18th International Conference
ENGINEERING MECHANICS 2012
Svratka, Czech Republic, May 14 – 17, 2012
pp. 915–926
Paper #32

GENERALIZED LINEAR MODEL WITH AERO-ELASTIC FORCES VARIABLE IN FREQUENCY AND TIME DOMAINS

J. Náprstek, S. Pospíšil [1]

Abstract: *Behavior of slender aero-elastic systems in a sub-critical domain including position of the lowest critical state is commonly investigated using double degree of freedom (DDOF) linear model. The most frequently used are neutral models treating aero-elastic forces as certain constants corresponding to system parameters and stream velocity. Although this approach is working well, it shows a number of shortcomings. For this reason modeling by flutter derivatives or indicial functions has been launched. However, these two groups of models have been developed separately one from each other. It seems they are rather isolated until now. Moreover they mostly suffer from various gaps in mathematical formulations and further treatment. The paper tries to put all three groups together on one common basis and to demonstrate linkage of them. This approach allows formulate more sophisticated models combining main aspects of all groups in question keeping the DDOF basis. These models correspond by far better to results of wind channel and full scale measurements.*

Keywords: ***Flutter derivatives, Indicial functions, Non-symmetric systems, Dynamic stability.***

1. Introduction

Slender prismatic structures exhibited to strong dynamic wind effects (bridge decks, towers, chimneys, etc.) are frequently analyzed using a double degree of freedom (DDOF) linear model working with heaving and torsional components of a cross-section, see e.g. Bartoli & Righi (2006). This aero-elastic model is often adequate to study the system response until the first critical state is reached. Relevant mathematical models appearing in literature differ in principle by way of composition of aero-elastic forces. This criterion enables to sort them roughly in three groups. The first group can be possibly called neutral models - aero-elastic forces are introduced as suitable constants independent from excitation frequency and time. The second one involves flutter derivatives - they respect the frequency dependence of aero-elastic forces, see Scanlan & Tomko (1971).

Finally the third is working with indicial functions - they are defined as kernels of convolution integrals formulating aero-elastic forces as functions of time, see Wagner (1929), Küssner (1954), Garrick (1938) and Scanlan, Beliveau & Budlong (1974). Second and third groups have been developing separately from each other and seem to be isolated until now, see **?** and Costa et al. (2007) for example. Moreover they mostly suffer from various gaps in mathematical formulations and further treatment. The paper tries to put all three groups together on one common basis and to demonstrate linkage of them. This approach allows formulate more sophisticated models combining main aspects of all groups keeping the DDOF basis. These models correspond by far better to results of wind channel and full scale measurements and seem to be very promising for the future investigation and practical applications.

For purposes of this study the bridge girder is considered as axially symmetric or almost symmetric with possible response components in heave u (vertical direction) and pitch φ (rotation around S point). An outline can be seen in Fig.1. In principle all types of above models have been investigating many years. Each of them has its advantages and shortcomings. However most of them suffer very often from mathematical gaps preventing their generalization and synthesis on formal basis in order to identify some special phenomena remaining hidden when dealing with heuristic approaches only. Let us characterize now briefly the groups of models mentioned above in forthcoming parts.

[1] Ing. Jiří Náprstek, DrSc., Doc.Ing. Stanislav Pospíšil, PhD.
Institute of Theoretical and Applied Mechanics ASCR, v.v.i.; Prosecká 76, 190 00 Prague 9,
tel. +420.286 892 515, e-mail: naprstek@itam.cas.cz, pospisil@itam.cas.cz

2. Neutral models

Neutral models are relatively the most simple and enable to provide many results analytically in a form of closed solution. These models have been extensively studied for instance in Pospíšil & Náprstek (2011).

Although there exist many versions of a basic formulation, in principle the most general model of neutral type can be expressed in the form:

$$\begin{aligned} \ddot{u} + b_m \cdot \dot{u} - hq \cdot \dot{\varphi} + \omega_u^2 \cdot u - p \cdot \varphi &= 0 \\ \ddot{\varphi} + q \cdot \dot{u} + b_I \cdot \dot{\varphi} + gp \cdot u + \omega_\varphi^2 \cdot \varphi &= 0 \end{aligned} \tag{1}$$

where we have denoted: $\omega_u^2, \omega_\varphi^2 -$ total eigen-frequencies in relevant components including stiffness and aero-elastic components; $b_m, b_I -$ total damping parameters including internal structural damping and aero-elastic contribution; $q[(ms)^{-1}]$ or $p[m \cdot s^{-2}]$ gyroscopic or non-conservative forces of aero-elastic origin respectively; $g[m^{-2}], h[m^2]$ auxiliary constants serving for dimensional compatibility of the above equations (they can be regarded as certain characteristics of the cross-section).

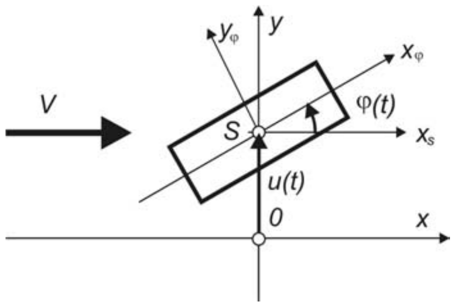

Fig. 1: Schematic DDOF model of a bridge symmetric cross-section under wind loading.

Parameters q, p in general don't include any static components which follow from elastic properties of the system itself, they consist only of aero-elastic terms vanishing for zero velocity of the air stream. So for stream velocity $V = 0$, the system (1) degenerates in two independent equations.

The main tool for stability investigation is, together with the system (1), its characteristic equation:

$$\begin{aligned} D = \lambda^4 + \lambda^3(b_m + b_I) + \lambda^2(\omega_u^2 + \omega_\varphi^2 + b_m b_I + hq^2) + \\ + \lambda(\omega_u^2 b_I + \omega_\varphi^2 b_m + (1 + gh)pq) + \omega_u^2 \omega_\varphi^2 + gp^2 &= 0 \end{aligned} \tag{2}$$

The resulting characteristic equation represents annuled polynomial of the fourth order $(n = 4)$ with roots $\lambda_1, \lambda_2, \lambda_3$ and λ_4. The trivial solution of system (1) is stable only if a real part of all four roots is negative. In other words, stability limits are given by conditions:

$$\mathrm{Re}(\lambda_i) = 0, \quad i \in (1, \ldots, 4) \tag{3}$$

Consequently, the trivial solution of system (1) is stable in a domain representing an intersection of sub-domains $\mathrm{Re}(\lambda_i) < 0, \quad i \in (1, \ldots, 4)$.

The system (1) and the characteristic equation (2) can provide a lot of information regarding motion stability, critical velocities V_{crit}, system response on stability limits, etc. Consequently, it enables to predict flutter/divergence onset velocity as well as to estimate their shapes in a particular case. However aero-elastic coefficients in Eqs (1) are introduced as constants corresponding to certain conditions ruling around the cross-section. Anyway, these coefficients are functions of V and ω, and therefore some iterative process should follow balancing these effects in order to harmonize velocity V with velocity V_{crit}. Despite these shortcomings the applicability of neutral models is quite wide if the variability of the aero-elastic terms is approximately linear. Otherwise one of more sophisticated models should be used, as we will see in next two parts.

Strategy of the stability investigation can be based on Routh-Hurwitz inspection of Eq. (2). The detailed analysis and relevant results can be found e.g. in Náprstek & Pospíšil (2001), Náprstek, J. (2007). The most important types of aero-elastic stability loss (flutter and divergence) and their possible interactions are there given together with the conditions of their existence.

3. Models with flutter derivatives

Flutter derivatives have been introduced many years ago, see for instance Theodorsen (1935) and more recently Poulsen, Damsgaard & Reinhold (1992). Their various aspects have been investigated extensively for a long time in the aircraft, civil and other branches of engineering. They have been introduced as functions in the frequency domain related to a particular cross-section without any link with other system parameters (inertia, elastic stiffness, internal damping). Nevertheless they can be understood as a certain extension of the damping and stiffness matrix elements. Flutter derivatives can be interpreted as amplitudes Q or M of the heaving forces or the pitching moments, respectively, which should reach a unit amplitude of one response component under harmonic external kinematic excitation, while remaining components are kept zero in the same time. Thus the flutter derivatives are the dimensionless functions of the excitation frequency ω, stream velocity V and geometric characteristic of the cross-section B $[m]$. They are combined in one dimensionless argument $\kappa = B\omega/V$. So the basic relations between kinematic and force components can be roughly outlined:

	$\dot{u}$	u	$\dot{\varphi}$	φ
Q :	$H_1(\kappa)$	$H_4(\kappa)$	$H_2(\kappa)$	$H_3(\kappa)$
M :	$A_1(\kappa)$	$A_4(\kappa)$	$A_2(\kappa)$	$A_3(\kappa)$
Q :	$A_{11}(\kappa)$	$A_{12}(\kappa)$	$A_{13}(\kappa)$	$A_{14}(\kappa)$
M :	$A_{21}(\kappa)$	$A_{22}(\kappa)$	$A_{23}(\kappa)$	$A_{24}(\kappa)$

$$; \qquad \kappa = \frac{B\omega}{V} \tag{4}$$

where following notation has been introduced: $H_i(\kappa)$ or $A_i(\kappa)$ - amplitudes of flutter derivatives corresponding to heaving forces Q or pitching moments M amplitudes due to individual sets of unit kinematic harmonic excitations of a proper cross-section in an aerodynamic tunnel (notation and indexing corresponds to literature referenced); $A_{ij}(\kappa)$ - alternative notification of flutter derivatives assigned with respect to the table in Eq. (4);

Since we try to write down final formulae of Q or M amplitudes, there appear expressions of the type $\kappa A_{11}(\kappa) \cdot \dot{u}(t)$, $\kappa A_{23}(\kappa) \cdot \dot{\varphi}(t)$, etc. They are to see everywhere since classical until contemporary literature, e.g. [Scanlan] and many others. However, they are inconsistent mixing both frequency and time variables together. Subsequent integral transform would be unapplicable. Therefore respecting harmonic regime of the flutter derivatives (functions of ω) also displacements $u(t), \varphi(t)$ and their time derivatives should be expressed correspondingly, for instance in the form of their Fourier transform. It means in particular as $\mathrm{i}\omega U, U, \mathrm{i}\omega\Phi, \Phi$. So that with reference to notification (4) the heaving and pitching aero-elastic forces in the frequency domain can be written as follows:

$$\begin{aligned} Q(\omega) &= \mu_m V^2 \left(\tfrac{\mathrm{i}\omega B}{V}\kappa A_{11} + \kappa^2 A_{12}\right) U + \mu_m V^2 \left(\tfrac{\mathrm{i}\omega B^2}{V}\kappa A_{13} + \kappa^2 B A_{14}\right) \Phi, \quad \mu_m = \varrho/m \\ M(\omega) &= \mu_I V^2 B^2 \left(\tfrac{\mathrm{i}\omega}{V}\kappa A_{21} + \tfrac{1}{B}\kappa^2 B A_{22}\right) U + \mu_I V^2 B^2 \left(\tfrac{\mathrm{i}\omega B}{V}\kappa A_{23} + \kappa^2 A_{24}\right) \Phi, \ \mu_I = 2\varrho/I \end{aligned} \tag{5}$$

where m or I are a mass or mass inertia moment of the vibrating body and ϱ is a specific mass of the air.

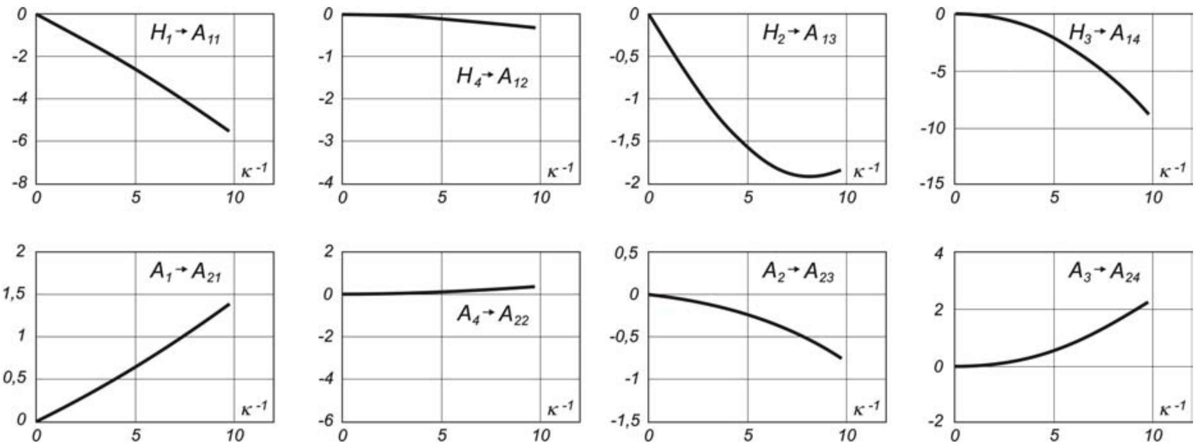

Fig. 2: Outline of flutter derivatives; rectangular cross-section, ratio 1:5; position of $A_{11} - A_{24}$ pictures correspond with table in Eq. (4).

The definition itself of flutter derivatives apparently implicates that they can serve only to develop a linear mathematical model as their application is based on the superposition principle. Flutter derivatives can be incorporate into the governing equations of type (1) only if these equations are expressed in the frequency domain. Hence the system (1) should be written in a form of the two-way Laplace transform (integration $t \in (-\infty, +\infty)$) to unify the basis of individual parts. Transformation exists if the system is stable and therefore influence of initial conditions disappear with increasing time. It means, however, that only steady state problems with explicit frequency $\omega = -\mathrm{i}\lambda$ can be investigated. Finally we write the complete system in the frequency domain, so that it has a character of an algebraic (unknowns $U,\ \Phi$) system:

$$\begin{array}{r} Q: \\ \\ M: \\ \\ \end{array} \left| \begin{array}{l} \lambda^2 + \lambda \cdot (b_m + \mu_m VB \cdot \kappa A_{11}) + (\omega_u^2 + \mu_m V^2 \cdot \kappa^2 A_{12})\ ; \\ \qquad\qquad \lambda \cdot \mu_m VB^2 \cdot \kappa A_{13} + \mu_m V^2 B \cdot \kappa^2 A_{14} \\ \lambda \cdot \mu_I VB^2 \cdot \kappa A_{21} + \mu_I V^2 B \cdot \kappa^2 A_{22}\ ; \\ \qquad\qquad \lambda^2 + \lambda \cdot (b_I + \mu_I VB^3 \cdot \kappa A_{23}) + (\omega_\varphi^2 + \mu_I V^2 B^2 \cdot \kappa^2 A_{24}) \end{array} \right| \cdot \left| \begin{array}{c} U \\ \\ \Phi \end{array} \right| = \left| \begin{array}{c} 0 \\ \\ 0 \end{array} \right| \tag{6}$$

The shape of flutter derivatives for the rectangular cross-section as they are plotted in Fig. 2 is commonly accepted. Let us go briefly through individual graphs in this figure. It can be observed that functions A_{ij} related with $\dot{u}, \dot{\varphi}$ are odd functions, while those related to u, φ are even with respect to the vertical axis. Indeed this fact can be shown also theoretically using Theodorsen functions, see Theodorsen (1935). Looking through Fig. 2, it is obvious that the courses of individual A_{ij} are not "dramatic". Hence with respect to the interval length needed on the $1/\kappa$ axis, only the first and the second terms of the odd or even polynomial expansions seems to be satisfactory to characterize A_{ij} in equations (6). Thus for instance:

$$A_{11} \approx a_{11}\frac{1}{\kappa} + b_{11}\frac{1}{\kappa^3}\ , \quad A_{12} \approx a_{12}\frac{1}{\kappa^2} + b_{12}\frac{1}{\kappa^4}\ , \quad \text{etc.} \tag{7}$$

where a_{ij} and b_{ij} are relevant dimensionless coefficients of the polynomial expansion. These coefficients can be obtained fitting relevant polynomials into experimental results.

Let us note that function values of A_{12} and A_{22} are markedly small. Indeed, dealing with a symmetrical cross-section and supposing perfectly uniform stream velocity in a wind tunnel, functions A_{12} and A_{22} should vanish identically. Their non-zero values presented in Fig. 2 are most probably the results of imperfections ruling in experiments. Despite this fact, these terms have been included to keep theoretical consistency of relevant matrices (many papers omit those and work with six derivatives only).

Let us introduce polynomial expansions Eqs (7) into Eqs (6). Being aware that $\omega^2 = \lambda^2$, one obtains a modified system:

$$\begin{array}{r} Q: \\ M: \end{array} \left| \begin{array}{l} q_{11},\ q_{12} \\ q_{21},\ q_{22} \end{array} \right| \cdot \left| \begin{array}{c} U \\ \Phi \end{array} \right| = \left| \begin{array}{c} 0 \\ 0 \end{array} \right| \tag{8}$$

$$\begin{aligned} q_{11} &= (\lambda^2 + \lambda(b_m + \mu_m VBa_{11}) + (\omega_u^2 + \mu_m V^2 a_{12})) - \mu_m(\tfrac{1}{\lambda}\tfrac{V^3}{B}b_{11} + \tfrac{1}{\lambda^2}\tfrac{V^4}{B^2}b_{12})\ , \\ q_{12} &= (\lambda\mu_m VB^2 a_{13} + \mu_m V^2 Ba_{14}) - \mu_m(\tfrac{1}{\lambda}V^3 b_{13} + \tfrac{1}{\lambda^2}\tfrac{V^4}{B}b_{14})\ , \\ q_{21} &= (\lambda\mu_I VB^2 a_{21} + \mu_I V^2 Ba_{22}) - \mu_I(\tfrac{1}{\lambda}V^3 b_{21} + \tfrac{1}{\lambda^2}\tfrac{V^4}{B}b_{22})\ , \\ q_{22} &= (\lambda^2 + \lambda(b_I + \mu_I VB^3 a_{23}) + (\omega_\varphi^2 + \mu_I V^2 B^2 a_{24})) - \mu_I(\tfrac{1}{\lambda}V^3 Bb_{23} + \tfrac{1}{\lambda^2}V^4 b_{24})\ . \end{aligned} \tag{9}$$

The neutral models following Eqs (1) include system parameters, which implicitly incorporate the influence of surrounding air, for instance $b_m = b_{m,syst} + b_{m,air}$, etc. Depending on a strategy of a particular analysis the additional part $b_{m,air}$ is subsequently considered as a function of the stream velocity V, but any relation with the frequency ω is always omitted. Anyway, terms containing coefficients a_{ij} in Eqs (9) can be considered as a certain "first approximation", e.g. $b_{m,air} = \mu_m VCa_{11}$, $\omega^2_{u,air} = \mu_m V^2 a_{12}$, etc. So that respecting terms with a_{ij}, one obtains result analogous with the neutral model Eqs (1), where the dependence on the stream velocity V is obvious approaching zero with $V \to 0$. It is apparent, see Fig. 2, that a_{ij} can be positive or negative. Therefore introduction of terms with a_{ij} can result (on the level of the "first approximation") in an increase or a decrease of effective system parameters due to the aero-elastic effects. In particular non-conservative and gyroscopic character of the system follows solely from these terms as the system itself doesn't contain extra-diagonal elements. Non-symmetric character of the system results from character of a_{ij} signs:

$$a_{13} < 0, \quad a_{14} < 0, \quad a_{21} > 0, \quad a_{22} > 0 \tag{10}$$

and therefore symbolical link of some coefficients in Eqs (1) and elements specified in Eqs (9) can be written:

$$\begin{array}{lll} -p = \mu_m V^2 B a_{14} < 0, & g = \mu_I a_{22}/\mu_m a_{14} < 0, & gp = \mu_I V^2 B a_{22} > 0, \\ q = \mu_I V B^2 a_{21} > 0, & h = \mu_m a_{13}/\mu_I a_{21} < 0, & -hq = \mu_m V B^2 a_{13} < 0. \end{array} \tag{11}$$

Coefficients b_{ij} represent the most simple quantification of the frequency ω influence within aero-elastic forces, see Eqs (9). Looking over Fig. 2 we can see, that the second terms in expansions Eqs (7) arithmetized by coefficients b_{ij} can be considered significantly smaller especially for rising ω.

In order to inspect the primary form of the differential system, let us make an inverse transform of Eqs (8), (9) back to the time domain. After tedious manipulation, the differential system including the influence of the flutter derivatives on the level of approximation Eqs (7) can be written as follows:

$$\begin{aligned} \ddot{u} \; &+ b_m(\dot{u} - \eta^2\beta_{11} \int_{-\infty}^{t} u(\tau)\mathrm{d}\tau) - hq(\dot{\varphi} - \eta^2\beta_{13} \int_{-\infty}^{t} \varphi(\tau)\mathrm{d}\tau) \\ &+ \omega_u^2(u - \eta^2\beta_{12} \int_{-\infty}^{t} (t-\tau)u(\tau)\mathrm{d}\tau) - p(-\eta^2\beta_{14} \int_{-\infty}^{t} (t-\tau)\varphi(\tau)\mathrm{d}\tau) \quad = \; 0 \\ \ddot{\varphi} \; &+ q(\dot{u} - \eta^2\beta_{21} \int_{-\infty}^{t} u(\tau)\mathrm{d}\tau) + b_I(\dot{\varphi} - \eta^2\beta_{23} \int_{-\infty}^{t} \varphi(\tau)\mathrm{d}\tau) \\ &+ gp(\dot{u} - \eta^2\beta_{22} \int_{-\infty}^{t} (t-\tau)\varphi(\tau)\mathrm{d}\tau) + \omega_\varphi^2(\varphi - \eta^2\beta_{24} \int_{-\infty}^{t} (t-\tau)\varphi(\tau)\mathrm{d}\tau) \; = \; 0 \end{aligned} \tag{12}$$

Both of the systems (8), (12) can be immediately used for further investigation. The system (12) is an extension of (1). It demonstrates memory properties due to convolution integrals. In principle the integrals could be avoided differentiating twice both equations of the system. However resulting equations of the fourth order are less suitable for further analysis than the form of Eqs (12). Especially stability of the numerical solution of Eqs (12) is far better.

Nevertheless, let us focus to a main tool of the dynamic stability analysis. It follows from the system (8) representing a condition of its zero determinant. Provided the matrix of the system Eq. (8) is multiplied by a factor λ^2, the condition of the stability gets a form of the characteristic equation of the eight degree of the parameter λ:

$$a_0\lambda^8 + a_1\lambda^7 + a_2\lambda^6 + a_3\lambda^5 + a_4\lambda^4 + a_5\lambda^3 + a_6\lambda^2 + a_7\lambda + a_8 = 0 \tag{13}$$

$$\begin{aligned} a_0 &= 1\,, && \text{(a)} \\ a_1 &= (b_m + b_I) + (\mu_m V B a_{11} + \mu_I V B^3 a_{23})\,, && \text{(b)} \\ a_2 &= (\omega_u^2 + \omega_\varphi^2 + b_m b_I) + (\mu_m V^2 a_{12} + \mu_I V^2 B^2 a_{24} + b_m \mu_I V B^3 a_{23} + b_I \mu_m V B a_{11} && \text{(c)} \\ &\quad + \mu_m \mu_I V^2 B^4 (a_{11}a_{23} - a_{13}a_{21}))\,, \\ a_3 &= b_m\omega_\varphi^2 + b_I\omega_u^2 + \mu_m\omega_\varphi^2 V B a_{11} + \mu_I \omega_u^2 V B^3 a_{23} + \mu_I b_m V^2 B^2 a_{24} + \mu_m b_I V^2 a_{12} && \text{(d)} \\ &\quad + \mu_m\mu_I V^3 B^3 (a_{11}a_{24} + a_{12}a_{23} - a_{13}a_{22} - a_{14}a_{21}) \\ &\quad - \mu_I V^3 B b_{23} - \mu_m \frac{V^3}{B} b_{11}\,, \end{aligned} \tag{14}$$

$$\begin{aligned}
a_4 &= \omega_u^2\omega_\varphi^2 + \mu_m\omega_\varphi^2 V^2 a_{12} + \mu_I\omega_u^2 V^2 B^2 a_{24} + \mu_m\mu_I V^4 B^2(a_{12}a_{24} - a_{14}a_{22}) && \text{(e)}\\
&\quad + \mu_m\mu_I V^4 B^2(-a_{23}b_{11} - a_{11}b_{23} + a_{21}b_{13} + a_{13}b_{21})\\
&\quad - \mu_I b_m V^3 B b_{23} - \mu_m b_I \tfrac{V^3}{B} b_{11} - \mu_I V^4 b_{24} - \mu_m \tfrac{V^4}{B^2} b_{12}\ ,\\
a_5 &= \mu_m\mu_I V^5 B(-a_{11}b_{24} - a_{23}b_{12} - a_{12}b_{23} - a_{24}b_{11} && \text{(f)}\\
&\qquad + a_{14}b_{21} + a_{22}b_{13} + a_{13}b_{22} + a_{21}b_{14})\\
&\quad - \mu_I b_m V^4 b_{24} - \mu_m b_I \tfrac{V^4}{B^2} b_{12} - \mu_I\omega_u^2 V^3 B b_{23} - \mu_m\omega_\varphi^2 \tfrac{V^3}{B} b_{11}\ ,\\
a_6 &= \mu_m\mu_I V^6(-a_{12}b_{24} - a_{24}b_{12} + a_{14}b_{22} + a_{22}b_{14}) && \text{(g)}\\
&\quad - \mu_I\omega_u^2 V^4 b_{24} - \mu_m\omega_\varphi^2 \tfrac{V^4}{B^2} b_{12} + \mu_m\mu_I V^6(b_{11}b_{23} - b_{21}b_{13})\ ,\\
a_7 &= \mu_m\mu_I \tfrac{V^7}{B}(-b_{14}b_{21} - b_{13}b_{22} + b_{11}b_{24} + b_{12}b_{23})\ , && \text{(h)}\\
a_8 &= \mu_m\mu_I \tfrac{V^8}{B^2}(b_{12}b_{24} - b_{14}b_{22})\ . && \text{(i)}
\end{aligned}$$

As it has been mentioned terms containing b_{ij} are relatively small and represent a certain "correction" of the main part which is given by terms with a_{ij}. Moreover b_{ij} related with $\dot{u}, u$ are even significantly smaller than those related with $\dot{\varphi}, \varphi$. So it can be put approximately: $b_{11} \approx 0,\ b_{12} \approx 0,\ b_{21} \approx 0,\ b_{22} \approx 0$. Therefore in Eqs (14) coefficients $a_7, a_8 = 0$ and $a_3 - a_6$ get simpler. Roots $\lambda_7, \lambda_8 \neq 0$ and hence the remaining part of the characteristic equation Eq. (13) can be divided by λ^2. Finally the degree of the characteristic equation drops from eight to six. Thus let us rewrite this one together with simplified coefficients $a_0 - a_6$:

$$a_0\lambda^6 + a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6 = 0 \tag{15}$$

$$\begin{aligned}
a_0 &= 1\ , && \text{(a)}\\
a_1 &= (b_m + b_I) + (\mu_m V B a_{11} + \mu_I V B^3 a_{23})\ , && \text{(b)}\\
a_2 &= (\omega_u^2 + \omega_\varphi^2 + b_m b_I) + (\mu_m V^2 a_{12} + \mu_I V^2 B^2 a_{24} + b_m\mu_I V B^3 a_{23} + b_I\mu_m V B a_{11} && \text{(c)}\\
&\quad + \mu_m\mu_I V^2 B^4(a_{11}a_{23} - a_{13}a_{21}))\ ,\\
a_3 &= b_m\omega_\varphi^2 + b_I\omega_u^2 + \mu_m\omega_\varphi^2 V B a_{11} + \mu_I\omega_u^2 V B^3 a_{23} + \mu_I b_m V^2 B^2 a_{24} + \mu_m b_I V^2 a_{12} && \text{(d)}\\
&\quad + \mu_m\mu_I V^3 B^3(a_{11}a_{24} + a_{12}a_{23} - a_{13}a_{22} - a_{14}a_{21}) - \mu_I V^3 B b_{23}\ , && (16)\\
a_4 &= \omega_u^2\omega_\varphi^2 + \mu_m\omega_\varphi^2 V^2 a_{12} + \mu_I\omega_u^2 V^2 B^2 a_{24} + \mu_m\mu_I V^4 B^2(a_{12}a_{24} - a_{14}a_{22}) && \text{(e)}\\
&\quad + \mu_m\mu_I V^4 B^2(-a_{11}b_{23} + a_{21}b_{13}) - \mu_I b_m V^3 B b_{23} - \mu_I V^4 b_{24}\ ,\\
a_5 &= \mu_m\mu_I V^5 B(-a_{11}b_{24} - a_{12}b_{23} + a_{22}b_{13} + a_{21}b_{14}) && \text{(f)}\\
&\quad - \mu_I b_m V^4 b_{24} - \mu_I\omega_u^2 V^3 B b_{23}\ ,\\
a_6 &= \mu_m\mu_I V^6(-a_{12}b_{24} + a_{22}b_{14}) - \mu_I\omega_u^2 V^4 b_{24}\ . && \text{(g)}
\end{aligned}$$

Although Eq. (15) is approximate only, it is obvious that higher degree of the stream velocity influence is focused rather on the rotating component and its velocity: $\varphi,\ \dot{\varphi}$, while heaving component and its velocity corresponds rather with the neutral model in Eqs (1). The structure of Eqs (16) suggests some more possible simplifications canceling remaining b_{ij}. Such step would lead to full analogy with the neutral model (1).

4. Generalized Routh-Hurwitz method

Dynamic stability of MDOF systems is closely related with eigen values of the characteristic matrix. If their real parts are all negative, the system is stable. To carry-out a general and careful analysis, a strategy based on an inspection of the characteristic polynomial $P(\lambda)$ is preferable. In such a case properties of the characteristic matrix are reflected in polynomial roots. So that limits separating their negative and positive real part values should be found. A large group of methods for searching these limits is based on properties of the Hurwitz matrix $\mathbf{H} \in \mathbb{R}^{n\times n}$ and its diagonal sub-determinants. Then the basic principle requests that all diagonal sub-determinants should be positive. Thus zero value of sub-determinants indicate individual stability limits. Here only a technique of this procedure will be outlined. For a rigorous mathematical proof, see monographs, i.e. Gantmacher (1966). Following scheme can be outlined:

$$P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_{n-1}\lambda + a_n \tag{17}$$

$$\mathbf{H} = \left\| \begin{array}{l} a_1,\ a_3,\ a_5\ ,\ldots,\ a_{2n-1} \\ a_0,\ a_2,\ a_4\ ,\ldots,\ a_{2n-2} \\ \ 0,\ a_1,\ a_3\ ,\ldots,\ a_{2n-3} \\ \ 0,\ a_0,\ a_2\ ,\ldots,\ a_{2n-4} \\ \ 0,\ \ 0,\ a_1\ ,\ldots,\ a_{2n-5} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array} \right\| \quad ; \qquad \begin{array}{c} a_{2k+1} = 0; \quad k > (n-1)/2 \\ \\ a_{2k} = 0; \quad k > n/2 \end{array} \tag{18}$$

In other words, elements of the 1st row are coefficients of the odd powers λ completed with zeroes in the right part of the row. Similarly the 2nd row consists of coefficients of the even powers λ. The 3rd and 4th rows correspond with 1st and 2nd rows being shifted one element to the right. Similarly 5th and 6th rows, etc., as far as the last non-zero element a_{2k+1} or a_{2k} reaches right boundary of the row.

In order to facilitate and make more transparent the sub-determinants evaluation, the matrix $\mathbf{H}$ is subjected now to triangulation using the Routh algorithm (remembers the Gauss elimination process). In the first step: The odd rows remain intact. From the even rows are deducted the respective odd rows multiplied by the factor a_0/a_1. Further steps are analogous until the $(n-1)$th step is done. The whole process can outlined as follows:

$$\mathbf{H}_{(1)} = \left\| \begin{array}{l} a_1,\ a_3,\ a_5,\ a_7,\ \ldots,\ a_{2n-1} \\ \ 0,\ b_1,\ b_3,\ b_5,\ \ldots,\ b_{2n-3} \\ \ 0,\ a_1,\ a_3,\ a_5,\ \ldots,\ a_{2n-3} \\ \ 0,\ \ 0,\ b_1,\ b_3,\ \ldots,\ b_{2n-5} \\ \ 0,\ \ 0,\ a_1,\ a_3,\ \ldots,\ a_{2n-5} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array} \right\| \quad \Longrightarrow \quad \ldots\ \mathbf{H}_{(n-1)} = \left\| \begin{array}{l} a_1,\ a_3,\ a_5,\ a_7,\ \ldots \\ \ 0,\ b_1,\ b_3,\ b_5,\ \ldots \\ \ 0,\ \ 0,\ c_1,\ c_3,\ \ldots \\ \ 0,\ \ 0,\ \ 0,\ d_1,\ \ldots \\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ldots \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \end{array} \right\| \tag{19}$$

where in $\mathbf{H}_{(1)}$ has been denoted: $b_1 = a_2 - a_3 \cdot a_0/a_1,\ b_3 = a_4 - a_5 \cdot a_0/a_1$, etc. The matrix $\mathbf{H}_{(n-1)}$ emerging after the $(n-1)$th step is triangular. Its diagonal sub-determinants can be evaluated easily:

$$\Delta_1 = a_1, \quad \Delta_2 = a_1 \cdot b_1, \quad \Delta_3 = a_1 \cdot b_1 \cdot c_1,\ \ldots \tag{20}$$

Hence we can state that the system is stable if it holds:

$$\Delta_i > 0, \quad i \in (1,n),\ a_0 > 0 \tag{21}$$

Although formulae Eq. (20) look to be simple, diagonal elements $a_1, b_1, c_1, d_1, ...$ are more and more complex. If a further analysis suffices to be numerical, then following a certain strategy of system parameter series, numerical value of all Δ_i can be evaluated for each system parameter combination. Then going throughout Δ_i values, a number of sign changes within $i \in (1,n)$ is inspected. If all Δ_i are positive, there are no sign changes and the system represented by the polynomial Eq. (17 is stable.

Let us add that the diagonal elements in the $\mathbf{H}_{n-1}$ matrix after the elimination process can be written in a closed form:

$$a_1 = \Delta_1, \quad b_1 = \Delta_2/\Delta_1, \quad c_1 = \Delta_3/\Delta_2, ... \quad x_1 = \Delta_{n-1}/\Delta_{n-2}, \quad z_1 = \Delta_n/\Delta_{n-1} = a_n \tag{22}$$

where x_1 or z_1 means diagonal element in the $n-1$ or n row, respectively. Very cumbersome but elementary process leading to formulae (22) has been omitted. On the other hand formulae (22) can be directly deduced from Eqs (20).

Routh-Hurwitz (RH) conditions Eq. (21) are necessary and satisfactory and therefore giving unique results. However, when an analytic investigation is necessary, then to use solely sub-determinants Δ_i in an analytic form is realistic until let say $n = 4$. Nevertheless RH conditions Eq. (21) can be combined with Descartes rule. This theorem requests positive value of all coefficients of the polynomial Eq. (17), i.e. $a_i > 0$, to keep real part of all roots negative. Because Descartes rule represents the set of necessary and not satisfactory conditions, it should be combined with RH conditions Eq. (21). The combination of both approaches makes possible to simplify the process of the stability analysis, as polynomial coefficients are incomparably simpler than RH determinants. Putting both sets together it can be shown that many of particular conditions are consequences of the others. Therefore as much minimized set of satisfactory conditions as possible should be selected. This selection can be done by sight until $n = 4$. For higher degree of a polynomial a formalized tool is necessary.

One possibility of an effective selection is offered by Liénard theorem. It is based on a knowledge that fulfilling conditions $a_i > 0$, RH conditions Eq. (21) are no more independent. For instance at $n = 4$ only one condition $\Delta_3 > 0$ is independent. With reference to Gantmacher (1966) for the proof and other mathematical details, we provide only instructions of the theorem application. Relevant conditions of negative real parts of polynomial (17) roots can be formulated in one of four versions:

$$\begin{array}{lll} \text{(a)} & a_n > 0,\ a_{n-2} > 0, \quad \dots, & \Delta_1 = a_1 > 0,\ \Delta_3 > 0, \dots \\ \text{(b)} & a_n > 0,\ a_{n-2} > 0, \quad \dots, & \Delta_2 > 0,\ \Delta_4 > 0, \dots \\ \text{(c)} & a_n > 0,\ a_{n-1} > 0,\ a_{n-3} > 0,\ \dots, & \Delta_1 = a_1 > 0,\ \Delta_3 > 0, \dots \\ \text{(d)} & a_n > 0,\ a_{n-1} > 0,\ a_{n-3} > 0,\ \dots, & \Delta_2 > 0,\ \Delta_4 > 0, \dots \end{array} \tag{23}$$

It follows from formulations Eq. (23) that positivity either of all coefficients a_i or one of subsets $a_n, a_{n-2}, ..., a_n, a_{n-1}, a_{n-2}$ invalidates the full independency of the determinant conditions (21). Namely positiveness of the odd Hurwitz sub-determinants implicates positiveness of the even Hurwitz sub-determinants and vice versa.

Let us demonstrate the above algorithm for a system $n = 4$. Respective condition sets read:

$$\begin{array}{lll} \text{coefficients:} & a_0 > 0, \quad a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad a_4 > 0, & \text{(a)} \\ \text{sub-determinants:} & \Delta_1 = a_1 > 0, \qquad \Delta_2 = a_1 \cdot a_2 - a_3 \cdot a_0 > 0, & \text{(b)} \\ & \Delta_3 = a_3 \cdot \Delta_2 - a_4 \cdot a_1^2 > 0, \qquad \Delta_4 = a_4 \cdot \Delta_3 > 0. & \end{array} \tag{24}$$

The condition $\Delta_1 > 0$ is included in Eq. (24a), condition $\Delta_2 > 0$ must be fulfilled if $\Delta_3 > 0$ should be in force and $\Delta_4 > 0$ follows from (24a) and $\Delta_3 > 0$. Thus if conditions (24a) are valid then among RH sub-determinants only Δ_3 is independent and must be taken into account. So we can see, that such arrangement of conditions comply with the third version of the Liénard theorem, see Eq. (23c). This set of conditions is popular to process problems upto $n = 4$. Anyway, such condition sets can be proceed also by way of visual assemblage. However problems $n \geq 6$, which are under consideration, should be discussed using one version of the Liénard theorem, see Eq. (23).

Let us try to configure conditions for $n = 6$. First of all following coefficients of the polynomial should be positive, see Eq. (23c) or (23c):

$$a_0 = 1 > 0, \quad a_1 > 0, \quad a_3 > 0, \quad a_5 > 0, \quad a_6 > 0 \tag{25}$$

With respect to Eq. (23c) it should hold:

$$\Delta_1 > 0, \quad \Delta_3 > 0, \quad \Delta_5 > 0 \tag{26}$$

The condition $a_0 = 1 > 0$ is explicit and conditions $a_1 > 0$ and $\Delta_1 = a_1 > 0$ are identical. Consequently, considering conditions (25), only sub-determinants Δ_3 and Δ_5 are independent. Together with (25) they make the close set of satisfactory conditions determining the negative real part of polynomial $P(\lambda)$ roots for $n = 6$. For $n = 6$, respective sub-determinants have the form as follows, see basic form of the **H** matrix (18):

$$\begin{aligned}
\Delta_2 &= a_1 \cdot a_2 - a_0 \cdot a_3, \\
\Delta_3 &= a_3\Delta_2 - a_1^2 a_4 + a_0 a_1 a_5 \\
\Delta_4 &= a_4\Delta_3 - a_2(a_5\Delta_2 - a_1^2 a_6) + a_0(a_1 a_4 a_5 - a_0 a_5^2 - a_1 a_3 a_6) \\
\Delta_5 &= a_5\Delta_4 + a_1 a_5 a_6 \Delta_2 - a_3 a_6 \Delta_3 - a_1^3 a_6^2 \\
\Delta_6 &= a_6\Delta_5
\end{aligned} \tag{27}$$

Redundancy of sub-determinants (27) (or independency of Δ_3 and Δ_5) when conditions (25) are taken into account is obvious. Going through Eqs (25) and (27) once again, we can see that also other versions of Liénard theorem are applicable, in particular (23d).

5. Numerical experiments related to real example

Let us recall conditions (25) and (26) together with Eqs (27). They should be carefully analyzed. Apparently the most transparent illustration of their character and interaction can be outlined in the plane $\omega_u^2 \times \omega_\varphi^2$. With the help of this pictures one can see the influence of individual parameters on the stability of the basic system while creating limits identifying the change in stability character. Conditions $a_i \geq 0$ and $\Delta_i \geq 0$, treated usually separately in the literature have now gained general meaning. Stability conditions may intersect mutually and thus create separated instability domains in which individual generalized forces need not be necessarily positive due to non-conservative and gyroscopic influences. Therefore traditionally discussed types of dynamic stability loss appear here as special cases of one general mechanism treated above. So that going through one can detect all types of instability dealing with the advanced "neutral" model proposed here.

The Fig. 3 shows the result of the analysis for the rectangular cross-section that has been analyzed in the wind tunnel. The characteristics of this cross-section is described in Table 1.

Tab. 1: Characteristics of the rectangular cross section and measured values of critical wind speed. $V_c^{(1)}$ is taken from the experiments, see Král et al. (2011) and $V_c^{(2)}$ results from the presented method respectively.

mass	inertia	width	frequency	frequency	damping	damping	speed	speed
m	I	B	ω_u	ω_φ	ζ_u	ζ_φ	$V_c^{(1)}$	$V_c^{(2)}$
$[kg/m]$	$[kgm]$	$[m]$	$[s^{-1}])$	$[s^{-1}]$	$[(\%]$	$[\%]$	$[m/s]$	$[m/s]$
4.02	0.0023	0.30	4.900	4.048	0.9	1.3	9.0	8.5

The graphs are actually the stability diagrams for certain value of the wind speed. For much simpler model that is described in the article by Pospíšil & Náprstek (2011) four planar curves divide the plane into several zones of stability and instability. By using the advanced neutral model the figure is more complicated, however the stability domains can be also observed. For example, the a_5 divide the plane $\omega_u^2 \times \omega_\varphi^2$ into two semi-planes where the right one is the stable zone. The Δ_4 is the most complicated condition being a result of combination of various a_i including Δ_3 and a_4, the latter one being a hyperbola with the axis in the second and fourth quadrant. This is know as a divergence stability conditions, see Pospíšil & Náprstek (2011).

Higher determinants Δ_5 and Δ_6 create complex parametric curves combining lower order determinants together, see Eqs (27). For example, in the figure with Δ_5, besides the lines, the parabolic shapes standing for the flutter condition, see Pospíšil & Náprstek (2011), are visible. The zone between those parabolas is the stable one, however with increase of the wind speed V this zone changes, the parabolic shapes may merge or undergo other transformation so that the stability domain is narrower in the general term of the word.

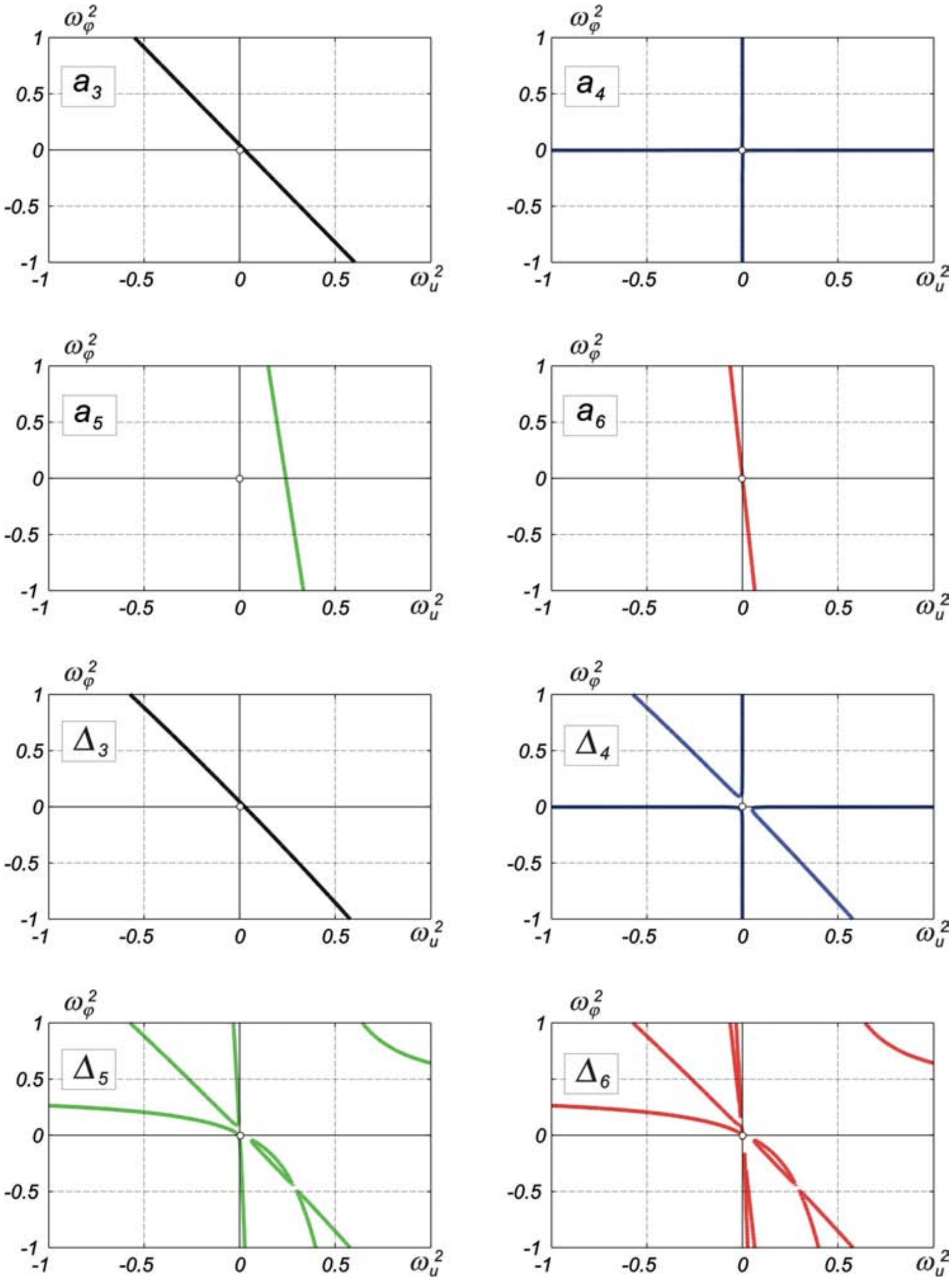

Fig. 3: Conditions of stability depicted in the frequency plain, $\omega_u^2 \times \omega_\varphi^2$.

The influence of the wind velocity can be seen more transparently at the different type of diagrams. The Fig. 4 illustrates the use of the described analysis for the evaluation of the critical wind speed, i.e. the speed when one of the conditions (25) and (26) is violated. Only the independent conditions are presented. In particular: $a_1 - a_6$, see Eq. (25) and two determinants Δ_3 and Δ_5, see Eq. (26). It can be seen from the Fig. 4 that the condition Δ_5 crosses the zero at the speed $V = 8.5\, m/s$. This corresponds quite well to the experimentally verified value $V = 9.0\, m/s$. Obviously, the critical speed depends much upon the measured value of structural damping.

Generally, the coefficients a_5, a_6 and the determinants Δ_5 and Δ_6 are very sensitive to precise determination of the flutter derivatives, which is usually very complicated task for both very low values of κ (higher values of the reduced wind velocity V) as well as for the values where the reduced frequency is very high. These regions determine the signs of the coefficients b_{ij}, which are used for the fitting of the flutter derivatives. Higher order expansion would improve the analysis. On the other hand this improvement would be lost in doubling of the order of the characteristic polynomial (13).

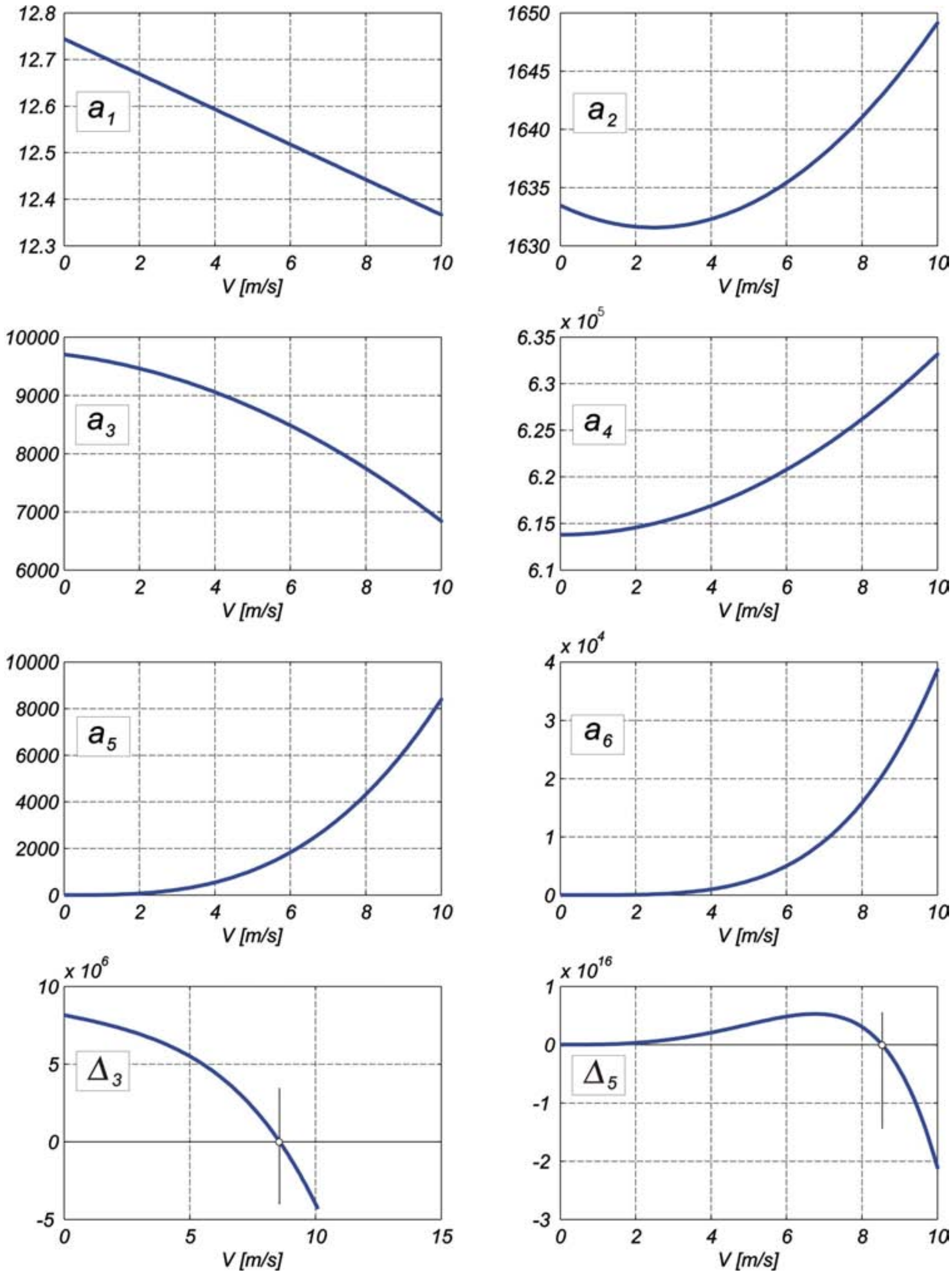

Fig. 4: Course of the some coefficients a_i and determinants Δ as function of velocity V. The point of the zero crossing at individual coefficients determines the loss of the stability, which may be of any kind.

6. Conclusion

Two types of double degree of freedom (DDOF) linear systems interacting with aero-elastic forces have been investigated and compared. The DDOF system under study describes inherent dynamic features of a slender prismatic beam attacked by a cross wind stream of a constant velocity (long bridge decks, guyed masts, towers, etc.). Relevant mathematical models of aero-elastic forces appearing in literature differ in principle by way of composition of aero-elastic forces. From this point of view two groups have been investigated: neutral models, where aero-elastic forces are introduced as suitable constants independent from excitation frequency and time and models using flutter derivatives for modelling the aero-elastic forces.

The second group respects explicitly the stream velocity and the frequency of the system response. It succeeded to put both groups together on one common basis to demonstrate their linkage. The platform of qualitative investigation of aero-elastic critical states in a frequency plain has been significantly expanded with respect to the stream velocity. Memory effects ruling in aero-elastic DDOF system have been substantiated and compared in frequency and time domains. The approach presented allows to formulate more flexible models combining main aspects of both groups keeping the DDOF basis. This approach can be used for the analysis of practical flow-structure interaction problems. However, teh attention should be paid to the precise flutter derivatives measurement, especially in the both very low and very high frequency domains.

Acknowledgement

The support of the Czech Scientific Foundation No. 103/09/0094, Grant Agency of the ASCR No. A200710902 as well as the support of RVO 68378297 are gratefully acknowledged.

References

Bartoli, G., Righi, M. (2006) Flutter mechanism for rectangular prisms in smooth and turbulent flow. *J. Wind. Eng. Ind. Aerodyn.* 94, 275–291.

Caracoglia, L., Jones N.P. (2003) Time domain vs. frequency domain characterization of aeroelastic forces for bridge sections. *J. Wind. Eng. Ind. Aerodyn.* 91, 1135–1150.

Costa C., Borri C., Flamand O., Grillaud G. (2007): Time-domain buffeting simulations for wind-bridge interaction. *J. Wind. Eng. Ind. Aerodyn.* 95, (9-11), 991–1006.

Gantmacher, F.R. (1966) *Theory of Matrices* (in Russian). Nauka, Moscow.

Garrick, I.E. (1938) On Some Reciprocal Relations in the Theory of Nonstationary Flows. *NACA Technical Report No. 629*, 1938.

Král, R., Pospíšil, Náprstek, J., 2011. Experimental analysis of frequency tuning influence on the response stability of bridge girders under wind action. In: Proceedings of 13th International Conference on Wind Engineering, Amsterdam. 8 pgs.

Küssner, H.G. (1954) A general method for solving problems of the unsteady lifting surface theory in the subsonic range. *J. Aeronautical Sci.*, 21(1).

Náprstek, J., Pospíšil, S. (2001) Aeroelastic instability of a system with nonconservative and gyroscopic forces. In: *Proc. 11th Engineering Mechanics Conference* (A.Poživilová, J.Masák eds), IT ASCR, Svratka, , 8 pgs. CD ROM.

Náprstek, J. (2007) Domains and types of aeroelastic instability of slender beam. In: *Proc. Biennial Conference Mech. Vibration and Noise 2007*, Las Vegas: ASME, CD paper #DETC2007-34132, 10 pgs.

Náprstek, J., Pospíšil, S. (2012) Relation of flutter derivatives and indicial functions in linear aero-elastic models. In: *Proc. Dynamics of Machines 2012* (L. Pešek edt.). Inst. Termomechanics ASCR, Prague, pgs 95–102.

Pospíšil, S., Náprstek, J., (2011) General formulation of linear aeroelastic system stability and respective post-critical response types. In: *Proc. 13th Int. Conf. On Wind Engineering* (C. Geurts, B. Blocken eds). Tech. Univ. Eindhoven, Amsterdam, paper 177, 8 pgs.

Poulsen, N.K., Damsgaard, A. and Reinhold, T.A. (1992) Determination of flutter derivatives for the Great Belt Bridge. *J. Wind Eng. Ind. Aerodyn.* 41, 153–164.

Scanlan, R.H., Tomko, J.J. (1971) Airfoil and Bridge Deck Flutter Derivatives. *J. of Eng. Mech. ASCE* 97, 1717–1737.

Scanlan, R.H., Beliveau, J.G., Budlong, K. (1974) Indicial aerodynamic functions for bridge decks. *J. of Engrg. Mech. ASCE* 100, 657–672.

Theodorsen, T. (1935) General Theory of Aerodynamic Instability and the Mechanism of Flutter. *NACA Technical Report No. 496.*

Wagner, H. (1929) Über die Entstehung des dynamischen Auftriebes von Tragflügeln. *Zeit. Angew. Math. und Mech.*, 5(1), 17–35.