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# RESONANCE BEHAVIOUR OF SPHERICAL PENDULUM – INFLUENCE OF DAMPING

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**Abstract:** *Experimental and numerical model of a uni-directionally driven pendulum-based tuned mass damper is presented in the paper. Stability of the motion in a vertical plane is analysed in the theoretically predicted resonance region. For the experimental part, special experimental frame is used, allowing independent change of linear viscous damping in the both perpendicular directions. The introduced damping is. Mathematical model respects the non-linear character of the pendulum and allows to introduce asymmetrical damping. Sensitivity of the resonance behaviour on the change of damping in both directions is studied and commented in the paper. The stability of the system is analysed experimentally and compared with numerical and theoretical results.*

**Keywords:** ***spherical pendulum, tuned mass damper, damping***

## 1. Introduction

A typical tuned mass damper is a pendulum. This low cost passive device used at tall masts and towers is very popular for its reliability and simple maintenance, see e.g. (Haxton, 1974; Náprstek and Pirner, 2002). However, dynamic behaviour of such substructure is significantly more complex than it is supposed by widely used simple linear single degree-of-freedom (SDOF) models working in one vertical plane. Such a conventional linear model is satisfactory only if the amplitude of kinematic excitation at the suspension point is very small and if its frequency remains outside a resonance frequency domain, which is possible only at the cost of lower efficiency of the damper. To improve the design of pendulum, a spherical pendulum should be considered.

The detailed review of the topic has been published by the authors recently in (Pospíšil et al., 2011). The mathematical model follows the approach presented in (Náprstek and Fischer, 2009). However, some historical remarks is worth to mention.

Auto-parametric systems have been intensively studied for the last four decades. The horizontally forced spherical pendulum was first studied by Miles (1962), who considered this problem of the stability of planar oscillations with respect to non-planar perturbations for small amplitude forcing in a neighbourhood of resonance using truncated equations. He found that planar solutions become unstable with respect to non-planar perturbations in particular parameter ranges and that there are stable non-planar oscillations. A more detailed later study (Miles, 1984) analysed a number of bifurcation diagrams for planar and non-planar motions as well as chaotic motion. Some experimental results have been presented by (Triton, 1986), where a good agreement with the theoretical results is demonstrated.

The present article exploits the analytical approach to the subject described in (Náprstek and Fischer, 2009) and compares it with some experimental findings. The movement of the pendulum is described in two coordinates $\theta, \varphi$ on a spherical surface respecting the non-linear interaction of both components, or in two cartesian coordinates $\xi, \zeta$, representing the projection of the pendulums bob to the $x, y$ plane. It means that the pendulum response is described by a system of two second-order non-linear ordinary differential equations. Interaction of both equations follows from non-linear terms only.

In the experimental as well in numerical approach the uni-directional harmonic excitation is supposed. If the excitation frequency belongs to the resonance region, post-critical states can emerge. These

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states are characterized by either highly increased in-plane response, or by more or less complicated space trajectories of various types. The shape of this motion stabilizes for increasing frequency in a nearly elliptic “horizontal” trajectory. Above the upper limit of the resonance domain an existence of a stable deterministic solution in the vertical plane resumes. The existence and stability level of individual solutions or response types are dependent on pendulum geometry and excitation structure. It is obvious that such a type of response destroys effectiveness of the tuned mass damper.

In this article, a pendulum is examined using a specially developed experimental rig. It contains kinematically driven pendulum suspended from the Cardan joint. The damping can be arbitrarily adjusted by means of two independent magnetic units attached to the frame and to the supporting axes of rotation. These units are able to reproduce the linear viscous damping for both degrees of freedom. The stability of the system is analysed experimentally and numerically for several values of the damping.

## 2. Mathematical model

The spherical pendulum will be considered as a strongly non-linear dynamic system with kinematic external excitation in the suspension point, see Fig. 1.

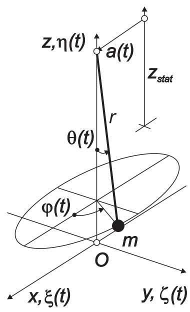


*Fig. 1: Outline of the idealized model*

The mathematical model follows from the balance of kinetic and potential energies. Using Hamilton’s principle, an approximate Lagrangian system in x, y-coordinates for components $\xi, \zeta$ can be obtained (see detailed derivation in (Náprstek and Fischer, 2009)):

$$\left.\begin{aligned}
\ddot{\xi} + \frac{1}{2r^2}\xi\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\xi^2+\zeta^2) + 2\beta_\xi\dot{\xi} + \omega_0^2\left(\xi + \frac{1}{2r^2}\xi(\xi^2+\zeta^2)\right) &= -\ddot{a} \qquad (a)\\
\ddot{\zeta} + \frac{1}{2r^2}\zeta\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\xi^2+\zeta^2) + 2\beta_\zeta\dot{\zeta} + \omega_0^2\left(\zeta + \frac{1}{2r^2}\zeta(\xi^2+\zeta^2)\right) &= 0 \qquad (b)
\end{aligned}\right\} \qquad (1)$$

The viscous damping has been introduced in a form of the Rayleigh function and denoted as $\beta_\xi, \beta_\zeta$ in (1). Accuracy of the mathematical model depends on the amplitude of the response, as the assumption of the small angle $\theta$ has been adopted. Natural frequency $\omega_0$ of the corresponding linear pendulum is given by $\omega_0^2 = g/r$, where $r$ is the suspension length of the pendulum and $g$ is the gravitational acceleration. Neglecting the non-linear terms in (1), the system would broke up into two independent linear equations. Thus, the interaction of both the equations is given by non-linear terms only.

As the harmonic excitation $a(t) = a_0 \sin(\omega t)$ acts in the $\xi$ direction only, the basic type of motion takes course in the vertical $(xz)$-plane if the time history starts under homogeneous initial conditions. With increasing amplitude of the excitation $a(t)$, the auto-parametric stability loss can occur and the post-critical state of the auto-parametric resonance arises.

## 3. Experimetnal analysis

The stability problems are, in general, very sensitive to boundary and initial conditions. Therefore, any simulation machine and its mechanical parts needs to be well prepared and manufactured to avoid creating parasitical influences, which are very difficult to eliminate. This applies not only to a complicated kinematic mechanism but also to the relatively simple spherical pendulum. The authors use an experimental pendulum, designed to comply the assumptions of the theoretical and numerical model. This pendulum is suspended from Cardan joint attached to a trolley moving on two parallel miniature rails. Two magneto-dynamic units allow to introduce viscous damping in the practically full range from (almost) zero to the critical value. For detail of the set-up see (Pospíšil et al., 2011).

The length of the pendulum was $0.41$ m, mass of its bob was significantly greater than it was in the previous experiments to increase inertia and thus to lower the minimal relative damping. Fundamental eigenfrequency of the pendulum was measured as $f_0 = 0.76$ Hz, i.e. $\omega_0 = 4.8\text{rad}\cdot\text{s}^{-1}$. Response of the pendulum was measured for excitation frequencies ranging from $f_l = 0.73$ Hz to $f_u = 0.97$Hz with increments $\Delta f = 0.002$ Hz, (i.e. range $\omega = 4.587\ldots6.095\text{rad}\cdot\text{s}^{-1}$). To cover the full range of the resonance interval, the each sweep was started for excitation frequency slightly higher than the eigenfrequency of the pendulum and small initial disturbance was given to the pendulum. Then was the excitation frequency gradually changed in small increments up or down to cover the whole frequency range. Each frequency was kept constant for three minutes and angles of the pendulum were measured and recorded. To eliminate the transition effects, only the last minute of each record was taken into account in the post-processing.

Figure 2 shows maximal and minimal measured amplitudes ($^\circ$) depending on the excitation frequency ($\omega$, rad.s$^{-1}$). Responses for several values of damping coefficients ($\beta_\xi = \beta_\zeta = 0.04, \ldots 1.2$) are shown in the individual rows. The alongside $\xi$ and transversal $\zeta$ components are in the left and right hand columns respectively. Three curves are present in each plot, they represent maximal, minimal and mean values of the amplitudes. When all three curves coincide, the response of the pendulum is harmonic. If the minimal and maximal curve form a stripe, multi-harmonic or chaotic type of the response takes place. However, this simple criterion is not able to distinguish chaotic and multi-harmonic response.

## 4. Numerical analysis

In order to get an overview concerning the system behaviour in the vicinity of resonance frequency intervals several numerical analyses using the governing differential system (1) have been performed. For numerical simulation, default numerical procedure NDSolve from package Wolfram Mathematica and $M = 2$ variant of implicit Gear method (routine `gear2`) from the GNU Scientific Software Library (Galassi et al., 2009) have proved themselves to be the most stable and efficient. Adaptive step control is used in both numerical methods. Due to a high number of necessary simulation to obtain a single resonance curve, the parallel algorithm has been developed to exploit the computational power of the 2 cpu / 16-thread computer.

To assess the correspondence of numerical simulation and experimental results, let us compare the Figs 2 and 3. The both figures show the resonance curves, obtained form measured and computed data respectively. The qualitative behaviour in the lower end of the resonance interval is rather comparable. On the other hand, a quite significant difference can be seen in the upper part of the studied frequency interval, namely for low damping coefficients $\beta_\xi = \beta_\zeta \in \{0.04, 0.05\}$. It appears, that the experimental pendulum was able to follow the (less stable) upper branch of the solution during the sweep-up simulation much better than it was the numerical solution. This behaviour is also different from the measurements published by the authors in (Pospíšil et al., 2011). It seems that the only explanation for such a "stability on upper branch" is the increased mass of the bob of the pendulum. It is worth to mention, that after some change in the excitation or other noticeable artificial disturbance the pendulum jumped to the standard (lower and planar) stable branch of the response.

It is necessary to confess, that quantitative agreement between numerical simulation and experimental results for a single set of input parameters and a single time history was not so good. Even for carefully selected initial conditions the numerical model was not able to follow the trajectory of the experimental pendulum. This behaviour is not surprising, as the mathematical model (1) has been derived with the assumption of small amplitude of the response. Moreover, the significant dependence of instant frequency of the response on its instant amplitude is clearly seen from the experimental data.

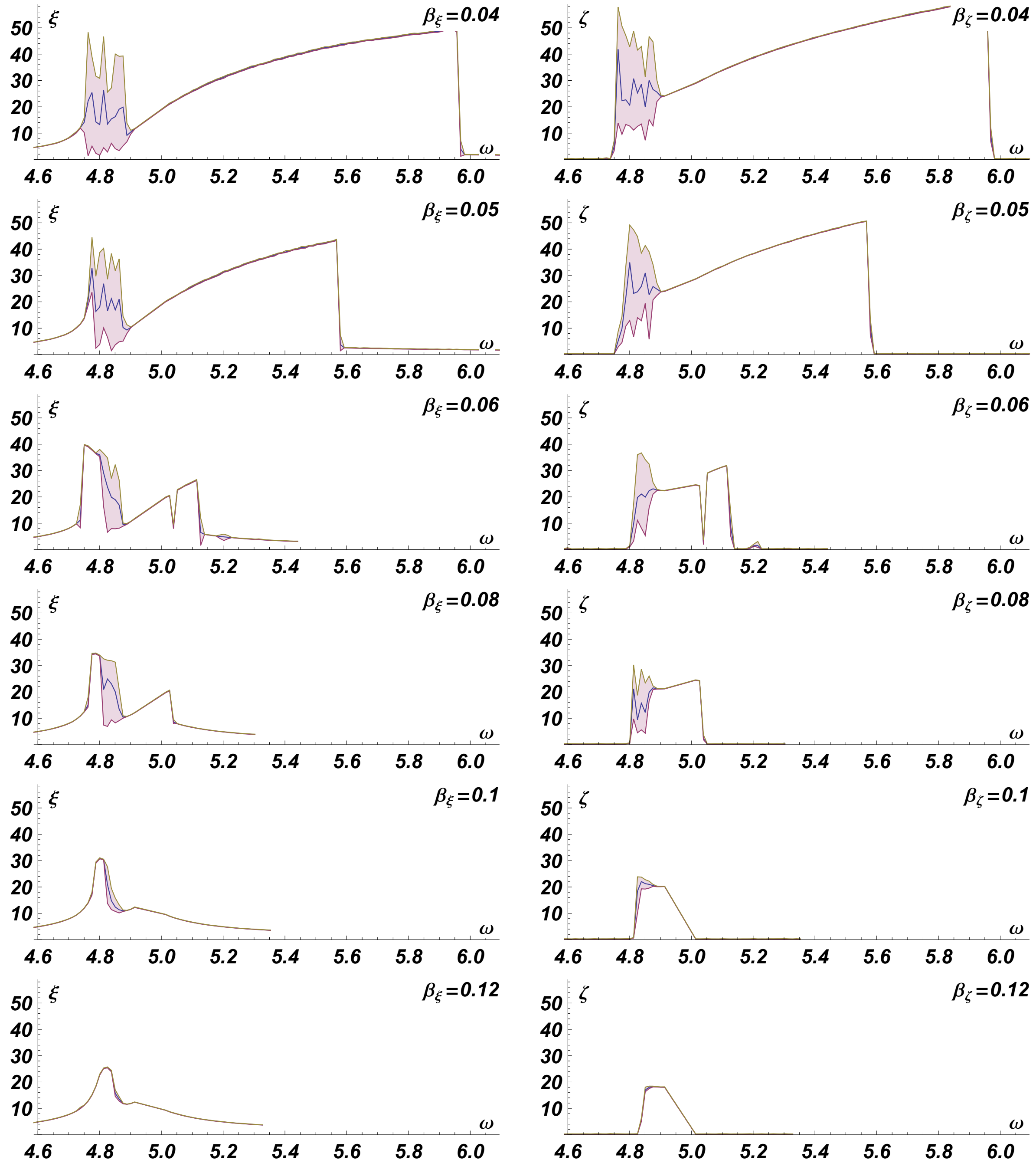


*Fig. 2: Experimental pendulum: measured amplitudes ($\cdot^\circ$) of the response depending on excitation frequencies $\omega = 4.6 \ldots 6.1\,\mathrm{rad.s^{-1}}$ for several values of damping coefficients, same in the both directions. Longitudinal movement ($\xi$) is on the left hand side, transversal response ($\zeta$) on the right hand side. For each plot, maximal, minimal and mean amplitudes are shown.*

Having roughly assessed the validity of the numerical model introduced by Eq. (1) let us study the influence of individual damping coefficients on the overall response of the system in the both directions. The figure 4 shows selected results obtained during the extensive parametric study. For the interval of excitation frequencies $\omega \in (4.6, 6.1)$ and the values of damping coefficients $\beta_\xi, \beta_\zeta \in (0.005, 0.12)$, the equation (1) was repeatedly integrated and the maximal amplitudes in both directions was recorded. There are 10 pairs of colour plots in figure 4. Each pair corresponds to a single excitation frequency $\omega \in \{4.7, 4.72, \ldots, 4.88\}$ to cover area surrounding the eigenfrequency of the pendulum. In each pair, left plot shows response in longitudinal direction ($\xi$) and right plot corresponds to the transversal direction. Values on the horizontal axis of each plot represent the damping coefficients $\beta_\xi$, whereas the vertical axis stands in values of the damping coefficients $\beta_\zeta$. Finally, the colour map shows the distribution of

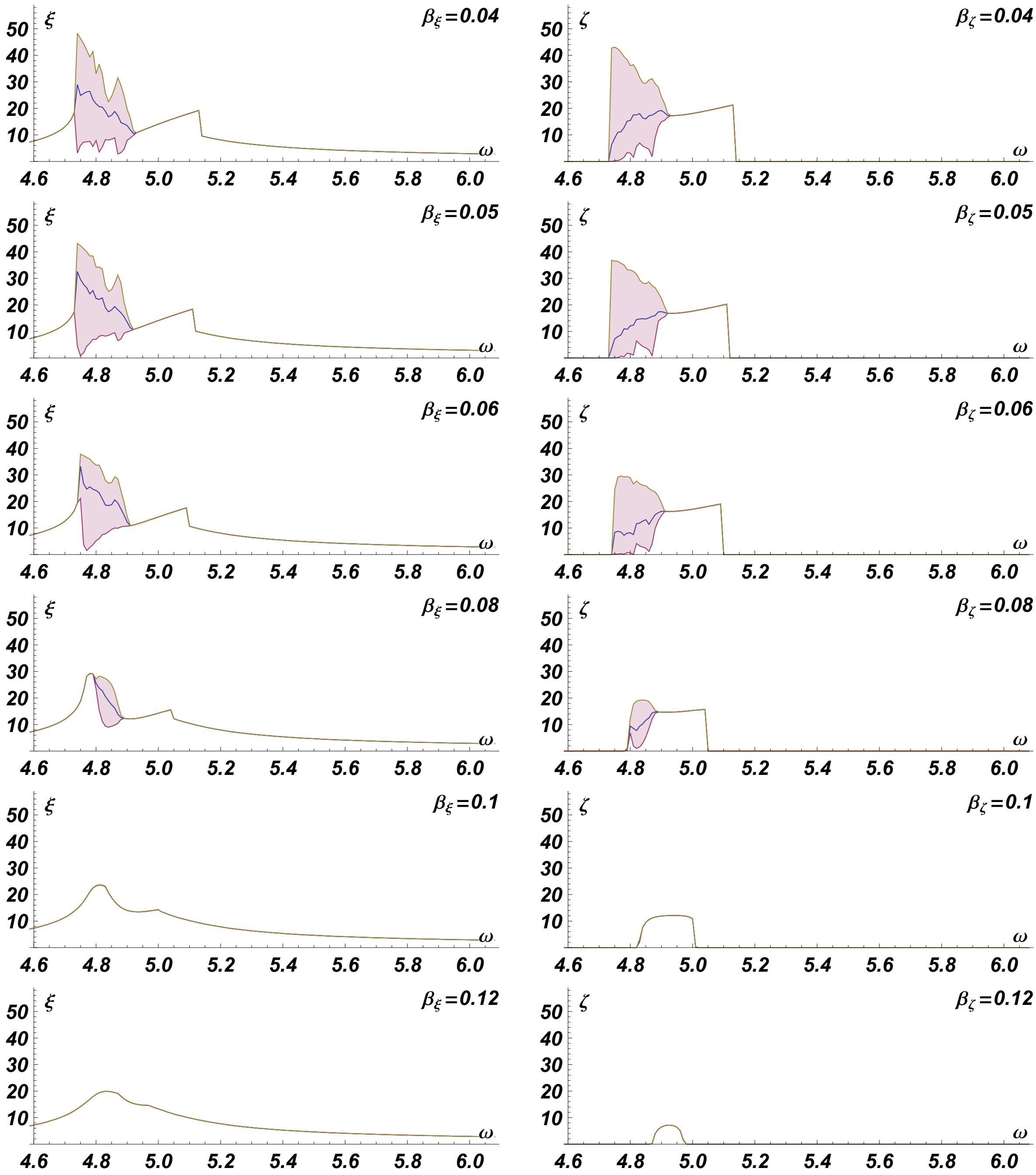


*Fig. 3: Numerical integration: computed amplitudes (·°) of the response depending on excitation frequencies* $\omega = 4.6 \dots 6.1\,\mathrm{rad.s^{-1}}$ *for several values of damping coefficients, same in the both directions. Longitudinal movement (ξ) is on the left hand side, transversal response (ζ) on the right hand side. For each plot, maximal, minimal and mean amplitudes are shown. Parameters of the model were chosen to meet geometrical properties of the experimental set-up.*

the maximal amplitudes of $xi$ and $\zeta$ in the left and right plot respectively. The dark blue colour indicates negligible or small amplitude of the response, whereas bright yellow and brown colours show the high response. The black dots in the each plot point at the discrete values of $\beta$ used in simulation.

Several remarks can arise from observation of the figure 4. Firstly, it appears, that the presence of the spatial character of the system response does not depend significantly on the value of damping coefficient $\beta_\zeta$ (transversal motion). Similarly, the overall amplitude of the response seems to be influenced mostly by $\beta_\xi$ (longitudinal motion) and far less by $\beta_\zeta$. Secondly, the spatial response in the lower part of the resonance interval have higher amplitudes, but can be suppressed by smaller values of damping $\beta_\xi$. The lower amplitudes which appear in the upper part of the resonance interval need higher damping $\beta_\xi$ to be

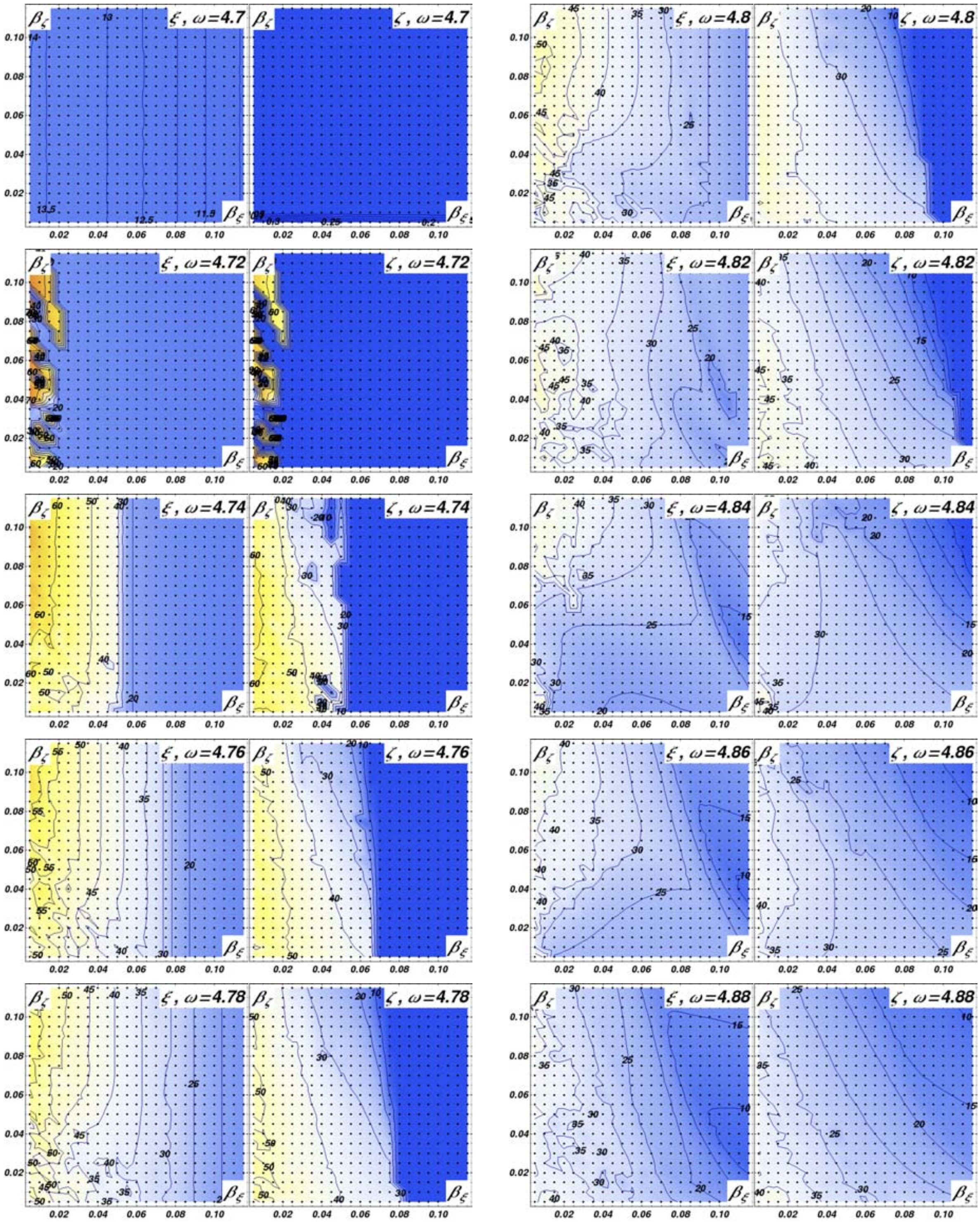


*Fig. 4: Maximal amplitude of the response depending on the values of damping coefficients in the both directions* $\beta_\xi, \beta_\zeta \in (0.005, 0.12)$ *for excitation frequencies* $\omega = \in \{4.7, 4.72, \ldots, 4.88\}$*. For each frequency the left plot shows response in longitudinal direction (*$\xi$*) and right plot corresponds to the transversal direction.*

wiped off. Third, it is not always true, that the higher damping (in transversal direction) automatically means the lower response (cf. $\xi$ plots for $\omega > 4.78$ in fig. 4).

There are some problematic points in the presented study. Firstly, the complete bunch of simulation was performed with fixed initial conditions, more or less randomly chosen. Thus, this numerical analysis was not able to cover up the variety of possible stable branches. Secondly, remarks from the previous paragraph can be roughly explained form the structure of the equation (1). It is not clear, whether they represent the behaviour of a real pendulum, or just its mathematical idealization. The further experimental study should cover up at least some cases of unsymmetrical damping.

## 5. Conclusions

It has been shown before that widely used linear model of the damping pendulum is acceptable only in a very limited extent of parameters concerning pendulum characteristics and excitation properties. In this work, two degrees of freedom experimental and non-linear numerical models was studied. The harmonic kinematic external excitation in the suspension point was applied in both cases. The viscous damping was varied in the analysis, independently in the numerical model and jointly in the present set of experimental results.

Various types of the response of the pendulum have been encountered for excitation frequency in the resonance frequency interval: in-plane, periodic or chaotic. The character of the response depends on structural parameters: frequency and amplitude of excitation, geometry of the pendulum and damping coefficients. It has been shown, that initiation of the spatial response is more sensitive to damping in the direction of excitation, whereas even a relatively high damping in the transversal direction does not prevent the spatial movement.

Dependence of the amplitude of the experimental pendulum on the values of damping is not surprising. On the other hand, reasonable correspondence between experimental measurements and numerical model has been confirmed. The qualitative correlation of numerical/experimental results has been observed. However, the assumption of small amplitude depreciate the quantitative relation of numerical and experimental results. For better results, the numerical model should be adopted to comprise dependence of instantaneous amplitude and frequency of the pendulum.

There are some open problems yet. The numerical analysis of the influence of damping should cover the additional (multiple) branches of the stable motion. The next experiments are necessary to validate the unsymmetrical influence of the values of damping coefficients.

From the practical point of view, it is highly recommended to design the damping pendulum absorber in such a way that any occurrence of non-linear resonance effects is avoided. If not, negative influence of the pendulum in the resonance domain is to be expected in both along-wind as well as in cross-wind directions. The experiments will continue with the application of more excitation amplitudes and damping values varying in both principal vibration planes.

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