18th International Conference
ENGINEERING MECHANICS 2012
Svratka, Czech Republic, May 14 – 17, 2012

pp. 1195–1200
Paper #26

ON APPLICATIONS OF DISTRIBUTIONS TO ANALYSIS OF CIRCULAR PLATE DESIGN ELEMENTS

J. Sobotka*

Abstract: *The mathematical model of a circular plate according to Kirchhoff's theory contains classical derivatives of internal forces, moments, slopes of a middle surface and deflection. However these derivatives are not defined at such internal points of middle plane where a line lateral loading, a line moment loading, a line support, or a rotational coupling between plate segments is situated. We have used the distributional derivatives for the unknown discontinuous functions, and developed a generalized mathematical model in the form of a system of ordinary differential equations in order that the mathematical model of the circular plate subjected to axisymmetric loading may be valid also along the lines of discontinuity mentioned, which are common in calculating experience. We have found a general solution to the generalized system of differential equations by means of a symbolic programming approach using Maple.*

Keywords: ***Circular plate, discontinuities, Dirac singular distribution, Heaviside step function.***

1. Introduction

Solving analytically a circular or annular plate subjected to an axisymmetric bending with discontinuous loading, support or geometry, we at first divide the plate into segments without discontinuities. Then, we find continuous solutions with integration constants for each plate segment separately. Finally, we determine integration constants using boundary conditions and continuity conditions among adjoining plate segments.

Applying distributional derivative (Schwartz, 1966) for the transverse shear force, the radial bending moment, and for the slope of a middle-surface in a meridian plane, we can derive a generalized mathematical model of the circular plate with discontinuities in loading, support and geometry that can be solved like only one differential problem without dividing plates into segments, and without using continuity conditions.

2. The classical mathematical model of an axisymmetric bending of the plate

According to the Kirchhoff's plate bending theory, a system of differential equations describing axisymmetric bending of circular plates may be composed of five ordinary differential equations of the first order (Reddy, 1999; Höschl, 1971; Timoshenko & Woinowsky-Krieger, 1959) as follows:

$$\frac{d}{dr}(r\, q_r(r)) = -r\, \mathrm{p}(r) \quad , \tag{1}$$

$$\frac{d}{dr}(r\, m_r(r)) = m_t(r) + r\, q_r(r), \tag{2}$$

$$m_r(r) = B\left(\left(\frac{d}{dr}\phi(r)\right) + \frac{\mu\,\phi(r)}{r}\right) \quad , \tag{3}$$

$$m_t(r) = B\left(\mu\left(\frac{d}{dr}\phi(r)\right) + \frac{\phi(r)}{r}\right) \quad , \tag{4}$$

* Ing. Jiří Sobotka, Ph.D., ČEZ, a.s., 67550 Dukovany, tel. +420 561 10 5234, e-mail: jiri.sobotka@cez.cz

$$\frac{d}{dr}\mathrm{w}(r) = -\phi(r) \quad , \tag{5}$$

for

$$\left(\frac{d}{dx}\mathrm{w}(x)\right)^2 << 1 \tag{6}$$

and

$$B = \frac{E\,h^3}{12\,(1-\mu^2)} \quad . \tag{7}$$

where

$q_r(r)$	the transverse shear force per unit length,
$m_r(r)$	the radial bending moment per unit length,
$m_t(r)$	the circumferential bending moment per unit length,
$\phi(x)$	the slope of a middle-surface in a meridian plane,
$\mathrm{w}(x)$	the deflection,
$\mathrm{p}(r)$	the surface loading in normal direction to the middle plane,
r	the radial coordinate.
B	the plate bending stiffness,
E	Young's modulus,
μ	Poisson's ratio,
h	the thickness of the plate.

Equations (1), (2) are equilibrium conditions of a plate element cut out in the undeformed shape. Bending moments (3), (4) have been derived for curvatures supposing (6).

3. The normal form of the ordinary differential equations system

In order to be able to express discontinuities of unknown dependently variable quantities mathematically, we at first express the system of differential equations (1), (2), (3), (5) in the normal form as follows where the circumferential bending moment has been excluded using equation (4):

$$\frac{d}{dr}Q_{rc}(r) = -2\,\pi\,r\,\mathrm{p}(r) \quad , \tag{8}$$

$$\frac{d}{dr}M_{rc}(r) = \frac{\mu\,M_{rc}(r)}{r} + \frac{2\,\pi\,B\,(1-\mu^2)\,\phi(r)}{r} + Q_{rc}(r) \quad , \tag{9}$$

$$\frac{d}{dr}\phi(r) = \frac{1}{2}\,\frac{M_{rc}(r)}{\pi\,B\,r} - \frac{\mu\,\phi(r)}{r} \quad , \tag{10}$$

$$\frac{d}{dr}\mathrm{w}(r) = -\phi(r) \quad , \tag{11}$$

where the total transverse shear force is:

$$Q_{rc}(r) = 2\,\pi\,r\,q_r(r) \quad , \tag{12}$$

and the total radial bending moment is:

$$M_{rc}(r) = 2\,\pi\,r\,m_r(r) \quad . \tag{13}$$

4. Dividing discontinuities of loading, support and geometry into groups

All the given discontinuities can be divided into two groups according to their influence on discontinuities of unknown dependently variable quantities, i.e. generalized internal forces and generalized deformations.

The first group contains such discontinuities which do not cause another ones, so they can be expressed by means of Heaviside step function denoted further by Heaviside(x).

The second group contains such discontinuities which cause another ones, so they can be expressed by means of Dirac singular distribution denoted further by Dirac(x).

E.g., the discontinuous surface loading p(r) occurring in equation (1) or (8) belongs into the first group of given discontinuities. A line lateral loading, a line support or a line bending loading acting at internal points of the middle-plane or a line joint between plate segments belong into the second group of the given discontinuities.

The equation (8) is not valid at such internal points of the middle-plane of the plate where a line lateral loading is applied, or a line support is situated. Likewise, the equation (9) is not valid at such internal points of the middle-plane of the plate where a line bending loading is applied. Similarly, the equation (10) is not valid at such internal points of the middle-plane of the plate where a line joint along a circle is placed connecting plate segments. In order that the mathematical model of the axisymmetric bending of the circular or annular plate may be valid also at internal points of the middle-plane in which jump discontinuities of the lateral shear force or the bending moment or the slope of the middle-surface may occur, we have to extend the right-hand side of the equations (8), (9), (10) by adding appropriate distributional part according to the distributional derivative definition (Schwarz, 1966; Kanwal, 2004) that is as follows:

$$\frac{d}{dx}\mathrm{f}(x) = \{\frac{d}{dx}\mathrm{f}(x)\} + [f]_{x_0}\ \mathrm{Dirac}(x - x_0)\ , \tag{14}$$

where

$\{\frac{d}{dx}\mathrm{f}(x)\}$	the classical derivative,
$[f]_{x_0}$	the magnitude of the jump discontinuity of the quantity f(x) at point $x = x_0$; $[f]_{x_0} = \mathrm{f}(x_0 + 0) - \mathrm{f}(x_0 - 0)$
$\mathrm{f}(x_0 + 0)$	the right-hand limit of f(x) at point x_0,
$\mathrm{f}(x_0 - 0)$	the left-hand limit of f(x) at point x_0,
$\mathrm{Dirac}(x - x_0)$	Dirac distribution with its singularity at point $x = x_0$.

5. The generalized mathematical model of axisymmetric bending of the circular plate

Considering n_1 line lateral loadings, n_2 line support reactions, and applying the equation (14) along with the equation (8), we can generalize the force equilibrium equation of the plate element into the form of the equation (15), which is valid also for a discontinuous diagram of the transverse shear force. Likewise, considering n_3 line bending loadings, and applying the equation (14) along with the equation (9), we can generalize the moment equilibrium equation of the plate element into the form of the equation (16), which is valid also for a discontinuous diagram of the bending moment. Similarly, considering n_4 line joints connecting plate segments, and applying the equation (14) along with the equation (10), we can generalize the first derivative of the slope into the form of the equation (17), which is valid also for a discontinuous diagram of the slope of the middle-surface of the plate in a meridian plane:

$$\frac{d}{dr} Q_{rc}(r) = -2\,\pi\, r\, \mathrm{p}(r) - \left(\sum_{i=1}^{n_1} F_i \,\mathrm{Dirac}(x - a_i) \right) + \left(\sum_{i=1}^{n_2} R_i \,\mathrm{Dirac}(x - b_i) \right) \quad , \tag{15}$$

$$\frac{d}{dr} M_{rc}(r) = \frac{\mu\, M_{rc}(r)}{r} + \frac{2\,\pi\, B\,(1 - \mu^2)\,\phi(r)}{r} + Q_{rc}(r) + \left(\sum_{i=1}^{n_3} C_i \,\mathrm{Dirac}(x - g_i) \right) \quad , \tag{16}$$

$$\frac{d}{dr} \phi(r) = \frac{1}{2}\,\frac{M_{rc}(r)}{\pi\, B\, r} - \frac{\mu\,\phi(r)}{r} + \left(\sum_{i=1}^{n_4} \Phi_i \,\mathrm{Dirac}(x - k_i) \right) \quad , \tag{17}$$

$$\frac{d}{dr} \mathrm{w}(r) = -\phi(r) \quad , \tag{18}$$

where

F_i	the i-th line lateral loading on a 360º basis,
R_i	the i-th line reaction force on a 360º basis,
C_i	the i-th line bending loading on a 360º basis,
Φ_i	the magnitude of a jump discontinuity of the slope, $\phi(x)$, at i-th line joint connection of plate segments,
a_i	the radius of the circle along which the line lateral loading F_i is applied,
b_i	the radius of the circle along which the plate is supported, and where a line reaction force R_i may occur,
g_i	the radius of the circle along which the line bending loading C_i is applied,
k_i	the radius of the circle along which two plate segments are joined, and where a jump discontinuity Φ_i of the slope may occur,
n_1	the number of line lateral loadings,
n_2	the number of line supports,
n_3	the number of line moment loadings,
n_4	the number of line joints connecting plate segments.

6. The general solution to the generalized system of differential equations (15) to (18)

Using equations (15) and (17), we can eliminate the shear force and the bending moment from the equation (16), receiving a final equation for only one unknown quantity, i.e. the slope of the middle-surface. We can integrate the final equation by using integrating factors.

In order that results gained may be easier to survey, we have computed the general solutions to the system of equations (15) to (18) for simple cases of discontinuous loading and geometry: i) a line lateral loading, ii) a line bending loading, iii) a line joint along a circle between plate segments.

A particular solution to the inhomogeneous system (15) to (18) for a combination of the line loadings, supports or joint couplings of the plate segments can be found as a sum of simple particular solutions presented further, where the particular solution for a line support is different from the particular solution for the line lateral loading by minus sign. Integration constants have been denoted by c_1, c_2, c_3, c_4. In order to determine the integration constants, we have to formulate boundary conditions. To determine support reactions at internal points of the middle-plane of the plate, we have to create deformation conditions of the line supports. Likewise, so as to be able to determine the magnitude of the jump discontinuity of the slope of the middle-surface at the line joint between plate segments, we have to put a deformation condition of the joint together.

6.1. The general solution to the system (15) to (18) for one line lateral loading

Let the line lateral loading be denoted by $F_1 = F$ acting along a circle of diameter $r=a_1=a$, i.e. $n_1=1$, $n_2=n_3=n_4=0$. In this case, the general solution can be expressed as follows:

$$Q_{rc}(r) = -F\,\mathrm{Heaviside}(r-a) - 16\,B\,\pi\,c_4 \tag{19}$$

$$M_{rc}(r) = (-4\pi\mu - 4\pi)\,r\,B\,c_2 + \frac{(2\pi - 2\pi\mu)\,B\,c_3}{r} + ((-8\pi - 8\pi\mu)\ln(r) - 8\pi)\,r\,B\,c_4 + \left(\left(\left(-\frac{\mu}{2}-\frac{1}{2}\right)\ln(r) + \frac{1}{2}\mu\ln(a) + \frac{1}{2}\ln(a) - \frac{1}{4} + \frac{\mu}{4}\right)r + \frac{-\frac{1}{4}\mu a^2 + \frac{1}{4}a^2}{r}\right)F\,\mathrm{Heaviside}(r-a) \tag{20}$$

$$\phi(r) = -2\,r\,c_2 - \frac{c_3}{r} - 4\,r\ln(r)\,c_4 + \frac{\left(\left(-\frac{1}{4}\frac{\ln(r)}{\pi} - \frac{1}{8}\frac{-2\ln(a)-1}{\pi}\right)r - \frac{a^2}{8\pi r}\right)\mathrm{Heaviside}(r-a)\,F}{B} \tag{21}$$

$$\mathrm{w}(r) = c_1 + r^2 c_2 + \ln(r)\,c_3 + r^2(2\ln(r)-1)\,c_4 + \frac{\left(\left(\frac{1}{8}\frac{\ln(r)}{\pi} + \frac{1}{8}\frac{-\ln(a)-1}{\pi}\right)r^2 + \frac{1}{8}\frac{a^2\ln(r)}{\pi} + \frac{1}{8}\frac{-a^2\ln(a)+a^2}{\pi}\right)F\,\mathrm{Heaviside}(r-a)}{B} \tag{22}$$

6.2. The general solution to the system (15) to (18) for one line bending loading

Let the line bending loading be denoted by $C_1 = C$ acting along a circle of diameter $r = g_1 = g$, i.e. $n_3=1$, $n_1=n_2=n_4=0$. In this case, the general solution can be expressed as follows:

$$Q_{rc}(r) = -16\,\pi\,B\,c_4\ , \tag{23}$$

$$M_{rc}(r) = (-4\pi - 4\pi\mu)\,r\,B\,c_2 + \frac{(-2\pi\mu + 2\pi)\,B\,c_3}{r} + ((-8\pi\mu - 8\pi)\ln(r) - 8\pi)\,r\,B\,c_4 + \left(\frac{\left(-\frac{\mu}{2}+\frac{1}{2}\right)g}{r} + \frac{\left(\frac{\mu}{2}+\frac{1}{2}\right)r}{g}\right)C\,\mathrm{Heaviside}(r-g)\ , \tag{24}$$

$$\phi(r) = -2\,r\,c_2 - \frac{c_3}{r} - 4\,r\ln(r)\,c_4 + \frac{\left(-\frac{g}{4r\pi} + \frac{r}{4\pi g}\right)\mathrm{Heaviside}(r-g)\,C}{B}\ , \tag{25}$$

$$\mathrm{w}(r) = c_1 + r^2 c_2 + \ln(r)\,c_3 + r^2(2\ln(r)-1)\,c_4 + \frac{\left(\left(\frac{1}{4}\frac{\ln(r)}{\pi} + \frac{1}{8}\frac{-2\ln(g)+1}{\pi}\right)g - \frac{r^2}{8\pi g}\right)C\,\mathrm{Heaviside}(r-g)}{B}\ . \tag{26}$$

6.3. The general solution to the system (15) to (18) for one line joint between plate segments

Let the magnitude of the jump discontinuity of the slope be denoted by $\Phi_1 = \Phi$ occurring along a circle of diameter $r = k_1 = k$, i.e. $n_4 = 1$, $n_1=n_2=n_3=0$. In this case, the general solution can be expressed as follows:

$$Q_{rc}(r) = -16\,\pi\,B\,c_4 \tag{27}$$

$$\begin{aligned} M_{rc}(r) = {} & -\pi\,(4\,\mu + 4)\,r\,B\,c_2 - \frac{\pi\,(-2 + 2\,\mu)\,B\,c_3}{r} + (-\pi\,(8 + 8\,\mu)\ln(r) - 8\,\pi)\,r\,B\,c_4 \\ & + \left(-\frac{\pi\,(1-\mu^2)\,k}{r} - \frac{\pi\,(\mu^2 - 1)\,r}{k}\right) B\,\Phi\,\mathrm{Heaviside}(r-k) \end{aligned} \tag{28}$$

$$\phi(r) = -2\,r\,c_2 - \frac{c_3}{r} - 4\,r\,c_4 \ln(r) + \left(\frac{\left(\frac{\mu}{2} + \frac{1}{2}\right)k}{r} + \frac{\left(-\frac{\mu}{2} + \frac{1}{2}\right)r}{k}\right)\mathrm{Heaviside}(r-k)\,\Phi \tag{29}$$

$$\begin{aligned} \mathrm{w}(r) = {} & c_1 + r^2\,c_2 + \ln(r)\,c_3 + r^2\,(2\ln(r) - 1)\,c_4 \\ & + \left(\left(\left(-\frac{\mu}{2} - \frac{1}{2}\right)\ln(r) + \frac{1}{2}\ln(k) + \frac{1}{4} - \frac{\mu}{4} + \frac{1}{2}\ln(k)\,\mu\right)k + \frac{\left(-\frac{1}{4} + \frac{\mu}{4}\right)r^2}{k}\right)\Phi\,\mathrm{Heaviside}(r-k) \end{aligned} \tag{30}$$

7. Conclusions

The contribution of this paper is that the generalized mathematical model of the plate, Eq. (15) to (18), is valid also for discontinuous graphs of the transverse shear force, the radial bending moment, and the slope of the middle-surface in a meridian plane caused by line lateral loadings or line supports, line moment loadings situated along the circles lying inside the middle-plane of the plate, and by line joints connecting plate segments, resp.. The jump discontinuities of the unknown dependently variable quantities have been expressed using Dirac singular distribution at the right side of Eq. (15) to (17).

In order to determine magnitudes of the unknown jump discontinuities owing to the internal along-circular supports or joints, we have to use deformation conditions at points of these geometric discontinuities.

The general solution to the system of ordinary differential equations (15) to (18) has been computed using symbolic programming approach, and has been partly presented in Eq. (19) to (30).

References

Höschl, C. (1971) *Pružnost a pevnost ve strojnictví*. SNTL, Praha.

Kanwal, R. P. (2004) *Generalized Functions*. Birkhäuser, Boston.

Němec, J., Dvořák, J. & Höschl, C. (1989) *Pružnost a pevnost ve strojírenství*. SNTL, Praha.

Reddy, J. N. (1999) *Theory and Analysis of Elastic Plates*. Taylor & Francis, Philadelphia.

Schwartz, L., (1966) *Théorie des distributions*. Hermann, Paris.

Štěpánek, J. (2001) *Distribuce a diferenciální rovnice*. Karolinum, Praha.

Timoshenko, S. & Woinowsky-Krieger, S. (1959) *Theory of plates and shells*. McGraw-Hill, N.Y.