}\end{aligned} \tag{1}$$

and the maximally perturbed system as

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned} . \tag{2}$$

The meaning of the particular terms is following:

$\mathbf{x}$ represents the vector of states, $\dot{\mathbf{x}}$ represents the vector of the time derivations of the states, $\mathbf{u}$ is the vector of inputs, $\mathbf{y}$ is the vector of outputs, matrices $\bar{\mathbf{A}},\bar{\mathbf{B}},\bar{\mathbf{C}},\bar{\mathbf{D}}$ represent state matrices of the nominal system and finally $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ represent the state matrices of the perturbed system.

State matrices of the perturbed system (2) may be also defined as

$$\mathbf{A} = \bar{\mathbf{A}} + \mathbf{A}_\Delta , \tag{3}$$

the uncertainty contribution is then $\mathbf{A}_\Delta = \mathbf{A} - \bar{\mathbf{A}}$. There are similarly derived $\mathbf{B}_\Delta$, $\mathbf{C}_\Delta$ and $\mathbf{D}_\Delta$.

The upper linear fractional transformation is described as

$$\mathbf{F}_u\left(\mathbf{M},\boldsymbol{\Delta}_u\right) = \mathbf{M}_{22} + \mathbf{M}_{21}\boldsymbol{\Delta}_u\left(\mathbf{I} - \mathbf{M}_{11}\boldsymbol{\Delta}_u\right)^{-1}\mathbf{M}_{12} . \tag{4}$$

It is then obtained $\mathbf{M}_{21}\mathbf{M}_{12} = \mathbf{A}_\Delta$, $\mathbf{M}_{11} = \mathbf{0}$, $\mathbf{M}_{12} = \mathbf{I}$, $\mathbf{M}_{21} = \mathbf{A}_\Delta$ and $\mathbf{M}_{22} = \bar{\mathbf{A}}$ by comparing (4) with (3). The approach is same also for other state matrices.

The interconnection transfer function matrix is then $\mathbf{M} = \begin{bmatrix}\mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22}\end{bmatrix}$.

Consequently, according to the scheme of the system containing particular transfer function matrices (Fig. 2), it is obtained

$$\begin{bmatrix} \mathbf{y}_{\Delta A} \\ \dot{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{A}_\Delta & \overline{\mathbf{A}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\Delta A} \\ \mathbf{x} \end{bmatrix}, \begin{bmatrix} \mathbf{y}_{\Delta B} \\ \dot{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{B}_\Delta & \overline{\mathbf{B}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\Delta B} \\ \mathbf{u} \end{bmatrix}, \tag{5}$$

$$\begin{bmatrix} \mathbf{y}_{\Delta C} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{C}_\Delta & \overline{\mathbf{C}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\Delta C} \\ \mathbf{x} \end{bmatrix}, \begin{bmatrix} \mathbf{y}_{\Delta D} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{D}_\Delta & \overline{\mathbf{D}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\Delta D} \\ \mathbf{u} \end{bmatrix}, \tag{6}$$

where $\mathbf{u}_{\Delta A}, \mathbf{u}_{\Delta B}, \mathbf{u}_{\Delta C}, \mathbf{u}_{\Delta D}$ represents inputs to the perturbation matrices $\mathbf{\Delta}_{\Delta A}, \mathbf{\Delta}_{\Delta B}, \mathbf{\Delta}_{\Delta C}, \mathbf{\Delta}_{\Delta D}$, $\mathbf{y}_{\Delta A}, \mathbf{y}_{\Delta B}, \mathbf{y}_{\Delta C}, \mathbf{y}_{\Delta D}$ are then outputs from the perturbation matrices.

The compact form of the matrix representation of the general uncertain model with the global perturbation matrix is then

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{y}_{\Delta A} \\ \mathbf{y}_{\Delta B} \\ \mathbf{y}_{\Delta C} \\ \mathbf{y}_{\Delta D} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{A}} & \mathbf{A}_\Delta & \mathbf{B}_\Delta & \mathbf{0} & \mathbf{0} & \overline{\mathbf{B}} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \overline{\mathbf{C}} & \mathbf{0} & \mathbf{0} & \mathbf{C}_\Delta & \mathbf{D}_\Delta & \overline{\mathbf{D}} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{u}_{\Delta A} \\ \mathbf{u}_{\Delta B} \\ \mathbf{u}_{\Delta C} \\ \mathbf{u}_{\Delta D} \\ \mathbf{u} \end{bmatrix}, \begin{bmatrix} \mathbf{u}_{\Delta A} \\ \mathbf{u}_{\Delta B} \\ \mathbf{u}_{\Delta C} \\ \mathbf{u}_{\Delta D} \end{bmatrix} = \begin{bmatrix} \mathbf{\Delta}_{\Delta A} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Delta}_{\Delta B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Delta}_{\Delta C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{\Delta}_{\Delta D} \end{bmatrix} \begin{bmatrix} \mathbf{y}_{\Delta A} \\ \mathbf{y}_{\Delta B} \\ \mathbf{y}_{\Delta C} \\ \mathbf{y}_{\Delta D} \end{bmatrix}. \tag{7), (8}$$

Let's note that $\mathbf{-I} \le \mathbf{\Delta}_{\Delta A, \Delta B, \Delta C, \Delta D} \le \mathbf{I}$ for the symmetrical +/- perturbation of the uncertainty around the nominal value.

The typical form of an uncertain model for a robust controller design is according to (Gu, 2005) following

$$\mathbf{G} = \begin{bmatrix} \mathbf{A}_v & \mathbf{B}_{v1} & \mathbf{B}_{v2} \\ \mathbf{C}_{v1} & \mathbf{D}_{v11} & \mathbf{D}_{v12} \\ \mathbf{C}_{v2} & \mathbf{D}_{v21} & \mathbf{D}_{v22} \end{bmatrix}. \tag{9}$$

The form (9) corresponds with (7) for $\mathbf{A}_v = \overline{\mathbf{A}}$, $\mathbf{B}_{\mathbf{v1}} = [\mathbf{A}_\Delta \quad \mathbf{B}_\Delta \quad \mathbf{0} \quad \mathbf{0}]$, $\mathbf{B}_{v2} = \overline{\mathbf{B}}$, $\mathbf{C}_{\mathbf{v1}} = [\mathbf{I} \quad \mathbf{0} \quad \mathbf{I} \quad \mathbf{0}]^T$, $\mathbf{C}_{v2} = \overline{\mathbf{C}}$, $\mathbf{D}_{v11} = [\mathbf{0}]$, $\mathbf{D}_{v12} = [\mathbf{0} \quad \mathbf{I} \quad \mathbf{0} \quad \mathbf{I}]^T$, $\mathbf{D}_{\mathbf{v21}} = [\mathbf{0} \quad \mathbf{0} \quad \mathbf{C}_\Delta \quad \mathbf{D}_\Delta]$.

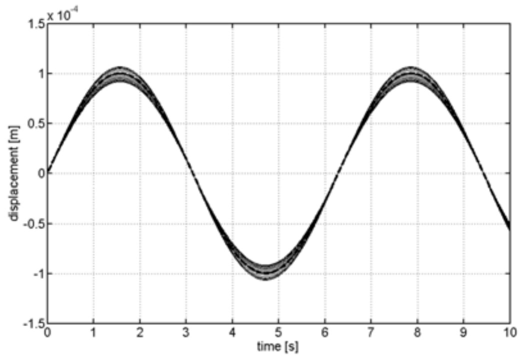
### 3. Example – Mass-Damper-Spring system

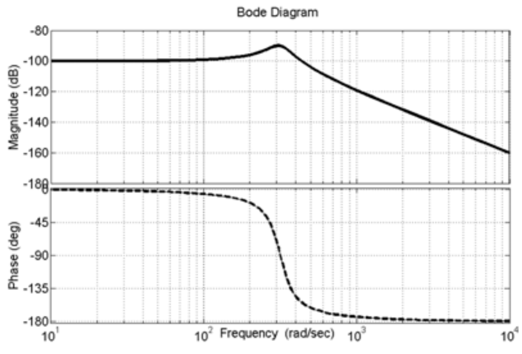
The example represents simple mass – damper – spring system which is in the nominal state-space form for $x_1 = x$ and $x_2 = \dot{x}$ described by matrices $\overline{\mathbf{A}}, \overline{\mathbf{B}}, \overline{\mathbf{C}}, \overline{\mathbf{D}}$ as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{\overline{k}}{\overline{m}} & -\dfrac{\overline{b}}{\overline{m}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{\overline{m}} \end{bmatrix} f, \; y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0] f. \tag{10}$$

The perturbed model defined by matrices $\mathbf{A,B,C,D}$ then contains parameters $m = \overline{m}(1+\delta_m)$, $b = \overline{b}(1+\delta_b)$, $k = \overline{k}(1+\delta_k)$ substituted to (10) instead of nominal parameters. There are consequently obtained matrices $\mathbf{A}_\Delta$, $\mathbf{B}_\Delta$, $\mathbf{C}_\Delta$ and $\mathbf{D}_\Delta$ according to (3). The compact uncertain model of the system and the perturbation matrix are then obtained according to (7) and (8).

The following results of the uncertain model behavior were obtained for the values of the nominal parameters $\overline{m} = 1kg$, $\overline{b} = 100Ns/m$, $\overline{k} = 1.10^5 N/m$. The perturbed model was obtained for the uncertainty 10% of each of the nominal values, i.e. $\delta_m, \delta_b, \delta_k = 0{,}1$. There were done twenty random samples of the model (i.e. $-\mathbf{I} \le \mathbf{\Delta}_{\Delta A,\Delta B,\Delta C,\Delta D} \le \mathbf{I}$) for the input force $f = F_0 \sin(\omega t)$, $F_0 = 10N$ and $\omega = 1 rad/s$.




*Dashed line for the nominal model, solid line for the uncertain model realizations.*

*Fig. 3: Random outputs from the uncertain model for the given output.*

*Fig. 4: Bode diagrams for the random realizations of the uncertain model.*

It may be easily determined that initial difference 10% of the nominal parameters causes for $\mathbf{\Delta}_{\Delta A,\Delta B,\Delta C,\Delta D} = \pm\mathbf{I}$ approximate maximal difference ±6,6% of the nominal output, Fig. 3. There is no significant change in the behavior in the frequency domain, Fig. 4.

## 4. Conclusions

The paper proposes a method for constructing of uncertain models which is based on obtaining of general uncertain model description in the analytic form. Once the general model is obtained by standard methods for modeling with the parametric uncertainty it may be used for constructing of any state-space model as uncertain. It is required knowledge of the nominal and perturbed model only. The method is suitable for models of high orders where modeling of the uncertainty for individual parameters might be very demanding or models with not precisely known structure (e.g. obtained from the identification). The advantage is its high versatility and the fact that obtained uncertain model respects the form for a robust controller design which might be a problem for commercial tools. The disadvantage of the method may be slightly wider uncertainty zone than necessary. This is for systems with nonsymmetrical balancing of matrices elements values in perturbed systems (with maximal and minimal uncertainty) around the corresponding values in the nominal system. The method was presented for sake of simplicity on a mass-damper-spring system.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