17th International Conference
ENGINEERING MECHANICS 2011
Svratka, Czech Republic, 9 – 12 May 2011

SHEAR STRESS ON ARBITRARY CROSS SECTION INCLUDING PLASTICITY

J. Kabeláč*

Abstract: *In this paper a method for determination of shear stress distribution over arbitrary cross section is described. Cross section is loaded by any combination of shear and torsion. Other inner forces are not considered at this presented method. PDE are formulated for this problem, which are transformed by variational principle to the deformational variant of the finite element method. The solution of overall internal forces and stiffness characteristic of cross section including plasticity for using in a beam element model is suggested in this paper.*

Keywords: ***Cross section, finite element method, plasticity, shear stress.***

1. Introduction

The motivation for the presented work is development of beam element with material nonlinearity. At the present time, a beam element with material nonlinearity for uniaxial stress is implemented in many of commercial FEM softwares. Its formulation is simple. Cross section is divided into particular tensile fibers with plasticity conditions. This element well describes plastic properties of cross section at tension and bending, but it does not respect the effect of shear and torsion. But in case of short or twisted beam it cannot be omitted in respect to their real plasticity behavior.

Formulation of beam element is based on stress distribution over the cross section and subsequent behavior of the cross section in response to beam deformation. Only nonlinear material behavior of cross section in response to a twist and skewness is assumed and condition of free warping is supposed. Other loads components are not considered. This simplified formulation is presented in this paper.

Deformation variant of FEM is used for solution of behavior of shear stress over cross section. The arguments of this use are discussed at the end of the paper. The primary function that is investigated is cross section warping and this problem is transformed by variational methods on solution by finite element method. The method is founded on the article by Gruttmann & Wagner (2001), where cross sections plasticity capacity in torsion is discussed. This method is extended to general combination of shear loading of cross section with possibility of finding out tangential stiffness of cross section including plasticity.

2. Method

For solution of shear loading of cross section the following simplifying assumptions are accepted:

– Beam is straight, prismatic and it is loaded only by shear and torsion.
– Overall rotations about y and z axis and displacement of cross section in beam axis x are zeros.
– On the basis of the assumptions 1. and 2. it can be stated that there are only shear strains γ_{xy} , γ_{xz} and thus there is shear stress τ_{xy} , $\tau_{xz.}$ in the cross section. The other members of stress and strain tensors are zeros.
– Cross section warping is free.
– Deformations are small.

* Ing. Jaromír Kabeláč: FEM Consulting, Veveří 331/94; 602 00, Brno, CZ, e-mail: jaromir.kabelac@seznam.cz

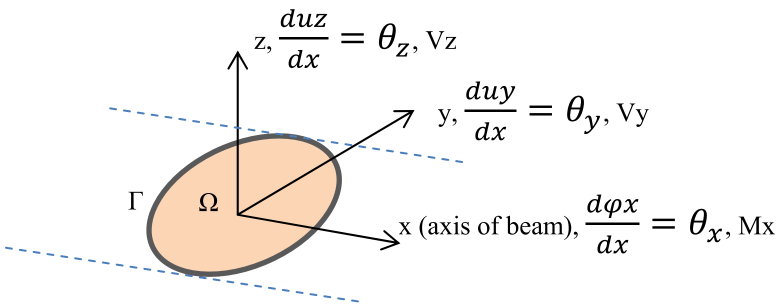

Fig. 1: Axis convention.

On the basis of the abovementioned assumptions, kinematic conditions for cross section deformation could be written as follows:

$$u_x(y,z) = w(y,z)$$
$$u_y(y,z) = \theta_z \cdot x - z.\theta_x \cdot x + \Phi_y(y,z) \qquad (1)$$
$$u_z(y,z) = -\theta_y \cdot x + y.\theta_x \cdot x + \Phi_z(y,z)$$

Where w(y,z) is the primarily searched warping function and functions $\Phi_y(y,z)$, $\Phi_z(y,z)$ are the functions of transversal contraction, which are caused by a axial stress. For shear strains over cross section it is valid:

$$\boldsymbol{\gamma} = \begin{bmatrix} \gamma_{xy} \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \\ \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{\partial w}{\partial y} + \theta_z - z \cdot \theta_x + \frac{\partial \Phi_y}{\partial x} \\ \frac{\partial w}{\partial z} - \theta_y + y \cdot \theta_x + \frac{\partial \Phi_z}{\partial x} \end{bmatrix} \qquad (2)$$

Because relation between bending moment and shear force is dM/dx = V, derivations of functions Φy(y,z), Φz(y,z) for real materials will be always nonzero. The impact of this effect on shear stress distribution for linear material is discussed for example in Gruttmann, Sauer & Wagner (1999). This effect is not considered in the presented method and the following assumption is accepted:

$$\frac{\partial \Phi_y}{\partial x} = 0; \frac{\partial \Phi_z}{\partial x} = 0 \qquad (3)$$

As further assumption the following constrains for overall rotation and displacement are accepted:

$$U_x = \int_\Omega u_x.d\Omega = 0$$
$$\varphi_y = \int_\Omega z.u_x.d\Omega = 0 \qquad (4)$$
$$\varphi_z = \int_\Omega -y.u_x.d\Omega = 0$$

Equilibrium equations have to be satisfied at every point of the cross section. Since external load does not have influence on the cross section, shear stresses have to be tangent to cross section boundary. In respect of assumption of the other zero elements of stress tensor the following boundary problem can be formulated.

$$\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \quad on \;\; \Omega$$
$$\tau_{xy} \cdot n_y + \tau_{xz} \cdot n_z = 0 \;\; on \;\; \Gamma \qquad (5)$$

Variational principle is used and the equation (5) is multiplied by test function and integrated across the area of the cross section Ω: The result has to be equal to zero in respect of variational principle.

$$a(w,\delta w) = \int_\Omega \left(\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right) \cdot \delta w \cdot d\Omega = 0 \qquad (6)$$

Equation (6) is integrated by parts.

$$a(w,\delta w) = -\int_\Omega \left(\tau_{xy} \cdot \frac{\partial \delta w}{\partial y} + \tau_{xy} \cdot \frac{\partial \delta w}{\partial y}\right) \cdot d\Omega + \oint_\Gamma (\tau_{xy} \cdot n_y + \tau_{xz} \cdot n_z) \cdot \delta w \cdot d\Gamma = 0 \qquad (7)$$

From the boundary condition (5) it is obvious that the value of the line closed integral in (7) is zero and final integral equation can be written:

$$\int_{\Omega} \boldsymbol{\delta\gamma^T} \cdot \boldsymbol{\tau} \cdot d\Omega = 0 \quad where \quad \boldsymbol{\tau} = \begin{bmatrix} \tau_{xy} \\ \tau_{xz} \end{bmatrix}, \boldsymbol{\delta\gamma} = \begin{bmatrix} \frac{\partial \delta w}{\partial y} \\ \frac{\partial \delta w}{\partial z} \end{bmatrix} \tag{8}$$

According to classic formulation of finite element method, the test function is assumed equal to the base function of finite element. In the following examples linear triangle elements and isoparametric elements are used. Formation of element and methods of integration are not described here, but they can be found for example in Němec et al. (2010) or Zienkiewicz & Taylor (2005).

$$\boldsymbol{\delta w} = \sum_{i}^{elem} \boldsymbol{N_i} \cdot \boldsymbol{\delta w_i} = (\sum_{i}^{elem} \boldsymbol{N_i}).\mathbf{E} \tag{9}$$

$\mathbf{N_i}$ are base functions, $\mathbf{E}$ is unit matrix and $\mathbf{w_i}$ are values of warping function at a grid of the mesh. From (9) it can be written:

$$\boldsymbol{\delta\gamma} = \sum_{i}^{elem} \begin{bmatrix} \frac{\partial \boldsymbol{N_i}}{\partial y} \\ \frac{\partial \boldsymbol{N_i}}{\partial z} \end{bmatrix} \cdot \delta \boldsymbol{w}_i = (\sum_{i}^{elem} \boldsymbol{B_i}) \cdot \mathbf{E} \tag{10}$$

Shear strains on elements can be formulated on the basis of (2) :

$$\boldsymbol{\gamma_i} = \boldsymbol{B_i} \cdot \boldsymbol{w_i} + \begin{bmatrix} -\boldsymbol{N_i} \cdot \boldsymbol{z_i} & 0 & 1 \\ \boldsymbol{N_i} \cdot \boldsymbol{y_i} & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} = \boldsymbol{B_i} \cdot \boldsymbol{w_i} + \boldsymbol{G_i} \cdot \boldsymbol{\theta} \tag{11}$$

where $\mathbf{y_i}$ and $\mathbf{z_i}$ are coordinates of mesh grids. On the basis of this the problem (8) can be transformed into system of nonlinear equations $\mathbf{F}$ with unknown values of warping $\mathbf{w_i}$ at mesh grid.

$$\boldsymbol{F} = \sum_{i}^{elem} \int_{\Omega_i} \boldsymbol{\delta\gamma}^T \cdot \boldsymbol{\tau}(\boldsymbol{\gamma_i}) \cdot d\Omega_i = \sum_{i}^{elem} \int_{\Omega_i} \boldsymbol{B_i}^{\boldsymbol{T}} \cdot \boldsymbol{\tau}(\boldsymbol{B_i} \cdot \boldsymbol{w_i} + \boldsymbol{G_i} \cdot \boldsymbol{\theta}) \cdot d\Omega_i = \boldsymbol{0} \tag{12}$$

A nonlinear relationship between shear stress and shear strain for isotropic material with HMH plastic condition is presented for example. However this is not a subject of this paper and using of the other formulations of plasticity is open and discussed for example in Chakrabarty (2006).

$$\boldsymbol{\tau}(\boldsymbol{\gamma}) = \mathbf{n}.\frac{1}{\sqrt{3}}\sigma(\varepsilon) \quad where\ \mathbf{n} = \frac{\gamma}{\|\gamma\|} \quad , \ \boldsymbol{\varepsilon} = \sqrt{\mathbf{3}}.\|\boldsymbol{\gamma}\| \tag{13}$$

$$\boldsymbol{C_{ij}} = \frac{\partial \tau_{xi}}{\partial \gamma_{xj}} \tag{14}$$

Constraints (4) can be rewritten by using finite element formulation:

$$\int_{\Omega} \begin{bmatrix} w(y,z) \\ z.w(y,z) \\ -yw(y,z) \end{bmatrix}.d\Omega = \left(\sum_{i}^{elem} \int_{\Omega_i} \boldsymbol{N_i}^T \cdot [1 \quad z_i \quad -y_i] \cdot d\Omega_i \right)^T \cdot \boldsymbol{w_i} = \boldsymbol{R^T}.\boldsymbol{w} = 0 \tag{15}$$

Newton method is used for solution of system of nonlinear equations (12). For this method it is necessary to determinate a tangent stiffness matrix:

$$\boldsymbol{K_{ij}} = \frac{\partial F_i}{\partial w_j} = \sum_{i}^{elem} \int_{\Omega_i} \boldsymbol{B_i}^{\boldsymbol{T}} \cdot \boldsymbol{C_i} \cdot \boldsymbol{B_i} \cdot d\Omega_I \tag{16}$$

Final iteration method for finding of warping at mesh grid is following with respect of constraints (15):

$$^{k+1}\boldsymbol{w} = {}^{k}\boldsymbol{w} + {}^{k}\boldsymbol{\Delta w} \tag{17}$$

$$\begin{bmatrix} {}^{k}\boldsymbol{K} & \boldsymbol{R} \\ \boldsymbol{R^T} & \boldsymbol{0} \end{bmatrix} \cdot \begin{bmatrix} {}^{k}\boldsymbol{\Delta w} \\ -{}^{k}\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} {}^{k}\boldsymbol{F} \\ \boldsymbol{0} \end{bmatrix} \tag{18}$$

After this solution of warping function $\mathbf{w}$ the last step remains. Resulting internal forces can be calculated in response to the deformations loading $\boldsymbol{\theta}$:

$$M_x = \int_{\Omega} (y.\tau_{xz} - z\tau_{xy}).d\Omega; V_y = \int_{\Omega} \tau_{xy}.d\Omega; V_z = \int_{\Omega} \tau_{xz}.d\Omega \tag{19}$$

$$\boldsymbol{V} = \begin{bmatrix} M_x \\ V_y \\ V_z \end{bmatrix} = \sum_{i}^{elem} \int_{\Omega_i} \boldsymbol{G_i}^{\boldsymbol{T}} \cdot \boldsymbol{\tau}(\boldsymbol{B_i} \cdot \boldsymbol{w_i} + \boldsymbol{G_i} \cdot \boldsymbol{\theta}) \cdot d\Omega_i \tag{20}$$

Tangential stiffness of cross section, which is necessary for formulation of beam element, is given by the following equation:

$$\boldsymbol{D_t} = \frac{\partial V_i}{\partial \theta_j} = \sum_i^{elem} \int_{\Omega_i} \boldsymbol{G_i}^{\boldsymbol{T}} \cdot \boldsymbol{C_i} \cdot \boldsymbol{G_i} \cdot d\Omega_i \qquad (21)$$

3. Example

The described method is illustrated by the following examples. Pure torsion of square cross section in plasticity state is shown in the left figure below. The color map shows intensity of shear stress and the vectors are directions of shear stress. Some combination of shear and torque of nonsymmetrical C profile is shown in the figure at the right side. This figure shows plastic strain as the color map and shear stress direction as vectors.

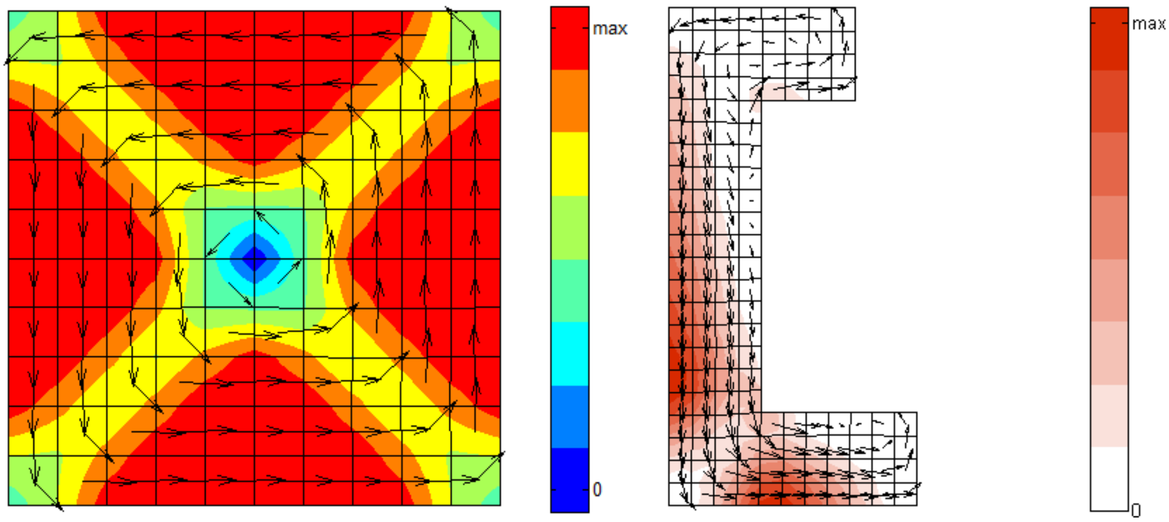

Fig. 2: Examples of solutions.

4. Conclusions

With regard to the use of deformation variant of the FEM it is not a problem to solve cross section with multiple internal loops without special constraints. Special conditions for the origin point of cross sections coordinate system are not required. The described method is directly applicable to simulation of plasticity free torsion. As it has already been discussed, shear does not occur without bending and due to this shear loading would be modeled with axial stress distribution. The condition of free warping is oversimplified for shear loading too. Nevertheless, this method is a contribution for formulation of a model of full plastic beam element.

Acknowledgement

This contribution has been prepared with the financial support of the company FEM Consulting.

References

Chakrabarty, J. (2006) Theory of plasticity. Elsevier Butterworth-Heinemann, Oxford.

Gruttmann, F. & Wagner, W. (2001) Finite Element Analysis of Saint-Venant Torsion Problem with Exact Integration of the Elastic-Plastic Constitutive Equations. Computer Methods in Applied Mechanics and Engineering, 190, pp. 3831-3848.

Gruttmann, F., Sauer, M. & Wagner, W. (1999) Shear Stresses in Prismatic Beams with Arbitrary Cross–Sections. International Journal for Numerical Methods in Engineering, 45, 7, pp. 865–889.

Němec, I. et al. (2010) Finite element analysis of structures. Shaker Verlag, Aachen.

Zienkiewicz, O. C. & Taylor R. L. (2005) The Finite Element Method for Solid and Structural Mechanics, 6th edn . Elsevier Butterworth-Heinemann, Oxford.