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B-SPLINE FINITE ELEMENT METHOD IN ONE-DIMENSIONAL ELASTIC WAVE PROPAGATION PROBLEMS

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Abstract: *In this paper, the spline variant of finite element method (FEM) is tested in one-dimensional elastic wave propagation problems. The special attention is paid to propagation of stress discontinuities as an outcome of the shock loading and also to spurious oscillations occurring near theoretical wave-fronts. Spline variant of FEM is a modern strategy for numerical solution of partial differential equations. This method is based on spline basic functions as shape, testing functions in FEM content. For examples, B-splines, T-splines, NURBS and more others could be applied. For one-dimensional problems, B-spline representation is sufficient. B-spline basis functions are piecewise polynomial functions. It was shown, that B-spline shape functions produce outstanding convergence and dispersion properties and also appropriate frequency errors in elastodynamics problems. In this initial work, accuracy, convergence and stability of the B-spline based FEM are studied in numerical modelling of one-dimensional elastic wave propagation of stress discontinuities. For the time integration, the Newmark method, the central difference method and the generalized-α method are employed.*

Keywords: ***elastic wave propagation, B-spline based finite element method, spurious oscillations.***

1. Introduction

A modern approach in the finite element analysis is the isogeometric analysis (IGA), see Cottrell et al. (2009), where shape functions are based on varied types of splines. For example, B-spline, NURBS, T-spline and others are used for spatial discretization. This approach has an advantage that the geometry and approximation of the field of unknown quantities is prescribed by the same technique. Another benefit is that the approximation is smooth.

It was shown for the IGA approach, that the optical modes did not exist unlike higher-order Lagrangian finite elements, see Cottrell et al. (2006); Hughes et al. (2008). Further, dispersion and frequency errors for the isogeometric analysis were reported to decrease with the increasing order of spline, see Cottrell et al. (2009). IGA, where continuous piecewise higher order polynomials are used as shape functions, improves the dispersion errors and frequency spectrum in comparison with Lagrangian finite elements. The spline based FEM with the small dispersion errors and the variation diminishing property, see Piegl and Tiller (1997), could eliminate the spurious oscillations, which are the outcome of the Gibb's effect and dispersion behaviour of FEM based on the continuous Galerkin's approximation method, see Chin (1975) and Belytschko and Mullen (1978). The convergence and accuracy of IGA in explicit elastodynamics have been shown in the paper Auricchio et al. (2012).

For one-dimensional problems, the B-spline basis functions could be used for spatial discretization. Generally, the B-spline basis functions for bounded solids are not uniform due to the end point interpolating property, see Fig. 1. For this reason, the non-homogeneity of basis functions near the boundary of the domain produces the dispersion and attenuation behaviour, see Kolman et al. (2011). The homogeneous spline shape functions are very important for wave propagation problems, because they produce repeating rows of mass and stiffness matrices. Thus the B-spline FEM in an unbounded domain does not produce optical modes unlike higher-order Lagrangian FEM, see Hughes et al. (2008).

A lot of methods for the numerical solution of wave propagation problems in elastic solids have been developed, for example finite difference method, finite volume method, front tracking algorithms, space-time treatment methods, oscillations filtering by postprocessing, finite element spatial discretization with

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the finite difference in time (semidiscretization), discontinuity Galerkin's method and variational construction method and more others. Details for references see Belytschko and Hughes (1986), Hughes (1983), Park et al. (2012). In this paper, only the semidiscretization method is tested in one-dimensional elastic wave propagation of sharp wave fronts and stress discontinuities. For the spatial discretization, the continuous Galerkin's approximation method is employed, see Hughes (1983). In this initial work, accuracy, convergence and stability of the B-spline based FEM is studied in numerical modelling of one-dimensional elastic wave propagation of stress discontinuities. For the time integration, the Newmark method, see Newmark (1959) or Subbaraj and Dokainish (1989), the central difference method, see Dokainish and Subbaraj (1989), and the implicit form of the generalized-α method , see Chung and Hulbert (1993), are employed.

2. B-SPLINE BASED FINITE ELEMENT METHOD

In this section, the basis of B-spline based finite element method is shortly introduced. In Computer-Aided Design (CAD), a B-spline curve is given by the linear combination of B-spline basis functions $N_{i,p}$, see book of Piegl and Tiller (1997),

$$\mathbf{C}(\xi) = \sum_{i=1}^{n} N_{i,p}\left(\xi\right) \mathbf{B}_i \tag{1}$$

where $\mathbf{B}_i, i = 1, 2, \ldots, n$ are corresponding coordinates of control points. B-spline basis functions $N_{i,p}(\xi)$ are prescribed by the Cox-de Boor recursion formula, see Piegl and Tiller (1997). For a given knot vector Ξ, $N_{i,p}(\xi)$ are defined recursively starting with piecewise constants ($p = 0$)

$$N_{i,0}\left(\xi\right) \quad = \begin{cases} 1 & \text{if} \quad \xi_i \leq \xi \leq \xi_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

For $p = 1, 2, 3, \ldots$, they are defined by

$$N_{i,p}\left(\xi\right) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}\left(\xi\right) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}\left(\xi\right) \tag{3}$$

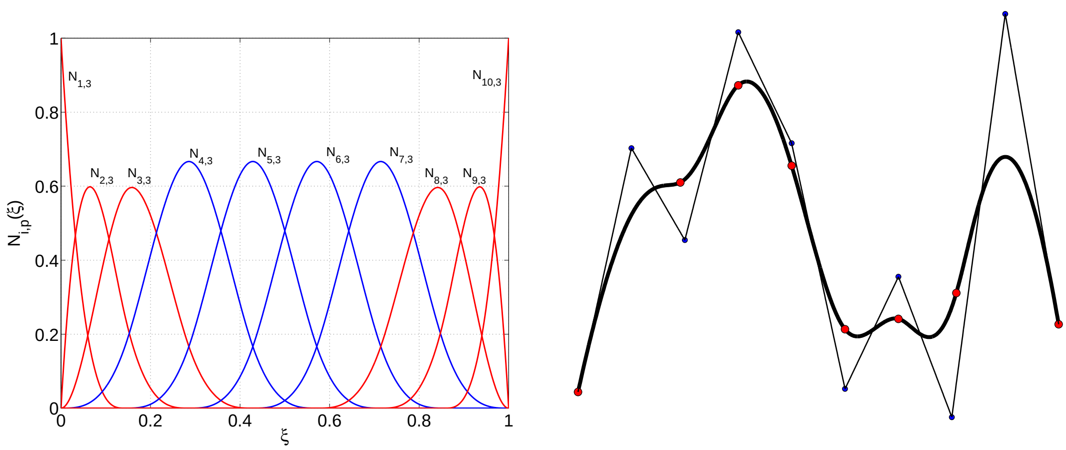

Fig. 1: Cubic B-spline basis functions (on the left) and an open cubic B-spline curve interpolating end points (on the right) for ten control points and the uniform knot vector. Red lines correspond to non-uniform basis functions and blue lines correspond to uniform (homogeneous) basis functions. The number of non-uniform basis functions depends on the polynomial order.

A knot vector in one dimensional case is a non-decreasing set of coordinates in the parameter space, written $\Xi = \{\xi_1, \xi_2, \ldots, \xi_m\}$, where $\xi_i \in R$ is the i-th knot, i is the knot index, $i = 1, 2, \ldots, m$, where $m = n + p + 1$, p is the polynomial order, and n is the number of basis functions. The main properties of B-spline basis functions are introduced in book Piegl and Tiller (1997). The knot vector for an open

B-spline curve interpolating end points should be in the form $\Xi = \{a, \ldots, a, \xi_{p+2}, \ldots, \xi_n, b, \ldots, b\}$, where values are usually set as $a = 0$ and $b = 1$. The multiplicity of the first and last knot value is $p+1$. If the values ξ_{p+1} up to ξ_{n+1} are chosen uniformly, the knot vector Ξ is called uniform, otherwise non-uniform, see Piegl and Tiller (1997). An example of cubic B-spline basis functions and an open cubic B-spline curve interpolating end points with its control polygon is displayed in Fig. 1.

In the B-spline based FEM, see book Cottrell et al. (2009), the approximation of the displacement field u^h is given by

$$u^h(\xi) = \sum_{i=1}^{n} N_{i,p}\left(\xi\right) u_i^{\mathrm{B}} \tag{4}$$

where u_i^{B} is the component of the vector of control variables – displacements corresponding to the control points. Remark, the linear B-spline FEM is identical with the standard linear FEM. In the following text, the continuous Galerkin's approximation method for the numerical solution of partial differential equations is employed, see book Hughes (1983). Spatial discretization of elastodynamics problems by the finite element method leads to, see Hughes (1983),

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{R} \tag{5}$$

Here, $\mathbf{M}$ is the mass matrix, $\mathbf{K}$ the stiffness matrix, $\mathbf{R}$ is the time-dependent load vector, $\mathbf{u}$ and $\ddot{\mathbf{u}}$ contain control point variables–displacements and accelerations. Mass matrix, stiffness matrix and load vector are defined by the same relationships as the standard FEM, see Hughes (1983). The element stiffness and mass matrices are given by

$$\mathbf{K} = \int_V E\mathbf{B}^{\mathrm{T}}\mathbf{B}\,\mathrm{d}V, \qquad \mathbf{M} = \int_V \rho\mathbf{N}^{\mathrm{T}}\mathbf{N}\,\mathrm{d}V \tag{6}$$

where E is the modulus of elasticity, ρ is the mass matrix, $\mathbf{B}$ is the strain-displacement matrix, $\mathbf{N}$ stores the displacement shape functions and integration is carried over the non-deformed domain V. Global matrices are assembled in the usual fashion. Mass matrix defined by the relationship (6) is called the consistent mass matrix. In this paper, the diagonal mass matrix is put together by 'row sum' method, see Hughes (1983). If the theory of linear elastodynamics is considered, then the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ are constant. These matrices are evaluated by the Gauss-Legendre quadrature formula, see Hughes (1983).

3. FINITE ELEMENT RESPONSE AND NUMERICAL INTEGRATION

For an undamped system in the current configuration at time t, we get the equations of motion in the form resulting from the finite element semidiscretization $\mathbf{M}\ddot{\mathbf{u}} = \mathbf{F}_{\text{res}}$ where the residual vector is $\mathbf{F}_{\text{res}} = \mathbf{R} - \mathbf{F}_{\text{int}}$. Generally, $\mathbf{F}_{\text{int}}$ is the vector of the internal control points forces corresponding to element stresses. In the linear elasticity theory, the interval forces are given $\mathbf{F}_{\text{int}} = \mathbf{K}\mathbf{u}$. The aim of computational procedures used for the solution of transient problems is to satisfy the equation of motion, not continually, but at discrete time intervals only. It is assumed that in the considered time span $[0, t_{max}]$ all the discretized quantities at times $0, \Delta t, 2\Delta t, 3\Delta t, \ldots, t$ are known, while the quantities at times $t + \Delta t, \ldots, t_{max}$ are to be found. The quantity Δt, being the time step, need not necessarily be constant throughout the integration process. We consider the constant time step Δt in the following text.

In this paper, the B-spline based FEM response of the elastic bar is computed numerically by the Newmark method with parameters $\beta = 1/4$, $\gamma = 1/2$, see the paper Newmark (1959), with the consistent mass matrix and the central difference method, see the paper Dokainish and Subbaraj (1989), with the lumped mass matrix by the 'row sum' method are employed. Implementation of the both methods is prescribed in work Okrouhlik (2008). The last method of the direct time integration is the implicit generalized-α method, see Chung and Hulbert (1993), with consistent mass matrix. The predictor-corrector explicit form of the implicit generalized-α method was implemented, see Hulbert and Chung (1996). The generalized-α method is an extension of the Newmark method that exhibits algorithmic damping. This second order algorithmic damping introduced to improve the shortcoming. This method achieves high-frequency dissipation while minimizing unwanted low-frequency dissipation. The dissipation effect of the generalized-α method is controlled by the spectral radius in the high-frequency limit

$\rho_\infty \in [0, 1]$. The value of $\rho_\infty = 0$ eliminates the high-frequency response (known as asymptotic annihilation). On the other side, setting $\rho_\infty = 1$ eliminates the algorithmic damping and the trapezoidal time-integration is considered. The value of $\rho_\infty = 0.8$ is recommended with respect to dissipation, dispersion and period time elongation, see paper Chung and Hulbert (1993). The values of $\rho = 0.8$ and $\rho = 0.5$ are tested in connection with B-spline discretization.

4. Problem description

In this contribution, the crucial test is a problem of axial elastic waves propagation in a free-fixed "thin" bar under the force loading prescribed by the Heaviside step function. The scheme of the task is depicted on Fig. 2. The shock loading generates stress and velocity jumps propagating by theoretical wave speeds. The parameters of the task are set: the bar length $L = 1\,m$, the cross-section $A = 1\,m^2$, Young's modulus $E = 1\,Pa$, the mass density $\rho = 1\,kg/m^3$ and the amplitude of impact pressure $\sigma_0 = 1\,Pa$. The analytical solution of this impact problem could be found in book Kolsky (1963), where the displacement field $u(x, t)$ without a wave reflection in the time range $t \in [0, L/c_0]$ is derived in the form

$$u(x, t) = v_0\,(t - x/c_0)\,H(t - x/c_0), \tag{7}$$

where the impact velocity is given by $v_0 = \sigma_0/\sqrt{E\rho}$ and $H(t)$ is the Heaviside time step function defined in book Kanwal (1998). Wave speed in an elastic bar is prescribed by the relationship $c_0 = \sqrt{E/\rho}$, see Kolsky (1963).

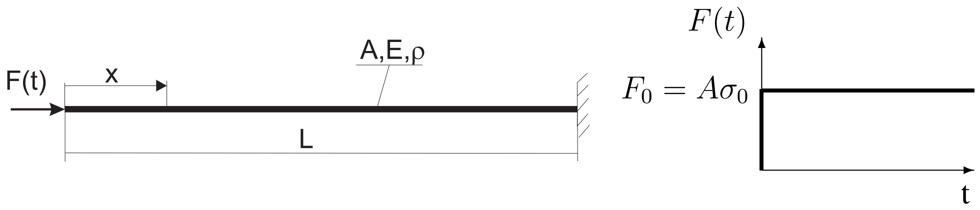

Fig. 2: Scheme of an elastic free-fixed bar under a shock loading.

5. Finite element response

The response of the elastic bar is computed numerically by the Newmark method, see Newmark (1959), with the consistent mass matrix , by the central difference method, see Dokainish and Subbaraj (1989), with the lumped mass matrix with respect to the 'row sum' method and by the implicit generalized-α method, see Hulbert and Chung (1996), with the consistent mass matrix. Details for the numerical implementation see the works Okrouhlik (2008) and Grosu and Harari (2007).

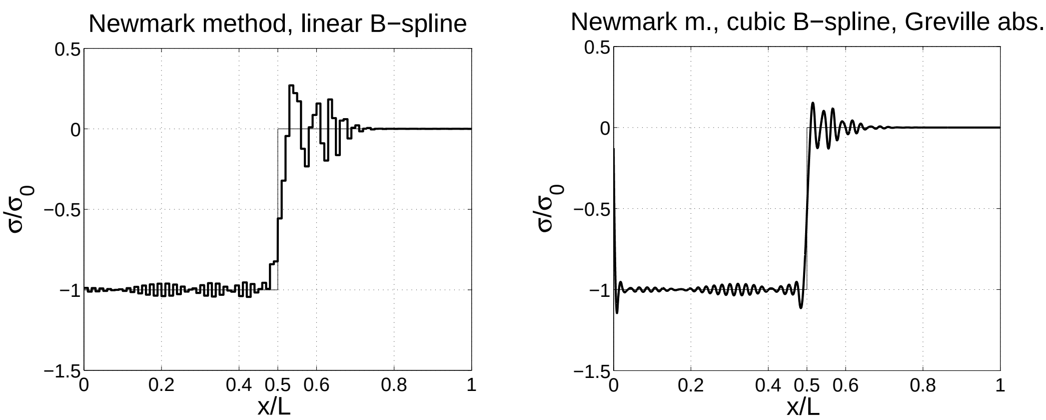

Fig. 3: Stress in an elastic bar under the shock loading at time $t = 0.5L/c_0$ computed by the Newmark method for linear (on the left) and cubic (on the right) B-splines.

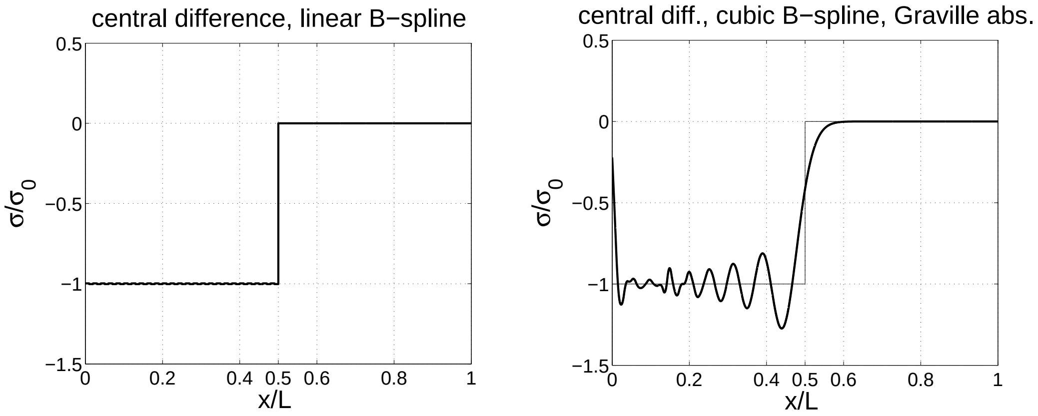

Fig. 4: Stress in an elastic bar under the shock loading at time $t = 0.5L/c_0$ computed by the central difference method for linear (on the left) and cubic (on the right) B-splines.

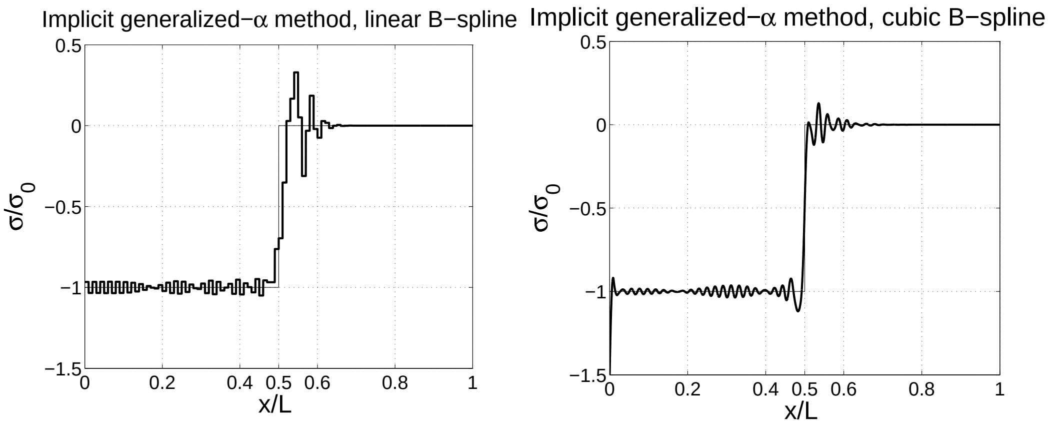

Fig. 5: Stress in an elastic bar under the shock loading at time $t = 0.5L/c_0$ computed by the implicit generalized-α method with $\rho_\infty = 0.8$ for linear (on the left) and cubic (on the right) B-splines.

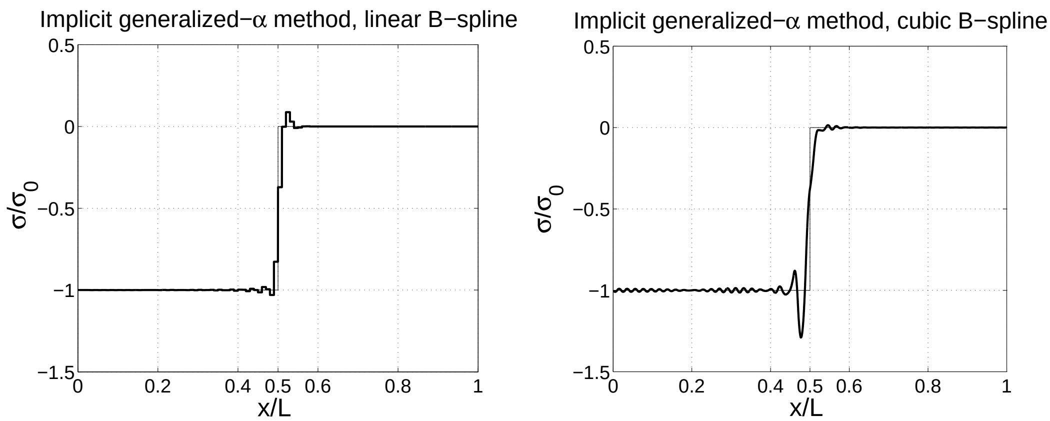

Fig. 6: Stress in an elastic bar under the shock loading at time $t = 0.5L/c_0$ computed by the implicit generalized α-method with $\rho_\infty = 0.5$ for linear (on the left) and cubic (on the right) B-splines.

Time step for the Newmark method is chosen as $\Delta t^{NM} = 1/8\, T_{min}$, where T_{min} is the minimal vibration period of the whole system (5). It is valid $T_{min} = 2\pi/\omega_{max}$, where ω_{max} is the maximum eigenfrequency of the whole system (5). The period elongation error for the Newmark method with this time step is smaller than 5%, see Hughes (1983). Time step for the central difference method is set with respect to the stability limit and good dispersion behaviour. Practically, time step is chosen as $\Delta t^{CDM} = 0.99999\Delta t_{crit}$, where the critical value is given by critical time step $\Delta t_{crit} = 2/\omega_{max}$, see paper Park (1977). The time step for the implicit form of generalized-α method is chosen with respect to the stability limit, see paper Hulbert and Chung (1996).

The bar is discretized by linear ($p = 1$) and cubic ($p = 3$) B-splines with $N = 101$ control points. For the linear B-spline discretization, the knot vector is used uniform, Piegl and Tiller (1997), and the control points are distributed uniformly with constant distances. For the cubic B-spline discretization, the knot vector is also employed uniform, but the control points are given by the Greville abscissa, see Greville (1967). Thus, this parameterization is linear. It means that the mapping from the parametric space to the geometrical one is linear and Jacobians of this transformation are constant values. This linear parametrization produces smaller dispersion and frequency errors than the uniform one, see paper Kolman et al. (2011). On the other side, the higher-order spline discretization with the linear parameterization shows the 'outlier frequencies' but with the smaller frequency errors that the non-linear parameterization, see Hughes et al. (2008). The 'outlier frequencies' correspond to the vibration of bar borders. These non-physical high frequencies influence the value of time step, and, in generally, also the accuracy and stability of the direct time integrations.

The courses of dimensionless stress σ/σ_0 along the bar computed by the Newmark method are depicted on Fig. 3 at time $t = 0.5L/c_0$. The results for the central difference method are shown on Fig. 4. For the implicit generalized-α method, the stress waveforms are presented for spectral radius $\rho = 0.8$ on Fig. 5 and for $\rho = 0.5$ on Fig. 6. The theoretical wavefront takes place in half of the bar and the stress value in the overlaying area should hold the magnitude $\sigma = -\sigma_0$.

6. Discussion and conclusions

In the numerical test of stress discontinuity propagation computed by B-spline variant of FEM, the oscillations near sharp wave-fronts are smaller than for the classical FEM due to the variation diminishing property and smaller dispersion errors. The post-shock oscillations are typical for the central difference method due to the 'row sum' diagonal mass matrix. This diagonal mass matrix is only of second order accuracy and also it produces unsuitable frequency spectrum. On the other side, the Newmark method and the implicit form of the generalized-α method produce the both types of oscillations, both post-shock and front-shock oscillations. However, the front-shock oscillations are dominant. Jumps in behaviour of a stress function obtained by the implicit form of the generalized-α method with high level of frequency dissipation are well approximated. Nevertheless, the total energy is not preserved. The best results have been obtained by the central difference method for the combination - uniform linear finite elements, lumped mass matrix and time step near the critical time step. In this case, dispersion effect and period elongation are reciprocally eliminated.

The post- and front-oscillations could be explained by dispersion behaviour of FEM for the consistent and diagonal mass matrices. The consistent mass matrix overestimates the wave speed. Thus, distortion of the wave-front comes up. Spurious oscillations occur in the front of the theoretical wave-front. On the other side, the diagonal mass matrix underestimates the wave speed and spurious oscillations occur behind of the theoretical wave-front. This effect for the central difference method is more intensified by higher-order non-homogeneous B-spline shape functions due to the end point interpolating property. The higher-order B-spline discretization with the 'row sum' diagonal mass matrix of second order accuracy produces frequency spectrum with extensive errors. The maximal frequency defines the global critical time step. But inside a bar, the local appropriate time step should be chosen considerably lower. Therefore, the central difference method integrates equations of motion with the time step performed for producing poor dispersion characteristics.

Acknowledgments

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