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A CLAMPED-CLAMPED BEAM STATIC SAG LIMITS UNDER PERPENDICULAR MAGNETIC FORCE

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Abstract: *There are mechatronic applications, where a slender beam or plate is subjected to static magnetic force generated by an electromagnetic actuator consisting of a solenoid on a ferromagnetic core and a yoke, fixed to the beam. The static magnetic force, acting perpendicularly onto the beam, causes sag (downwards bending) of the beam. If the magnitude of the magnetic force surpasses some threshold value the beam is buckled. For small deflections the mathematical expression of the magnetic force can be approximated by a polynomial dependence on the distance to the magnet. It is important to analyse the nature of the sag and to determine the limits of the linear approximation, as well as the limits leading to the buckled state. The mathematical generalisation of the sag is valid for electrostatic force between planar electrodes, too.*

Keywords: ***Clamped beam sag, electromagnetic actuator threshold current, sag approximation.***

1. Introduction

There are mechatronic applications, where a slender beam or plate of length L is subjected to static magnetic force F_M, generated by an electromagnetic actuator. The actuator consists of a solenoid wound on a pot-form ferromagnetic core and an armature (of length $L_m << L$), fixed to the beam at its midpoint (Fig. 1). The magnetic force F_M is acting in the middle of the beam at distance $L/2$ from rigid fixtures on both ends and induces a sag (downwards deflection) z_{max}. If the intensity of the magnetic force F_M exceeds certain threshold, the beam is permanently attracted to the end-stops (Bishop, 2002).

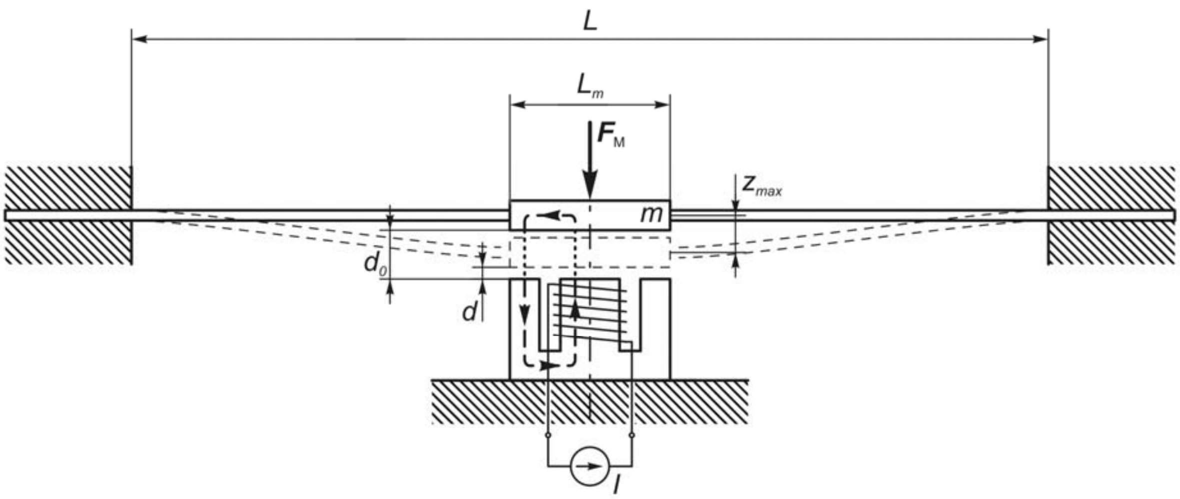

Fig. 1: Schematics of the clamped-clamped beam with electromagnet (flux line is denoted dashed).

2. Mathematical model of the equilibrium state

The sag, z_{max}, at the beam midpoint due to the magnetic force F_M is, according to (Rao, 2004) is:

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$$z_{\max}(F_{\mathrm{M}}) = \frac{F_M}{192} \cdot \frac{L^3}{E_{\mathrm{b}} I_{\mathrm{b}}}, \tag{1}$$

where E_b is the modulus of elasticity (Young's module) of the beam material and I_b is the second moment of inertia of the beam. This can be expressed as $F_M = z_{max} \cdot k_{ef}$, too. The theoretical value of the effective (lumped) stiffness k_{ef} of the clamped-clamped slender beam loaded at the midpoint is given as $k_{ef} = 192 \cdot (E_b I_b)/L^3$ (e.g. (Rao, 2004)).

Energizing the electromagnet with a steady state (DC) current I the magnitude of magnetic force F_M is described by a scalar equation (Giurgiutiu & Lyshewski, 2009; Mayer & Ulrych, 2009):

$$F_{\mathrm{M}}(d, I) = \frac{\mu_0 S N^2 I^2}{\left(2d + l_{\Phi}/\mu_{\mathrm{r}}\right)^2}. \tag{2}$$

The magnetic flux line is crossing twice the air gap, as shown in Fig. 1; μ_0 is the permeability of air and l_Φ is the flux line length in the ferromagnetic material of relative permeability μ_r. All parameters are known from vendor's data, were measured or calculated from set-up geometry (Darula, 2008).

The static equilibrium of the magnetic force $F_M(d, I)$ and the elastic force due to the beam deflection $z_{max} \cdot k_{ef}$ is described by Eq. (3). From the geometry follows: $z_{max} = d_0 - d$, where d_0 is the initial distance between electromagnet and the beam in de-energised state:

$$(d_0 - d)k_{\mathrm{ef}} = F_{\mathrm{M}}(d, I). \tag{3}$$

Let's introduce a non-dimensional air gap width α: $\alpha = (d_0 - d)/d_0$. (4)

From physical point of view, the quantity α is non-negative and cannot be larger than one. If $\alpha = 1$, beam is fully attracted by the electromagnet and would adhere to its poles.

Further the middle magnetic flux line path of length l_Φ will be considered. A more thorough magnetic field analysis by FEM approach would be beyond the scope of this contribution. The flux line length l_Φ can be transformed into an equivalent half flux line length in air d_{Fe}, assuming *linear properties* of the core magnetic material: $d_{\mathrm{Fe}} = \frac{1}{2} l_{\Phi}/\mu_{\mathrm{r}}$. This is also a simplifying assumption, because for common magnetic materials B-H relation is non-linear (Giurgiutiu & Lyshewski, 2009; Mayer & Ulrych, 2009). However, up to the saturation point, the concept of linear permeability can be used.

Introducing α into Eq. (3) and using Eq. (2), the equilibrium equation is:

$$\alpha \cdot k_{\mathrm{ef}} = \frac{F_{\mathrm{M}}}{d_0} = \left(\frac{\mu_0 S N^2}{4 d_0}\right) \cdot \frac{I^2}{\left[(d_0 - \alpha d_0 + d_{\mathrm{Fe}})\right]^2} = \left(\frac{\mu_0 S N^2}{4 d_0}\right) \cdot \frac{I^2}{d_0^2\left[(1+\delta_{\mathrm{M}}) - \alpha\right]^2}. \tag{5}$$

A relative measure $\delta_M = d_{Fe}/d_0$ can be introduced, while $\delta_M < 1$, because $\mu_r > 1$. Eq. (5) can be, after some algebraic manipulation, re-written as follows:

$$\frac{\alpha(I)}{(1+\delta_{\mathrm{M}})} \cdot k_{\mathrm{ef}} = \left(\frac{\mu_0 S N^2}{4 d_0^3}\right) \cdot \frac{I^2}{(1+\delta_{\mathrm{M}})^3 \left[1 - \frac{\alpha(I)}{(1+\delta_{\mathrm{M}})}\right]^2}, \tag{6a}$$

which calls for introduction of a normalised parameter β: $\beta = \alpha/(1+\delta_M)$. Parameter β relates the *air gap width change* (d - d_0) to the properties of the magnetic circuit δ_M, which are constant for the initial distance d_0: Obviously $\beta < 1$. The physically feasible limit is $\beta \leq 1/(1+\delta_M)$. Then Eq.(6a) is modified:

$$\beta(I) \cdot k_{\mathrm{ef}} = \left[\frac{\mu_0 S N^2}{4(d_0 + d_{\mathrm{Fe}})^3}\right] \cdot \frac{I^2}{[1-\beta(I)]^2} = K_{\mathrm{M}} \frac{I^2}{[1-\beta(I)]^2}. \tag{6b}$$

3. Solution of the equilibrium equation

The Eq. (6b) can be solved for variable $\beta(I)$ by an approximate approach using linear approximation, or in the exact way, applying analytical or numerical tools.

The denominator of the right hand side of Eq. (6b) can be approximated by a McLaurin's series:

$$\beta k_{ef} = K_{\mathrm{M}} I^2 \cdot \{1 + 2\beta + 3\beta^2 + ...\}. \tag{7}$$

Just the first two terms of the expansion are considered, i.e. the linear approximation is used. After some algebra the formula for approximate calculation of β' emerges:

$$\beta' = \frac{K_{\mathrm{M}} I^2}{k_{\mathrm{ef}} - 2K_{\mathrm{M}} I^2} . \tag{8}$$

The exact solution stems from the cubic equation obtained by rewriting Eq. (6b):

$$\beta \cdot [1-\beta]^2 = (K_{\mathrm{M}}/k_{\mathrm{ef}}) I^2 \text{ , i.e.: } \qquad \beta^3 - 2\beta^2 + \beta - (K_{\mathrm{M}}/k_{\mathrm{ef}}) I^2 = 0 \text{ .} \tag{9a, b}$$

The solution of Eq. (9b) calls for the use of Cardano's formulas for evaluation of cubic equations or rely on numerical solvers of algebraic equations, embedded in simulation programming environment, e.g. MATLAB®. The numerical solution leads, according to (Frank et al., 1973), to three different complex roots. In analogy to the quadratic equation there is a cubic discriminator D_3, furnishing for $D_3 > 0$ three real roots. This is the case here. By further analysis, two pairs of special real solutions of this cubic equation were found:

- a pair for $\beta = 0$ and $\beta = 1$, which is a result for $I = 0$;
- a pair for $\beta = 1/3$ and $\beta = 4/3$, which results if I attains a specific threshold value I_{crit}:

$$I_{\mathrm{crit}}^2 = \frac{4}{27}\left(\frac{k_{\mathrm{ef}}}{K_{\mathrm{M}}}\right). \tag{10}$$

The threshold current I_{crit} is determined by the beam stiffness k_{ef} and the magnetic circuit properties K_{M}. The value $\beta = 4/3$ corresponds to the *triple real root* at $D_3 = 0$. For $I > I_{\mathrm{crit}}$ (when $D_3 < 0$) there is only a single real root and two complex conjugate roots.

Let us introduce a generalized variable q_{N}, which is physically the current I normalized by the value of threshold current, $q_{\mathrm{N}} = I/I_{\mathrm{crit}} \leq 1$. Then Eqs. (8) and (9b) can be re-formulated and simplified:

$$\beta' = \frac{1}{\left(\frac{27}{4}\right)\frac{1}{q_{\mathrm{N}}^2} - 2} , \tag{11a}$$

$$\beta^3 - 2\beta^2 + \beta - (4/27)\, q_{\mathrm{N}}^2 = 0 \text{ .} \tag{11b}$$

For calculation of the exact solutions of β the MATLAB® function 'roots' was used, returning a complex three element vector for each q_{N} value. Then the roots are ordered in ascending order and plotted in the form of line graphs (Fig. 2 – medium lines). Note, that this is not a plot of a function, because for any positive value of $q_{\mathrm{N}} < 1$ three different values are possible. The course of the approximate solution β' (Eq. (11a)) is plotted as a thin line.

Physically feasible values of numerical solution of the cubic Eq. (11b) are bound to the interval $[0, \beta \leq 1/(1+\delta_{\mathrm{M}})]$ (white area in Fig. 2); hence the solutions above the bold limit (in the grey area) have no physical meaning (the beam would have to move within the electromagnetic core!). The dashed course is not physically realistic either, because this would assume that the elastic beam was buckled prior to energising the field. The physically plausible course is the lowest curve, starting at zero and reaching for $q_{\mathrm{N}} = 1$ the value of $\beta = 1/3$. However, for the value $q_{\mathrm{N}} = 1$ two different solutions do exist: $\beta = 1/3$ and $\beta = 4/3$. This can by interpreted as the limit of stability: **at the threshold current the beam buckles** from the value of $\beta = 1/3$ to $\beta = 1/(1+\delta_{\mathrm{M}})$, as denoted by the red vertical line.

If the current would revert from a value of $I > I_{\mathrm{crit}}$ the beam would follow the same trajectory, i.e. as soon as the value of q_{N} drops below unity the beam, firstly adhering to the magnet core, would attain (after extinction of the transient phenomenon) a position corresponding to the $\beta = 1/3$.

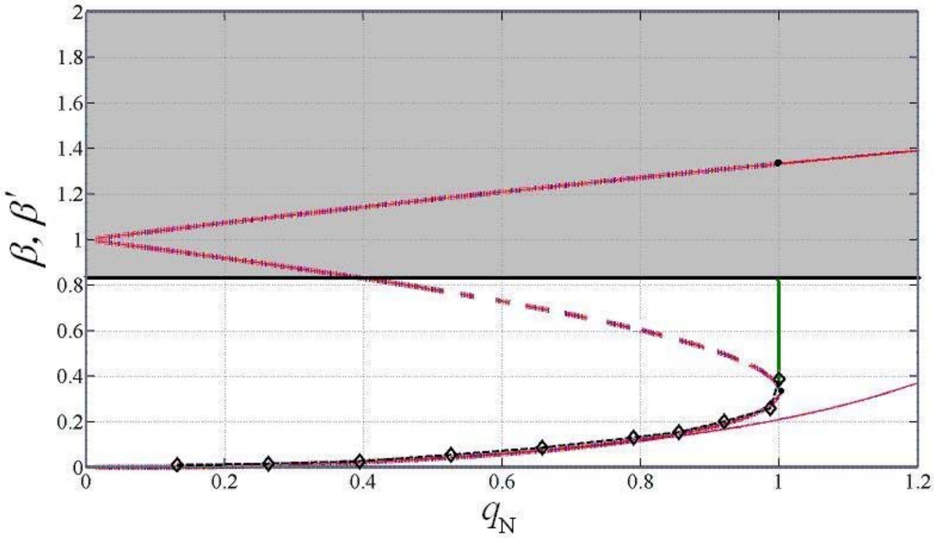

Fig. 2: The solution of the cubic equation (11b) in the generalised coordinates (medium line) and of the approximate solution (11a) (thin). The non-realistic solutions are in the gray area. Measured data, after (Darula, 2008), are superimposed as black dots.

From Fig. 2 it is seen that when $q_N < 0.80$ there is no marked difference between the exact solution and the approximate solution. For a specific case of initial air-gap width $d_0 = 1.0$ mm and magnetic circuit properties corresponding to $\delta_M = 0.15$ for $q_N = 0.80$ the difference between the exact solution $\beta = 0.123$ and the approximate solution $\beta' = 0.117$ is -5 %, i.e. still technically acceptable.

Preliminary experimental data, reported in (Darula, 2008) are superimposed onto the theoretical curves. The agreement is very good. Further experiments are under way to illustrate the deliberations.

4. Practical conclusions

The threshold value of the current I, I_{crit} is crucial for discrimination between bending and buckling of the elastic slender beam fixed on both ends. If a current larger than I_{crit} energises the solenoid the beam is permanently attracted to the electromagnet. This finding is supported by experimental results, too.

The linearised form of the magnetic force in the force equilibrium expression (Eq. (8) or (11a)) can be used up to 80 % of the threshold current with an error not exceeding –5 %.

For plane parallel electrodes driven by DC voltage U a similar formula to the Eq. (2) holds ($\delta_M = 0$). Using the above generalisation, it follows that there exists a threshold voltage U_{crit}, described by a similar formula to the formula (10), when the electrodes are completely attracted and short-circuited by the exerted electro static force acting perpendicular to the electrode planes.

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