18th International Conference
ENGINEERING MECHANICS 2012
Svratka, Czech Republic, May 14 – 17, 2012
pp. 433–448
Paper **#41**

APPLICABILITY OF EXISTING INDEXES OF NON-PROPORTIONALITY OF DAMPING IN CASE OF THEORETICAL MODEL OF SLENDER STRUCTURE WITH INSTALLED TMD

S. Hračov*, S. Pospíšil, J. Náprstek*****

Abstract: *The paper analyzes an applicability of to date published indexes of non-proportionality in the case of a linear viscously damped numerical model of slender structure equipped with tuned mass damper (TMD). The installation of TMD into the structure not only reduces the level of undesired vibration, but it can also cause due to damping element of TMD a significant increase of damping non-proportionality. The paper recommends the most suitable indexes for such a type of structure and points out to impropriety of the others. The point of view of is also focus on the validity of the existing criterions for neglecting of non-diagonal terms of a modal damping matrix. Only indexes and criterions based on the properties of the modal damping matrix were taking into account. The verifications of validity and recommendations for usage of particular indexes and criterions were performed using analysis of the dynamic response of an existing structure on harmonic excitation with and without neglecting of non-diagonal terms of modal damping matrix. The applicability was also checked using analysis of particular complex eigen-modes.*

Keywords: ***Indexes of non-proportionality of damping, tuned mass damper, slender structure***

1. Introduction

Numerically efficient solution of a dynamic response of linear viscously damped numerical models of real structures using modal superposition method (MSM) see e.g. Hart & Wong (2000) motivates many authors to set boundaries, to which the inaccuracy of a solution with neglecting of non-diagonal terms of a modal damping matrix is still acceptable. Simultaneously, they attempted to quantify an extent of non-proportionality of the damping by means of indexes of various types.

The first group of indexes of non-proportionality is related to complex eigen-modes. Prater & Singh (1986) defined two indexes based on calculations of surfaces that form individual components of complex eigen-modes in the complex plane. Another two indexes suggested by these authors are functions of phase differences between individual complex components. Similarly in Bhaskar (1999), an index related to a modal area, which creates components of the complex eigen-modes in the complex plane was proposed. The second index defined in this paper is based on a placement of components of complex eigen-modes in the complex plane and on their relative position to the position of components of eigen-modes of the same proportionally (classically) damped system. Three indexes based on relation of a real and a complex part of eigen-modes formulated Liu et al. (2000). Prells & Friswell (2000) proposed an index of non-proportionality equal to a norm of difference between orthonormal matrix generated from complex eigen-modes and a unit matrix. All above mentioned indexes require calculation of complex eigen-values and eigen-modes. Thus almost all the main numerical advantages of subsequent and prospective using of MSM are lost. More accurate results of the calculation of the response with negligible added computational time in comparison with MSM could be reached by complex mode superposition method

* Ing. Stanislav Hračov, Ph.D.: Institute of Theoretical and Applied Mechanics AS CR, v.v.i.; Prosecká 76; 140 00, Prague; CZ, e-mail: hracov@itam.cas.cz

** doc. Stanislav Pospíšil, Ph.D.: Institute of Theoretical and Applied Mechanics AS CR, v.v.i.; Prosecká 76; 140 00, Prague; CZ, e-mail: pospisil@itam.cas.cz

** Ing. Jiří Náprstek, DrSc.: Institute of Theoretical and Applied Mechanics AS CR, v.v.i.; Prosecká 76; 140 00, Prague; CZ, e-mail: naprstek@itam.cas.cz

see Hurty & Rubinstein (1964). This method utilizes the previously calculated complex eigen-modes to uncouple the system of differential equation of the numerical model.

More useful from the point of view of the computational time is the second group of indexes and criterions, which do not require the previous calculation of complex eigen-modes. The indexes are derived from a distribution of damping elements in the modal damping matrix, from a mutual frequency distances between the dominant frequency of a loading and individual eigen-frequencies and between eigen-frequencies themselves. In this paper short summary of till now published indexes and criterions of this type are presented. The focus is aimed especially at their applicability for numerical models of slender structures equipped with a tuned mass damper. On an example of real structure errors in response on harmonic excitation that are caused by neglecting of non-diagonal terms of the modal damping matrix are investigated together with indexes and criterions for selected damping ratios of absorber. On the basis of mutual relation of errors and criterions and of indexes and calculated complex eigen-modes the most appropriate ones are recommended.

2. Theoretical background

The discrete mathematical model of structure and its response can be described by very well-known system of differential equations of the second order:

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{p}(t) \tag{1}$$

K, **M** and **C** are the stiffness, mass and damping matrices respectively; **x***(t)* and **p***(t)* are the displacement and force vectors. The key to analysis of the response of the governing system (1) using MSM is the transformation:

$$\mathbf{x}(t)=\mathbf{X}\mathbf{q}(t) \tag{2}$$

where **q***(t)* is a vector of principal (modal) co-ordinates and **X** is the matrix, columns of which are real eigen-modes of the undamped system. The substitution (2) leads to set of uncoupled differential equations, if the modal damping matrix:

$$\tilde{\mathbf{C}}=\mathbf{X}^{\mathbf{T}}\mathbf{C}\mathbf{X} \tag{3}$$

is diagonal i.e. it fulfills the relation:

$$\mathbf{K}\mathbf{M}^{-1}\mathbf{C}=\mathbf{C}\mathbf{M}^{-1}\mathbf{K} \tag{4}$$

In this case the mathematical expression of viscous damping is called proportional or classical. When the modal damping matrix is not diagonal the equations in principal coordinates are coupled. The simplest method to obtain uncoupled equations is to neglect the non-diagonal terms of this modal damping matrix. Nevertheless this method could lead to significant error in calculation of the response due to omitting of a presence of a mechanical interaction between eigen-modes.

3. Indexes of non-proportionality and criterions for neglecting of non-diagonal terms of modal damping matrix

In this chapter to date published indexes of non-proportionality and criterions which are based on properties of modal damping matrix are summarized. Short comments on their limitations and usability are attached to their mathematical formulations.

The basic and the most general requirement and criterion for the neglecting of non-diagonal terms of modal damping matrix is a diagonal dominance of the modal damping matrix:

$$\left|\tilde{C}_{ii}\right|>\sum_{\substack{j=1\\ j\neq i}}\left|\tilde{C}_{ij}\right| \qquad pro\,\forall i \tag{5}$$

The first relevant criterion for possible omitting of the mechanical interaction between eigen-modes was suggested by Hasselman (1976). He expressed the condition for possible ignoring of non-diagonal terms in a form:

$$\sqrt{\frac{2\zeta_r}{(\omega_s/\omega_r)^2-1}} \ll 1 \tag{6}$$

where ζ_r is a damping ratio of the r-th eigen-mode given by:

$$\zeta_r = \tilde{C}_{rr}/2\omega_r \tag{7}$$

and ω_s is the s-th and ω_r is the r-th eigen-frequency of the undamped system. However it is supposed, that a ratio of both examined eigen-frequencies is smaller than unity:

$$\omega_s/\omega_r > 1 \tag{8}$$

Next assumption of a usability of the condition (6) is:

$$\tilde{C}_{rs}/\tilde{C}_{rr} < 1 \tag{9}$$

Generally, the Hasselman's criterion removes the main difference between non-classically and classically damped systems. It supposes also high mechanical interaction of eigen-modes of classically damped systems which lies close to each other.

Similar to condition (6) Warburton & Soni (1977) defined a criterion based on a solution of the response on harmonic excitation in a form:

$$\zeta_r < \varepsilon \left| \frac{\tilde{C}_{rr}}{2\tilde{C}_{rs}} \left(\frac{\omega_s^2}{\omega_r^2} - 1 \right) \right|_{\min s} \tag{10}$$

Coefficient ε expresses the maximal value of desired and acceptable error in determination of the response. For problems in practice the authors recommended the value of ε equal:

$$\varepsilon = 0,05 \tag{11}$$

This value should correspond to relative error up to 10 percent. In comparison to criterion (6) the condition (10) is taking into account a ratio of diagonal and non-diagonal terms of modal damping matrix.

In the paper of Prater & Singh (1986) three indexes non-proportionality are published. For the purposes of their mathematical expressions authors divided the modal damping matrix into a sum of diagonal matrix $\mathbf{C}_d$ and matrix $\mathbf{\Gamma}$ which has zero diagonal terms:

$$\tilde{\mathbf{C}} = \mathbf{C}_d + \mathbf{\Gamma} \tag{12}$$

The first generalized index is defined as the quotient between the sum of non-diagonal terms of the transformed damping matrix and the sum of all its terms:

$$\delta_1 = \sum_{i=1}^{n}\sum_{j=1}^{n} \left|\Gamma_{ij}\right| / \sum_{i=1}^{n}\sum_{j=1}^{n} \left|\tilde{C}_{ij}\right| \tag{13}$$

Index (13) indicated the degree of non-proportionality of the damping of the system as a whole. In problems where the response is investigated only in given frequency range or when only few eigen-modes are coupled due to the damping, it is recommended to use the modified index (13) in a form:

$$\delta_{1i} = \sum_{j=1}^{n} \left|\Gamma_{ij}\right| / \sum_{j=1}^{n} \left|\tilde{C}_{ij}\right| \tag{14}$$

which is valid for i-th eigen-mode. The second index proposed by Prater & Singh (1986) is given by ratio of the determinants of matrices $\mathbf{\Gamma}$ and $\mathbf{C}_d$:

$$\delta_2 = \frac{|\mathbf{\Gamma}|}{|\tilde{\mathbf{C}}|} \tag{15}$$

Since the determinant is scalar, no index of particular eigen-mode can be formulated. The third proposed index is based on comparison of the response of a system with full modal damping matrix and the same system with modal damping matrix with diagonal terms only. The response of both systems on harmonic load is given by solution of following system of differential equations:

$$\left(-\omega^2\mathbf{I} + \mathrm{i}\,\omega\tilde{\mathbf{C}} + \mathbf{\Lambda}\right)\mathbf{q} = \mathbf{f} \tag{16}$$

The vector of amplitudes of harmonic forces is assumed as a unity vector **f**:

$$\mathbf{f} = [1,1,\ldots,1]^{\mathrm{T}} \tag{17}$$

The loading frequencies ω are chosen equal to the damped eigen-frequencies with damping ratios given by equation (7) in which one can expect the highest response:

$$\omega = \omega_{di} = \sqrt{1-\zeta_i^2}\,\omega_i \tag{18}$$

The proposed response based index of i-th eigen-mode is then given by ratio of difference between amplitude of steady state response of approximate $\hat{q}_i$ and full modal system q_i and amplitude q_i:

$$\delta_{3i} = \frac{\left|\left(|q_i| - |\hat{q}_i|\right)\right|}{|q_i|} \tag{19}$$

To quantify the amplitude q_i the whole matrix system (16) must be solved. In the case of system with neglected non-diagonal terms of modal damping matrix for amplitude $\hat{q}_i$ it holds:

$$\hat{q}_i = \frac{1}{-\omega_{di}^2 + \omega_i^2 + \mathrm{i}\,\omega_{di}\tilde{C}_{ii}} \tag{20}$$

It is also possible to define the overall index as an arithmetic mean:

$$\delta_3 = \frac{1}{n}\sum_{i=1}^{n}\delta_{3i} \tag{21}$$

Authors recommended omitting of non-diagonal terms of modal damping matrix if following conditions are fulfilled:

$$\delta_1 \ll 1 \qquad \delta_2 \ll 1 \qquad \delta_3 \ll 1 \tag{22}$$

From the point of view of numerical difficulty the index δ_1 is the most appropriate. Index δ_2 was determined by authors as the less useful. The third index δ_3 could serve as indicator of error rate of the response of variously modified damped systems.

Tong et al. (1994) defined an index for quantifying the damping non-proportionality as:

$$I = \left(\sigma_{\max} - \sigma_{\min}\right) / \left(\sigma_{\max} + \sigma_{\min}\right) \tag{23}$$

σ_{max} and σ_{min} are the maximum and minimum eigen-values of matrix **H**:

$$\mathbf{H} = \tilde{\mathbf{C}}_d^{-1}\tilde{\mathbf{C}} \tag{24}$$

Coefficient I is zero for classically damped system. With increasing of the non-proportionality of the damping the coefficient approximates the unity. Authors defined an upper bound of the relative error of the response caused by neglecting of non-diagonal terms in the case of harmonic load as follows:

$$E(\omega) \le I\left(\tilde{\mathbf{C}}\right) + \left(\frac{\max(\tilde{C}_{ii})}{\min(\tilde{C}_{ii})}\right)^{1/2} \tag{25}$$

The degree of non-proportionality of the damping based on relations of the non-diagonal and diagonal terms of the modal matrix was presented by Venancio-Filho et.al. (2001):

$$\chi_r = \max\left|\frac{\tilde{C}_{rs}^2}{\tilde{C}_{rr}\tilde{C}_{ss}}\right| \tag{26}$$

However, the limiting value of coefficient (26) for approximative solution by neglecting of non-diagonal terms of the modal damping matrix was not given.

Bhaskar (1995) proposed an index which originates from the solution of the response of the system on harmonic excitation. For i-th eigen-mode it has a form:

$$\kappa_i = \frac{\sum_{j=1}^{n}\left|\Gamma_{ij}\right|}{\left(\left(1/\omega^2\right)\left(\omega_i^2-\omega^2\right)^2+\tilde{C}_{ii}^2\right)^{1/2}} \tag{27}$$

For diagonally dominant modal matrix the coefficient κ_i should fulfill a condition:

$$0 \le \kappa_i < 1 \tag{28}$$

Coefficients (27) include not only the terms of modal damping matrix but also a frequency of the loading and a relation of the loading frequency to the eigen-frequencies. If one is interested only in upper bound of non-proportionality of the damping, the coefficient (27) could be simplified in a form:

$$\kappa_i \le \frac{\sum_{j=1}^{n}\left|\Gamma_{ij}\right|}{\tilde{C}_{ii}} \tag{29}$$

Gawronski & Sawicki (1997) derived an upper bound of the relative error of the response in modal coordinates for the case of neglecting of non-diagonal terms of modal damping matrix. The condition of the error of i-th modal coordinate is given by:

$$\frac{\left|q\left(\omega_i\right)-\hat{q}\left(\omega_i\right)\right|}{\left|q\left(\omega_i\right)\right|} \le \frac{\nu_i\sigma_i}{2\xi_i\omega_i} \tag{30}$$

where coefficient σ_i is given by a sum of terms of the *i*-th row of the non-diagonal matrix **Γ**:

$$\sigma_i = \sum_k \left|\Gamma_{ik}\right| \tag{31}$$

Coefficient ν_i is the maximum from a series:

$$\nu_i = \max_{k\neq i}\left(\nu_{i,k}\right) \tag{32}$$

Coefficient $\nu_{i,k}$ is scalar, which is given by ratio of modal amplitudes of *i*-th and *k*-th eigen-mode in the case of harmonic load with frequency ω_k:

$$\nu_{i,k} = \frac{\left|f_i\right|/\left|f_k\right|}{\xi_i/\xi_k\left(\left(\omega_i/\omega_k\right)^2+\left(\left(\omega_i/\omega_k\right)^2-1\right)^2/4\xi_i^2\right)^{1/2}} \tag{33}$$

The prerequisite of using the condition (30) is a small ratio of modal loadings:

$$\frac{\left|f_i\right|}{\left|f_k\right|} \ll 1 \tag{34}$$

The requirement (34) is very conservative and for many system impossible to fulfill.

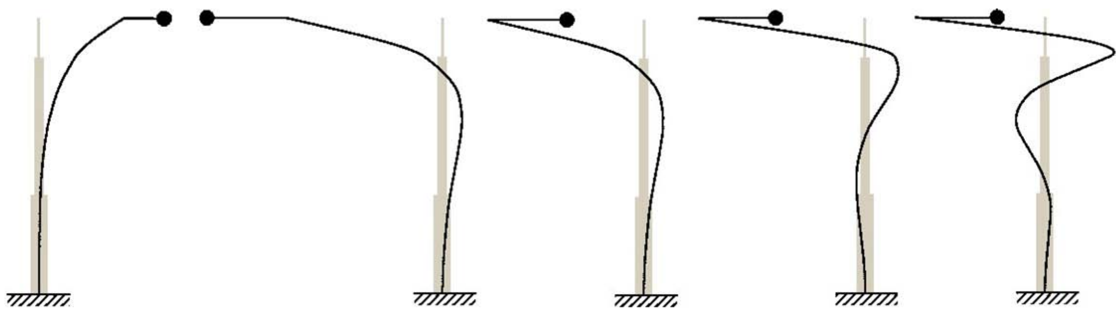
4. Applicability of indexes and criterions in case of real structure equipped with TMD

4.1. Specification of structure

The indexes and criterions from previous chapter were calculated and analyzed for the case of a linear discrete numerical model of existing TV tower equipped with TMD. Absorber in a form of pendulum was installed into a laminate extension of a top of the tower, due to possible excessive vibrations caused by a wind load. The vortex-shedding effect on the cylindrical extension without absorber could cause danger stresses in laminate from the point of view of material fatigue and life-time of the structure. TMD was designed and tuned to be the most effective in vibrations in the second eigen-mode i.e. eigen-frequency of the tower. The weight of absorber is 1 tone, which is 1/10 of a generalized (effective) mass of the second eigen-mode. TMD has usually higher damping properties than the part of the structure, where it is installed, and thus, it could represent an significant origin of non-proportionality of the damping.

4.2. Relative errors of response of numerical model on harmonic excitation due to approximate solution

The discrete numerical model of structure was created in CALFEM, which is Matlab toolbox for computing by the finite elements method. Basic model without TMD had 15 nodes, each with 3 degrees of freedom. The absorber was subsequently modeled as a concentrated mass connected to the top of the tower with Kelvin-Voigt damping term and had one degree of freedom in horizontal direction. Adding this degree of freedom into the model resulted in increasing of total number of eigen-modes by one. This eigen-mode was associated due to tuning of absorber to the second eigen-mode of the basic system. It means, that model with absorber had two eigen-modes, which are similar in shape. However, they differ especially in a phase between absorber and the top of the tower see Figure 1.



$f_1 = 0{,}378$ Hz ($f_{1P} = 0{,}394$ Hz) $f_2=0{,}679$ Hz $f_3=0{,}931$ Hz ($f_{2P} = 0{,}806$ Hz) $f_4=1{,}275$Hz ($f_{3P} = 1{,}213$ Hz) $f_5=2{,}120$Hz ($f_{4P} = 2{,}109$ Hz)

Fig. 1: The first five eigen-modes and eigen-frequencies of the structure with absorber (in parentheses the corresponding eigen-frequencies of the basic system without absorber)

At first, the steady state response of the structure with absorber on a harmonic force located at the top of the tower was calculated. The response were analyzed for an interval of the frequency of the loading force (f = 0 ÷ 3 Hz), where the first four eigen-modes of the basic system lie. Damping matrix of the basic system was proportional to the combination of the mass and stiffness matrix. Multiplicative coefficients related to these stiffness and mass matrices were calculated from the given structural damping ratio ($\zeta = 0{,}005$) for the first two eigen-frequencies of the basic system. Factor of the damping non-proportionality of the system has been examined as follows: a set of various damping ratio of the TMD (dashpot absorber) was used ($\zeta_{TMD} = 0 \div 0{,}8$), while constant structural damping was kept. First seven eigen-modes were used for reduction of the matrix system (1) using transformation (2) and two different solutions of response of this reduced model were assumed. The first one concerned the direct solution of the reduced system with full modal damping matrix. The second one took into account only the diagonal terms of the modal damping matrix. Relative errors of amplitudes of steady state response of the top of the tower ε_{Ytop} and absorber ε_{Ytmd} and of the phase difference between the top and the absorber ε_{φ} caused by the second approximate solution were calculated for a set of ζ_{TMD}. Specifically, the relative errors were determined for the first four dominant peaks of frequency-amplitude curves see Figure 2, which lay near the eigen-frequencies of the undamped system and where the highest response was noticeable. On Figure 2 only the part of investigated frequency interval with the second and third peak is shown, for which the errors were the most significant. The peak corresponding to the third eigen-frequency was not identifiable for nonzero damping ratio ζ_{TMD} and it wasn't included in the analysis. The relative errors in determination of the frequency of the peaks ε_f together with all previously defined errors are summarized in the Table 1.

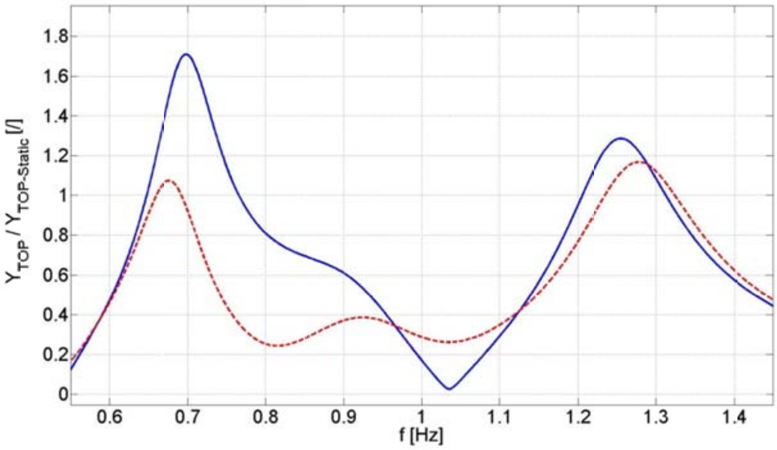


Fig. 2: Dynamic magnification factor of amplitude of the top of the tower as a function of driving frequency of the load for exact and approximate solution (Blue solid line – exact solution, red dashed line – approximate solution; ζ_{TMD} = 0,2)

Tab. 1: Relative errors of the response caused by approximate solution for various values of ζ_{TMD} (In parentheses the corresponding eigen-frequencies of the structure with absorber)

Peak n.	**1** (f_1= 0,378 Hz)				**Peak n.**	**2** (f_2= 0,679 Hz)			
ζ_{TMD} [/]	0	0,2	0,5	0,8	ζ_{TMD} [/]	0	0,2	0,5	0,8
ε_f [%]	< 0,1	< 0,1	< 0,1	0,26	ε_f [%]	< 0,1	3,15	9,65	10,48
ε_{Ytop} [%]	0,01	2,11	12,69	29,61	ε_{Ytop} [%]	0,03	37,15	79,40	89,61
ε_{Ytmd} [%]	0,01	-0,7	0,38	2,68	ε_{Ytmd} [%]	0,01	19,09	56,23	68,18
ε_{φ} [%]	0,7	9,12	12,76	26,33	ε_{φ} [%]	0,28	10,89	17,50	10,09
Peak n.	**3** (f_4= 1,275 Hz)				**Peak n.**	**4** (f_5= 2,12 Hz)			
ζ_{TMD} [/]	0	0,2	0,5	0,8	ζ_{TMD} [/]	0	0,2	0,5	0,8
ε_f [%]	< 0,1	-1,75	-6,46	-5,47	ε_f [%]	< 0,1	-0,28	-1,04	-2
ε_{Ytop} [%]	0,02	9,16	45,74	68,60	ε_{Ytop} [%]	0,01	-1,97	0,38	9,33
ε_{Ytmd} [%]	0,02	2,66	20,30	34,28	ε_{Ytmd} [%]	0,03	-3,99	-10,44	-13,82
ε_{φ} [%]	0,57	19,32	33,04	31,24	ε_{φ} [%]	0,27	19,81	33,21	40,57

The table shows, that neglecting of mechanical interaction between eigen-modes resulted in significant errors in amplitudes of the top of the tower especially for the frequencies corresponding to the second and the third peak for nonzero ζ_{TMD}. These resonant frequencies lie very close to the tuning frequency of the absorber and thus are more influenced by it than the frequencies of the first and fourth peak.

With increasing of the damping of the absorber the errors also increase. This fact isn't generally valid for every numerical model of the structure with absorber. It depends as well as on a chosen model of classical viscous damping of the basic structure and numerical model of the absorber as on the difference of damping ratios of the eigen-modes of the structure and ζ_{TMD}. The ζ_{TMD} of a real absorber installed into the existing tower was from practical point of view set equal 20%. This value of ζ_{TMD} is optimal for reduction of vibrations with frequencies close to the second eigen-frequency of the basic structure. However, this value of ζ_{TMD} is 40 times higher than the damping ratio of this second eigen-mode. The relative error of the amplitude of the response of the top of the tower is in this case significant and is equal almost 40% for the second peak see Table 1 and Figure 2. Similarly, the maximal and appreciable error of amplitude of the absorber is equal 20%. It follows, that the non-proportionality of the damping is in this practical case substantial. Amplitudes of approximate solution are for almost every studied case lower than for exact solution. This fact could be explained by the mechanical interaction between eigen-modes for non-classically damped system. It could be also interpreted by different modal damping of eigen-modes for both solutions see Table 2. The final results of the response are strongly affected by a number of eigen-modes of undamped system being taken into the consideration. The controlling calculation of the response of the full system (1) on the same excitation showed, that using a set of first seven eigen-modes resulted to acceptable maximal absolute error less than 0,1%.

The influence of non-proportionality of the damping on the individual eigen-modes could be also illustrated by different phases of their components. The components of the first six complex eigen-modes, which correspond to displacements of the tower, are for the practical value of ζ_{TMD} depicted in the complex plane on Figure 3. The most influenced are the second, the third and the fourth complex eigen-mode. Their components have different phases i.e. they don't lie in the complex plane on one line. On the other hand, the fifth, the sixth and especially the first eigen-mode almost correspond to the real undamped eigen-modes, which are characterized by same phase of all components.

Tab. 2: Damping ratios of eigen-values of approximate (ζ_{APP}) and real (ζ) numerical model for various values of ζ_{TMD}

1st eigen-value					**2nd eigen-value**				
ζ_{TMD} [/]	0	0,2	0,5	0,8	ζ_{TMD} [/]	0	0,2	0,5	0,8
ζ [/]	0,0047	0,0067	0,0087	0,0093	ζ [/]	0,0027	0,0597	0,0585	0,0381
ζ_{APP} [/]	0,0047	0,0068	0,0100	0,0132	ζ_{APP} [/]	0,0027	0,0651	0,1586	0,2522
3rd eigen-value					**4th eigen-value**				
ζ_{TMD} [/]	0	0,2	0,5	0,8	ζ_{TMD} [/]	0	0,2	0,5	0,8
ζ [/]	0,0033	0,1105	0,4505	0,8485	ζ [/]	0,0054	0,0551	0,0574	0,0385
ζ_{APP} [/]	0,0033	0,0986	0,2415	0,3844	ζ_{APP} [/]	0,0054	0,0594	0,1404	0,2215
5th eigen-value					**6th eigen-value**				
ζ_{TMD} [/]	0	0,2	0,5	0,8	ζ_{TMD} [/]	0	0,2	0,5	0,8
ζ [/]	0,0093	0,0182	0,0266	0,0282	ζ [/]	0,0156	0,0178	0,0208	0,0230
ζ_{APP} [/]	0,0093	0,0187	0,0327	0,0467	ζ_{APP} [/]	0,0156	0,0179	0,0213	0,0247

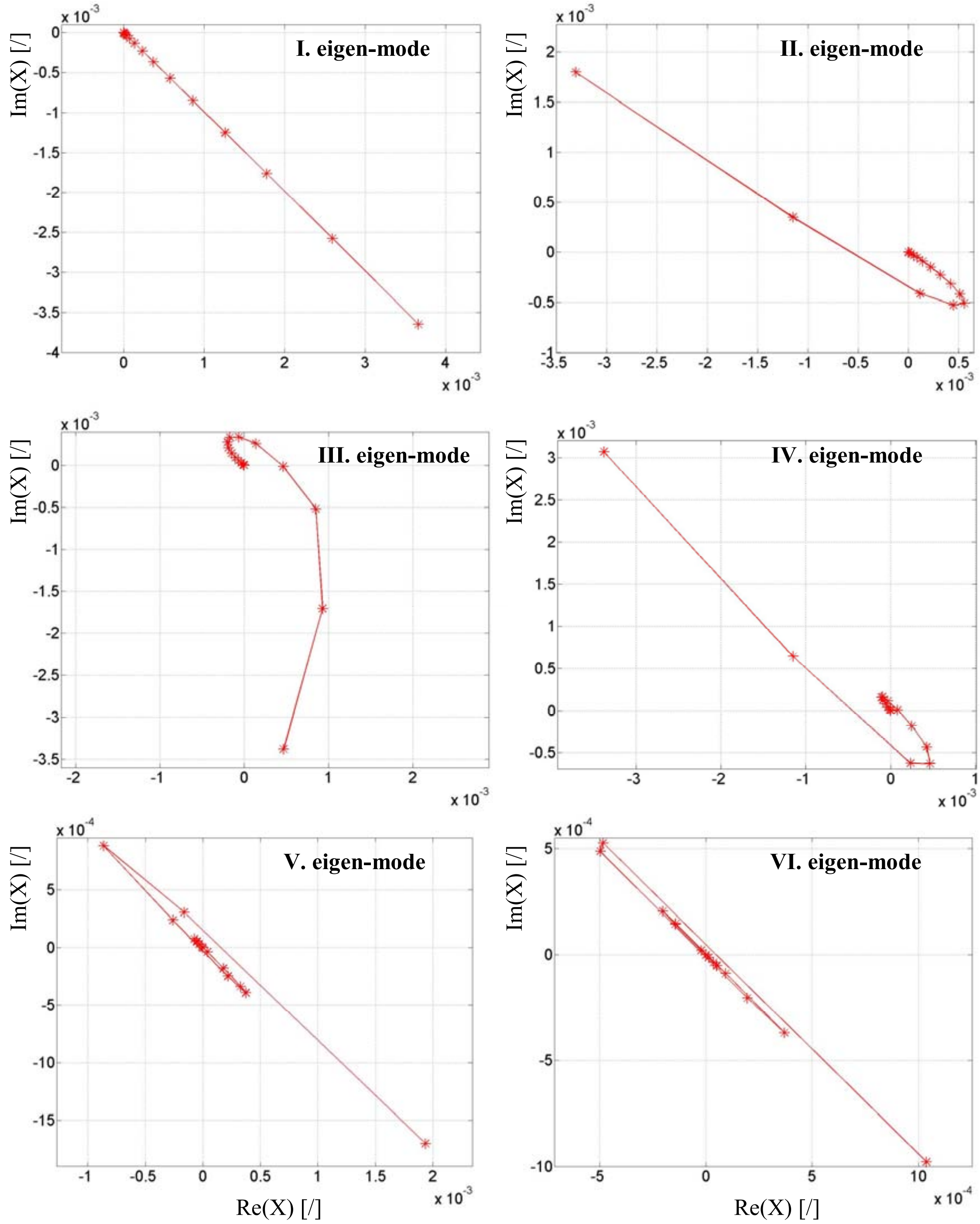


Fig. 3: Components of first six complex eigen-modes of structure in complex plane (ζ_{TMD} = 0,2)

4.3. Calculation of indexes of non-proportionality and criterions as functions of damping ratio of the absorber

For the investigated numerical model and for a set of ζ_{TMD} the previously defined indexes and criterions were calculated. In this chapter their values and recommendations of their applicability in case of structures with absorber are given. The guidelines are based on comparison of the indexes and criterions with complex character of the eigen-modes and relative errors given in Table 1. The focus was aimed especially to errors of amplitude of the top of the tower.

The most general indication of non-proportionality of the damping is the non-fulfillment of the diagonal dominance of the modal damping matrix. In Table 3 the ratio of absolute value of diagonal term and the sum of absolute values of non-diagonal terms for each row of modal damping matrix as a function of ζ_{TMD} is evaluated. The cases for which the requirement of dominance is fulfilled are printed in bold. From the table it follows, that the modal damping matrix isn't diagonally dominant for all analyzed values of ζ_{TMD}. The requirement of dominance is fulfilled only for zero ζ_{TMD} and for eigen-modes, which aren't associated with the second eigen-mode of the basic system. The decrease of ratios with increase of ζ_{TMD} corresponds with increase of the extent of the non-proportionality of the damping.. Nevertheless, the diagonal dominance of the modal matrix could serve only as additional not the decisive criterion for neglecting the non-diagonal terms. It follows from the fact, that although relative errors for zero ζ_{TMD} are small, the modal damping matrix isn't diagonally dominant.

Tab. 3: Ratio of absolute value of diagonal term and the sum of absolute values of non-diagonal terms for each row of modal damping matrix as function of ζ_{TMD}

ζ_{TMD} [/]	Eigen-mode (row) n.					
	1	2	3	4	5	6
0	**1,453**	0,422	0,583	**1,539**	**7,917**	**34,642**
0,2	0,084	0,235	0,373	0,334	0,286	0,691
0,5	0,049	0,226	0,361	0,312	0,198	0,326
0,8	0,040	0,223	0,358	0,307	0,176	0,236

The values of the left side of the condition (6) suggested by Hasselman (1976) is graphically presented for particular eigen-modes and for various ζ_{TMD} on Figure 4.

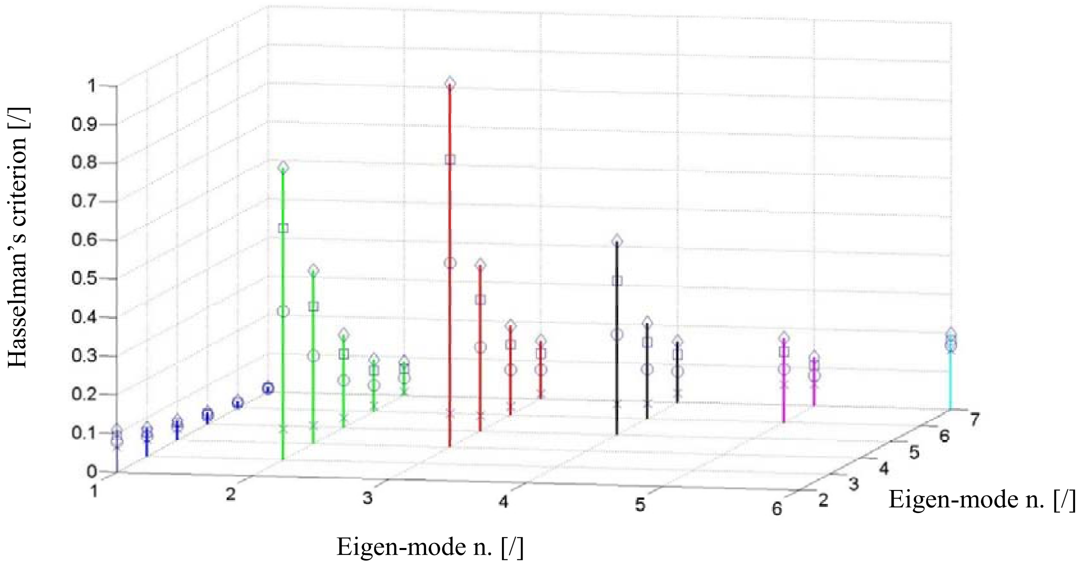


Fig. 4: Graphical expression of Hasselman's criterion for neglecting of mechanical interaction of eigen-modes for various ζ_{TMD} (×…ζ_{TMD} =0; o…ζ_{TMD} =0,2; □…ζ_{TMD} =0,5;◊ …ζ_{TMD} =0,8)

The significant mechanical interaction between the second and third, the third and fourth as well as between the fourth and the fifth eigen-mode was determined. On the other hand, small interaction between the first eigen-mode and the others were found out. However these results come from frequency proximity of these eigen-modes rather than from a distribution of damping in the system. It could be demonstrated on the example of the mechanical interaction of the sixth and the seventh eigen-mode. Relatively high value of interaction (0,2) didn't change significantly with increasing of the ζ_{TMD} and also didn't correspond to negligible error in calculation of the response using approximate solution. It should be also noted, that condition (9), which is required for using Hasselman's criterion and which is related to ratios of diagonal a non-diagonal terms of modal damping matrix, is fulfilled only for zero ζ_{TMD}. However, the main disadvantage and the reason of unusability of Hasselman's criterion is the fact that it finds out the mechanical interaction even between eigen-modes of classically damped models.

Criterion suggested by Wartburton & Soni (1977) for neglecting of non-diagonal terms of modal damping matrix was expressed by means of boundary value ε_b of the coefficient ε see Table 4 and Figure 5. For smaller values of ε than ε_b the condition (10) is fulfilled, for higher values it is not. In comparison to Hasselman's criterion the proposed criterion could identify the classically damped system due to ratio of diagonal and non-diagonal terms of the modal damping matrix. In this case the coefficient ε_b is theoretically zero. The authors proposed for problem in practice value of ε_b equal 5%. This value should correspond to maximal achievable error of response equal 10%. For the first eigen-mode the assumption of a small mechanical interaction based on analysis of the relative errors and complex eigen-modes was confirmed with exception of the highest ζ_{TMD}. For the lowest ζ_{TMD} a small non-proportionality of whole system was also confirmed. For higher values of ζ_{TMD} especially the practical one the coefficient ε_b for particular eigen-modes shows relatively good agreement with errors of the top of the tower given in Table 1. On Figure 5a there is depicted the coefficient ε_b as a function of ζ_{TMD} for the first six eigen-modes. It shows almost the linear dependency of ζ_{TMD} on ε_b i.e. increase of the ε_b with increase of ζ_{TMD}. The decreasing of the value of ε_b with increasing of ζ_{TMD} occurs only for values of ζ_{TMD} lower than a specific ζ_{TMD} for which the damping matrix is the best approximation of the classically damped one see Figure 5b. This specific value of ζ_{TMD} is in our case close to damping ratio of the first and second eigen-mode of structure without absorber.

Tab. 4: Boundary value ε_b of parameter ε of criterion of particular eigen-modes suggested by Wartburton & Soni (1977) for various ζ_{TMD}

ζ_{TMD} [/]	Eigen-mode n.					
	1	2	3	4	5	6
0	0,0010	0,0052	0,0071	0,0051	0,0009	0,0003
0,2	0,0129	0,1993	0,2734	0,2586	0,0536	0,0157
0,5	0,0336	0,5059	0,6942	0,6542	0,1354	0,0396
0,8	0,0543	0,8126	1,1151	1,0497	0,2172	0,0635

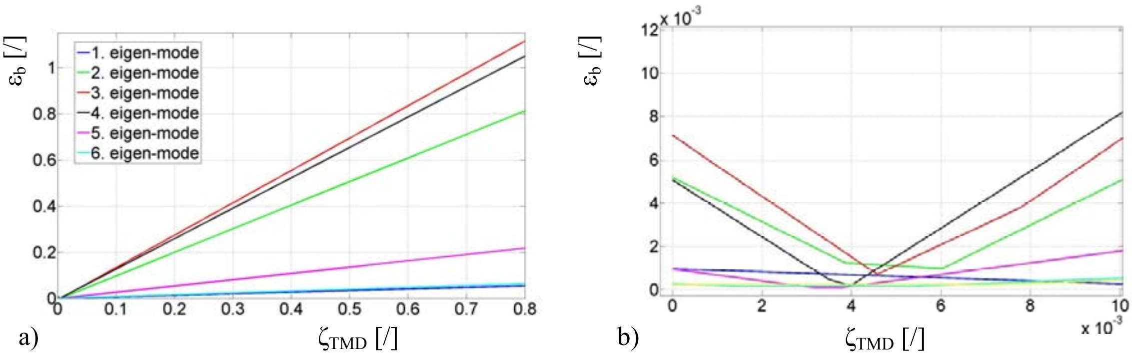


Fig. 5: Boundary values ε_b of parameter ε of criterion of particular eigen-modes suggested by Wartburton & Soni (1977) as function of ζ_{TMD}

The summation based indexes δ_{1i} of particular eigen-modes, which was proposed by Prater & Singh (1986), are depicted as a function of ζ_{TMD} on Figure 6. All of these indexes are increasing with increasing of ζ_{TMD}. Only for lower values of ζ_{TMD} than specific value of ζ_{TMD}, which was defined in previous paragraph, the indexes have decreasing trend see Figure 6b. The values of indexes δ_{1i} for chosen ζ_{TMD} are also quantified in Table 5. From the Table 5 and from the Figure 6b follow, that for zero ζ_{TMD} the first four eigen-modes are highly coupled. It doesn‘t correspond with errors of solution of the response obtained from the analysis of numerical model and also with a character of complex eigen-modes. The behaviour of indexes also shows, that for practical and higher values of ζ_{TMD} the most influenced eigen-mode by damping term of absorber is the first one. It is in contrast to an expected and confirmed presumption, that the most influenced eigen-modes are the second, the third and the fourth one.

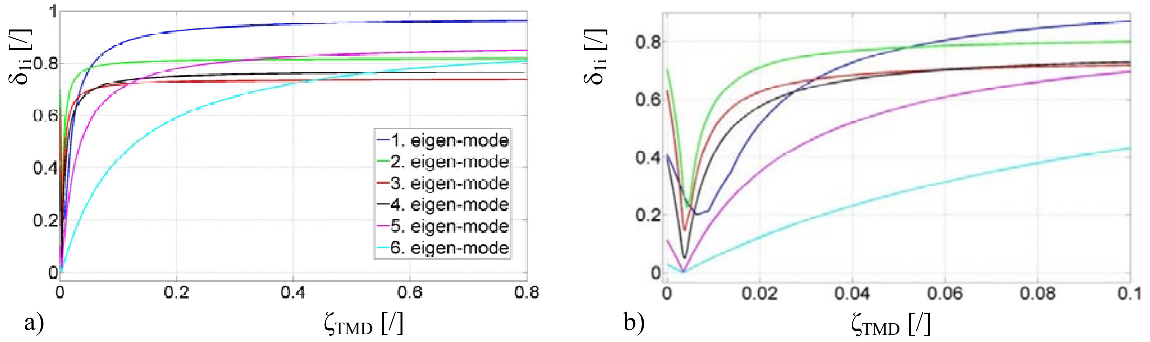


Fig. 6: Indexes δ_{1i} of particular eigen-modes as function of ζ_{TMD} (Prater & Singh (1986))

Tab. 5: Indexes of non-proportionality of damping δ_{1i} for various values of ζ_{TMD}

ζ_{TMD} [/]	Eigen-mode n.					
	1	2	3	4	5	6
0	0,408	0,703	0,632	0,394	0,112	0,028
0,2	0,922	0,810	0,729	0,750	0,778	0,592
0,5	0,954	0,816	0,735	0,762	0,835	0,754
0,8	0,962	0,817	0,737	0,765	0,850	0,809

Another calculated response based indexes δ_{3i} of particular eigen-modes see Figure 7 and Table 6 express very well the trend of behaviour of obtained errors of peaks of the response in frequencies near the corresponding eigen-frequencies. However, the calculation of indexes requires the solution of the response of the whole system with full modal damping matrix. And thus no advantage of calculation of indexes in comparison with the solution of the real and full problem is gained.

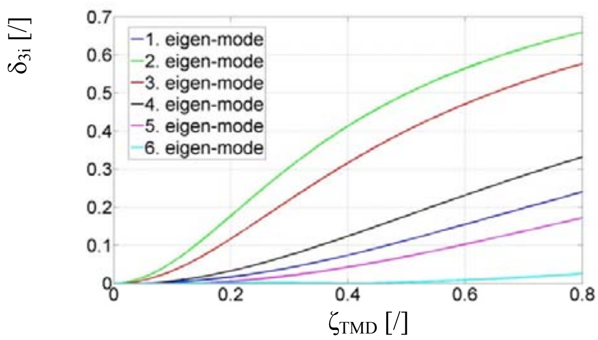


Fig. 7: Indexes δ_{3i} of particular eigen-modes as function of ζ_{TMD} (Prater & Singh (1986))

Tab. 6: Indexes of non-proportionality of damping δ_{3i} for various values of ζ_{TMD}

ζ_{TMD} [/]	Eigen-mode n.					
	1	2	3	4	5	6
0	3,8e-5	9,3e-4	1,1e-4	6,3e-5	1e-5	7,8e-6
0,2	0,017	0,177	0,117	0,033	0,005	0,002
0,5	0,113	0,498	0,401	0,177	0,071	0,003
0,8	0,240	0,659	0,577	0,332	0,172	0,025

The values of the generalized determinant based index δ_2 and generalized indexes δ_1 and δ_3 of the investigated numerical model suggested by Prater & Singh (1986) are for chosen ζ_{TMD} given in Table 7. Index δ_2 indicates the unrealistically high value of non-proportionality of the damping. Even for the realistic value of ζ_{TMD} equal 0,2 it exceeds the border of total non-proportionality of the system given by one. The other generalized indexes δ_1 and δ_3 are strongly affected by the number of eigen-modes taking into account. Assuming in the solution of indexes one additional eigen-mode, which is minimally influenced by absorber i.e. is almost classical, results in decreasing of both generalize indexes δ_1 and δ_3.

Tab. 7: Generalized indexes of non-proportionality of damping δ_{1-3} for various values of ζ_{TMD}

Index	ζ_{TMD} [/]			
	0	0,2	0,5	0,8
δ_1	0,089	0,693	0,762	0,781
δ_2	2,6e-8	1,773	466,7	8022
δ_3	1,7e-4	0,050	0,181	0,287

The values of index *I* defined by Tong et al. (1994) show very high non-proportionality of the system almost in the whole interval of investigated values of ζ_{TMD} see Figure 8 and Table 8. Only in a narrow interval of ζ_{TMD} in the neighborhood of specific ζ_{TMD} the index is minimized. This specific value of ζ_{TMD} corresponds as it was previously defined to the best approximation of the damping matrix to its classically damped form. The values of the index don't correspond to the results of analysis of the error. For all chosen ζ_{TMD} the value of index are almost one, which stands for the total non-proportionality of the damping. However, the errors caused by approximate solution are especially for zero ζ_{TMD} less significant than it could be expected from the value of index. This conclusion of unusability of this index is also supported by the excessive values of upper bounds of the relative errors given by (25) for all ζ_{TMD} see Table 8.

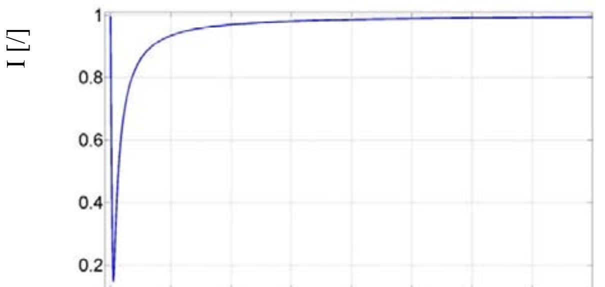


Fig. 8: Index of non-proportionality of damping I as function of ζ_{TMD} (Tong et al. (1994))

Tab. 8: Index of non-proportionality of damping I and upper bound of error of response for various values of ζ_{TMD}

ζ_{TMD} [/]	0	0,2	0,5	0,8
I	0,995	0,97	0,990	0,994
E	9,56	8,423	8,705	9,466

The prerequisite of the diagonal dominance of the modal damping matrix for a quantification of the non-proportionality of the individual eigen-mode using indexes defined by Bhaskar (1995) has been confirmed. The value of indexes is close to one which represents the absolute non-proportionality even for the ζ_{TMD}= 0,2 see Figure 9. For higher values of ζ_{TMD} the indexes are higher than one. On the figure the indexes of each eigen-mode were calculated for driving frequencies of the loading equal to the first six eigen-frequencies of the structure with absorber. Only indexes of eigen-modes for which driving frequencies are not equal to their corresponding eigen-frequencies are depicted. The indexes of particular eigen-modes and corresponding eigen-frequencies are summarized in Table 9.

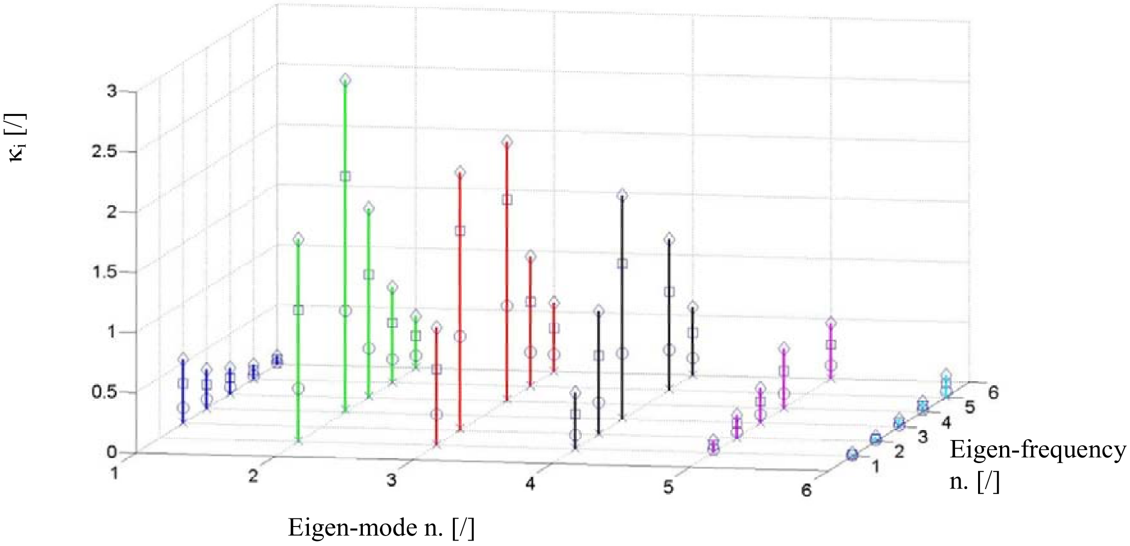


Fig. 9: Indexes of non-proportionality κ_i of particular eigen-modes for driving frequencies of loading equal to eigen-frequencies of the system with absorber for various ζ_{TMD} (Bhaskar (1995)) (×...ζ_{TMD} =0; o...ζ_{TMD} =0,2; □...ζ_{TMD} =0,5;◊ ...ζ_{TMD} =0,8)

Tab. 9: Maximal values of indexes of non-proportionality κ_i of particular eigen-modes and their corresponding eigen-frequencies for various values of ζ_{TMD} (Bhaskar(1995))

ζ_{TMD} [/]	Eigen-frequency and eigen-mode n.					
	1	2	3	4	5	6
0	0,688	2,370	1,715	0,650	0,126	0,029
0,2	11,843	4,262	2,683	2,992	3,501	1,448
0,5	20,609	4,433	2,773	3,202	5,052	3,068
0,8	25,123	4,477	2,796	3,258	5,671	4,239

The upper bounds of the relative errors in modal coordinates derived by Gawronski & Sawicki (1997) are given for investigated model in Table 10.

Tab. 10: Relative errors of modal displacements due to neglecting of non-diagonal terms of modal damping matrix for various values of ζ_{TMD} (Gawronski & Sawicki (1997))

ζ_{TMD} [/]	Eigen-mode n.					
	1	2	3	4	5	6
0	0,029	0,082	0,055	0,045	0,008	0,002
0,2	3,713	1,923	0,345	1,386	0,263	0,101
0,5	15,829	4,485	0,727	3,382	0,928	0,244
0,8	30,715	6,327	0,935	4,915	1,657	0,378

Calculated upper bounds of errors are for ζ_{TMD} equal and higher than 20% very high. The highest bound was obtained for the modal displacement which corresponds to the first eigen-mode of the system. Although the influence of the damping term of the absorber should be more significant in case of the second, the third and the fourth eigen-mode. We can't directly compare the errors of generalized displacements and errors of modal displacements. However we can't expect for these excessive upper bounds such a real level of relative errors of the response in generalized co-ordinates.

The coupling indexes χ_r of individual eigen-modes defined by Venancio-Filho et al. (2001) are given in Table 11. Under assumption, that zero stands for minimum and one for maximum non-proportionality, it follows that in all investigated cases index is very high. This fact doesn't correspond with calculated character of complex modes and relative errors.

Tab. 11: Indexes of non-proportionality of damping of particular eigen-modes for various values of ζ_{TMD} (Venancio-Filho et al. (2001))

ζ_{TMD} [/]	Eigen-mode n.					
	1	2	3	4	5	6
0	0,050	0,430	0,430	0,108	0,009	0,001
0,2	0,277	0,880	0,880	0,846	0,467	0,120
0,5	0,508	0,950	0,950	0,934	0,695	0,261
0,8	0,627	0,968	0,968	0,958	0,787	0,364

5. Conclusions

The article deals with an applicability of till now published indexes of non-proportionality of the damping in the case of slender structure (TV tower) equipped with absorber which is subjected to the harmonic excitation. The focus is also aimed at criterions for neglecting of non-diagonal terms of the modal damping matrix of its numerical model. These terms express the coupling of modal coordinates i.e. the mechanical interactions of individual eigen-modes. The applicability is assessed using comparison of indexes and criterions with character of complex eigen-modes and with relative errors of the response that are caused by neglecting of non-diagonal terms for selected values of damping ratios of absorber. From the practical point of view the errors corresponding to the dominant peaks of response curve of the top of the tower were taking into account. None of investigated indexes and criterions did fully correspond to the obtained solutions. The summation based index suggested by Prater & Singh (1986), indexes proposed by Bhaskar (1995) and Venancio-Filho et al. (2001) indicated very high values of non-proportionality particularly for the first four eigen-modes for all selected damping ratios of absorber. Even for zero damping ratio for which the calculated errors of the response were negligible and also for the first eigen-mode, which is almost identical with real eigen-mode of the undamped system. Also the generalized index suggested by Tong et al. (1994), determinant based index proposed by Prater & Singh (1986) indicated almost total non-proportionality of the system for all investigated cases. The analysis confirmed that criterion suggested by Hasselman (1976) couldn't be used because it supposes the mechanical interaction between individual eigen-modes even for classically damped structures. The relatively good agreement between calculated relative errors and proposed criterions was obtained for criterion proposed by Warburton & Soni (1977). The boundary values of its parameter ε for which the criterion is still fulfilled could also serve as an approximate index of non-proportionality of particular eigen-modes. Analysis of the response also highlights the necessity of prerequisite of non-proportional damping, when passive damping equipment is installed into the structure and when the detailed behavior is investigated.

Acknowledgement

The kind support of the Czech Science Foundation No. 103/09/0094, Grant Agency of the ASCR No. A200710902, Ministry of Industry and Trade No. MPO TIP FR-TI3/654 and RVO 68378297 research plan is gratefully acknowledged.

References

Bhaskar, A. (1995) Estimates of errors in the frequency response of non-classically damped system. *Journal of Sound and Vibration*, 184 (1), pp. 59–72.

Bhaskar, A. (1999) Mode shapes during asynchronous motion and non-proportionality indexes. *Journal of Sound and Vibration*, 224 (1), pp. 1–16.

Gawronski, W., Sawicki, J. T. (1997) Response errors of non-proportionally lightly damped structures. *Journal of Sound and Vibration*, 200 (4), pp. 543–550.

Hart, G. C. & Wong, K. (2000) *Structural Dynamics for Structural Engineers*. John Wiley & Sons, Inc., New York

Hasselsman, T. K. (1976) Modal coupling in lightly damped structures. *AIAA Journal*, 14, pp. 1627–1628.

Hurty, Walter C, & Rubinstein, Moshe F, (1964) *Dynamics of Structures*, Englewood Cliffs: Prentice-Hall,Inc,

Liu, K., Kujath, M. R. & Zheng, W. (2000) Quantification of non-proportionality of damping in discrete vibratory systems. *Computer and Structures*, 77 (5), pp. 557–569.

Prater, G. & Singh, R. (1986) Quantification of the extent of non-proportional viscous damping in discrete vibratory systems. *Journal of Sound and Vibration,* 104(1), pp. 109-125.

Prells, U. & Friswell, M. I. (2000) A measure of non-proportional damping. *Mechanical System and Signal Processing*, 14 (2), pp. 125–137.

Tong, M., Liang, Z., Lee, G. C. (1994) An index of damping non-proportionality for discrete vibrating systems. *Journal of Sound and Vibration*, 174, pp. 37–55.

Venancio-Filho, F., Wang, Y.K., Lin, F.B., Claret, A.M. & Ferreira, W.G. (2001) Dynamic Analysis of Nonproportional Damping Structural Systems Time and Frequency Domain Methods, *Transactions*, SMiRT 16, Washington DC.

Warburton, G. B. & Soni, S. R. (1977) Errors in response calculations for non-classically damped structures. *Earthquake Engineering and Structural Dynamics*, 5, pp. 365–376.