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THE SECULAR EQUATION FOR SURFACE WAVES IN 2D ANISOTROPIC ELASTODYNAMICS

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Abstract: *The secular equation for the surface (Rayleigh–edge) waves propagating in a thin semiinfinite anisotropic elastic continuum is derived. The secular equation is obtained as a quartic one for the squared wave velocity. Some numerical examples are shown.*

Keywords: ***Composite laminates, Crystals, 2D anisotropic elasticity***

1. Introduction

The traditional way of deriving the secular equation for Rayleigh-edge waves propagating in the direction of the x_1-axis in an anisotropic elastic half-plane $x_2 \geq 0$ is to find a general steady-state solution for the displacement components that vanishes at $x_2 = \infty$. This involves the computation of quartic equation roots that depend not only on material constants but also on wave velocity. The secular equation (explicit or implicit) is then obtained by vanishing of the surface traction at $x_2 = 0$. For the solution of such secular equation it is necessary to precompute some roots of characteristic quartic equation. The method shown in this paper leads to explicit secular equation that depends on material constants only.

2. Preliminaries

We suppose that material and body axes of the 2D anisotropic linear elastic medium in the state of plane stress are denoted by X_1, X_2 and x_1, x_2 respectively. Third axis x_3 is identical with material axis X_3 and constitutes axis of possible rotation of principal material axes X_1, X_2 from body axes x_1, x_2. Due to the plane stress it holds $\sigma_{33} = \sigma_{23} = \sigma_{13} = 0$. In this paper we will assume that principal material axes X_1, X_2 coincide with body axes x_1, x_2. For considered material the relationship between the stress σ_{ij} and strain ε_{ij} components is given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix}, \tag{1}$$

where C_{ij} denote the elastic stiffnesses. The strain components ε_{ij} are related to the displacement components u_1, u_2 through

$$2\varepsilon_{ij} = \left(u_{i,j} + u_{j,i}\right), (i, j = 1, 2). \tag{2}$$

The equations of motion, written in the absence of body forces, are

$$\sigma_{ij,j} = \rho \cdot \ddot{u}_i \,, \tag{3}$$

where ρ is the mass density and the comma denotes differentiation with respect to x_j (j=1,2).

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3. Rayleigh waves

The propagation of a Rayleigh wave along an edge of a semiinfinite 2D anisotropic medium is modeled. It is supposed that corresponding displacement and stress fields have the forms

$$u_s(x_1,x_2,t)=U_s(k\cdot x_2)\cdot e^{i\cdot k(x_1-c\cdot t)}\ ,\sigma_{rs}(x_1,x_2,t)=k\cdot\Sigma_{rs}(k\cdot x_2)\cdot e^{i\cdot k(x_1-c\cdot t)}\ (r,s=1,2)\quad , \tag{4}$$

where k is the wave number and c is the wave velocity. The boundary conditions of the problem are

$$\Sigma_{j2}(0)=0,\ U_j(\infty)=0\ ,\ (j=1,2). \tag{5}$$

Substituting (4) into (3), the equations of motion reduce to (the prime denotes differentiation with respect to $k{\cdot}x_2$)

$$i\cdot\Sigma_{11}+\Sigma'_{12}=-\rho\cdot c^2\cdot U_1\ ,\quad i\cdot\Sigma_{12}+\Sigma'_{22}=-\rho\cdot c^2\cdot U_2. \tag{6}$$

Since no boundary conditions are prescribed for σ_{11} and consequently also for Σ_{11} , this component may be eliminated. After some algebra we obtain a system of four ordinary differential equations of the first order for unknowns U_1, U_2, Σ_{12}, Σ_{22} . The system may be written in a matrix format

$$\begin{Bmatrix}U_1'\\U_2'\\\Sigma_{12}'\\\Sigma_{22}'\end{Bmatrix}=\begin{bmatrix}i\dfrac{d_3}{d_1} & -i & \dfrac{C_{22}}{d_1} & -\dfrac{C_{26}}{d_1}\\ -i\dfrac{d_2}{d_1} & 0 & -\dfrac{C_{26}}{d_1} & \dfrac{C_{66}}{d_1}\\ \dfrac{d}{d_1}-\rho\cdot c^2 & 0 & i\dfrac{d_3}{d_1} & -i\dfrac{d_2}{d_1}\\ 0 & -\rho\cdot c^2 & -i & 0\end{bmatrix}\cdot\begin{Bmatrix}U_1\\U_2\\\Sigma_{12}\\\Sigma_{22}\end{Bmatrix}, \tag{7}$$

where d, d_1, d_2, d_3 are coupled by the relation

$$d=C_{11}\cdot d_1-C_{12}\cdot d_2+C_{16}\cdot d_3\ . \tag{8}$$

It is easily seen that the symbol d represents the determinant of stiffness matrix $\mathbf{C}$ (see (1)) and d_1, d_2, d_3 are subdeterminants of $\mathbf{C}$. From the positive definiteness of stiffness matrix $\mathbf{C}$ it follows that d, d_1 are positive. So we have

$$d>0\ \ \&\ \ d_1>0\ . \tag{9}$$

Denoting the stress vector **T** and displacement vector **U** as

$$\mathbf{T}=\{T_1\quad T_2\}^T\ ,\mathbf{U}=\{U_1\ \ U_2\}^T\ , \tag{10}$$

where $T_1 = \Sigma_{12}$, $T_2 = \Sigma_{22}$ then the above system of four equations may be rewritten as

$$\begin{Bmatrix}\mathbf{U}'\\\mathbf{T}'\end{Bmatrix}=\begin{bmatrix}\mathbf{M}_1 & \mathbf{M}_2\\\mathbf{M}_3 & \mathbf{M}_4\end{bmatrix}\cdot\begin{Bmatrix}\mathbf{U}\\\mathbf{T}\end{Bmatrix}\quad . \tag{11}$$

The submatrices $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$ and $\mathbf{M}_4$ are given by

$$\mathbf{M}_1=-i\cdot\begin{bmatrix}-\dfrac{d_3}{d_1} & 1\\ \dfrac{d_2}{d_1} & 0\end{bmatrix}\quad ,\quad \mathbf{M}_2=\begin{bmatrix}\dfrac{C_{22}}{d_1} & -\dfrac{C_{26}}{d_1}\\ -\dfrac{C_{26}}{d_1} & \dfrac{C_{66}}{d_1}\end{bmatrix}\quad ,$$

$$\mathbf{M}_3 = \begin{bmatrix} \frac{d}{d_1} - \rho \cdot c^2 & 0 \\ 0 & -\rho \cdot c^2 \end{bmatrix} = \begin{bmatrix} \frac{d}{d_1} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \rho \cdot c^2 & 0 \\ 0 & \rho \cdot c^2 \end{bmatrix} \quad , \quad \mathbf{M}_4 = \mathbf{M}_1^{\ T} . \tag{12}$$

Besides this, it holds that

$$\mathbf{M}_1 = i \cdot \mathbf{N}_1 \ , \ \mathbf{M}_2 = \mathbf{N}_2 = \mathbf{N}_2^{\ T} \ , \ \mathbf{M}_3 = -\mathbf{N}_3 - \rho \cdot c^2 \cdot \mathbf{I} = \mathbf{M}_3^{\ T} \ , \ \mathbf{M}_4 = \mathbf{M}_1^{\ T} = i \cdot \mathbf{N}_1^{\ T} . \tag{13}$$

Symbols $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ appearing in the relation (13) are submatrices of the fundamental elasticity matrix **N** introduced by Ingebrigtsen and Tonning (1969). Symbol **I** is identity matrix of size 2. It is supposed that matrix $\mathbf{M}_3$ is not singular. It means that the Rayleigh wave propagates at a velocity distinct from that given by $\rho \cdot c^2 = d/d_1$. With this assumption, the derivative of the second vector line of the system (11) yields relation for $\mathbf{U}'$. Substituting for $\mathbf{U}'$ into first vector line of (11) we get the relation for $\mathbf{U}$. Inserting the relation for $\mathbf{U}$ into second vector line of the system (11) gives after some matrix manipulations

$$\boldsymbol{\alpha} \cdot \mathbf{T}'' - i \cdot \boldsymbol{\beta} \cdot \mathbf{T}' - \boldsymbol{\gamma} \cdot \mathbf{T} = \mathbf{0} , \tag{14}$$

where real and symmetric matrices $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are given by

$$\boldsymbol{\alpha} = \mathbf{M}_3^{\ -1} , \quad i \cdot \boldsymbol{\beta} = \mathbf{M}_1 \cdot \mathbf{M}_3^{\ -1} + \mathbf{M}_3^{\ -1} \cdot \mathbf{M}_1^{\ T} , \quad \boldsymbol{\gamma} = \mathbf{M}_2 - \mathbf{M}_1 \cdot \mathbf{M}_3^{\ -1} \cdot \mathbf{M}_1^{\ T} . \tag{15}$$

The system (14) for traction components $T_1 = \Sigma_{12}$, $T_2 = \Sigma_{22}$ is more convenient to work with than the corresponding system for displacement components, because the boundary conditions, instead of (5), are now homogeneous. It holds

$$T_j(0) = T_j(\infty) = 0 \ , \ (j = 1,2) . \tag{16}$$

The solution of (14) is assumed in the form

$$\mathbf{T}(k \cdot x_2) = \mathbf{T}_0 \cdot e^{i \cdot p \cdot k \cdot x_2} \quad , \tag{17}$$

where $\mathbf{T}_0 = \{T_{01} \ T_{02}\}^T$ is a constant vector and p is a complex number with $\mathrm{Im}(p) > 0$ to fulfil the boundary conditions at infinity. Introducing the solution (17) into the equation (14) we arrive at the following problem. It is necessary to solve the homogeneous system of two linear equations for unknowns T_{01} and T_{02} which are the components of the vector $\mathbf{T}_0$. The system has the form

$$\begin{bmatrix} -\alpha_{11} \cdot p^2 + \beta_{11} \cdot p - \gamma_{11} & \beta_{12} \cdot p - \gamma_{12} \\ \beta_{12} \cdot p - \gamma_{12} & -\alpha_{22} \cdot p^2 - \gamma_{22} \end{bmatrix} \cdot \begin{Bmatrix} T_{01} \\ T_{02} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} . \tag{18}$$

The homogeneous system (18) will have a nontrivial solution if and only if its determinant of the matrix is zero. This leads to a quartic characteristic equation in p. It has the form

$$\alpha_{11} \cdot \alpha_{22} \cdot p^4 - \alpha_{22} \cdot \beta_{11} \cdot p^3 + (\alpha_{11} \cdot \gamma_{22} + \alpha_{22} \cdot \gamma_{11}) \cdot p^2 - \beta_{11} \cdot \gamma_{22} \cdot p + \gamma_{11} \cdot \gamma_{22} = 0 , \tag{19}$$

where the real coefficients α_{ij} , β_{ij} , γ_{ij} correspond to matrices $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, respectively. If a quartic equation has real coefficients, then either i) all roots are real or ii) there is an even number of complex roots (i.e. 4 or 2 complex roots), in conjugate pairs, see Schwarz (1958). First case i) may be discarded due to assumption $\mathrm{Im}(p) > 0$. Second case ii) falls into three possibilities. There are two distinct roots $p_1 \neq p_2$ with positive imaginary parts. Then the general solution to (14) takes the form

$$\mathbf{T}(k \cdot x_2) = q_1 \cdot \mathbf{T}_0^{(1)} \cdot e^{i \cdot p_1 \cdot k \cdot x_2} + q_2 \cdot \mathbf{T}_0^{(2)} \cdot e^{i \cdot p_2 \cdot k \cdot x_2} \quad . \tag{20}$$

where $\mathbf{T}_0^{(1)}, \mathbf{T}_0^{(2)}$ correspond to p_1 , p_2 respectively. Symbols q_1, q_2 are arbitrary constants. Second possibility covers the case $p_1 = p_2$. It gives the general solution to (14) as

$$\mathbf{T}(k \cdot x_2) = q_1 \cdot \mathbf{T}_0^{(1)} \cdot e^{i \cdot p_1 \cdot k \cdot x_2} + q_2 \cdot k \cdot x_2 \cdot \mathbf{T}_0^{(1)} \cdot e^{i \cdot p_1 \cdot k \cdot x_2} \quad . \tag{21}$$

The case $p_1 = p_2$ seems to be not important from point of view of practical application. Third possibility is represented by only one root p_1 with positive imaginary part. Then the general solution to the equation of motion (14) has the form

$$\mathbf{T}(k \cdot x_2) = q_1 \cdot \mathbf{T}_0^{(1)} \cdot e^{i \cdot p_1 \cdot k \cdot x_2} \quad . \tag{22}$$

Due to boundary conditions at $x_2=0$ (see (16) where $T_1(0)= T_2(0)=0$ and the conditions that $T_{01}^{(1)}$, $T_{02}^{(1)}$ are not simultaneously zero we obtain $q_1=0$. It leads to trivial solution $\mathbf{T}(k \cdot x_2) = 0$, and therefore this possibility may now be safely discarded.

Applying zero boundary conditions at $x_2=0$ into (20) we get another homogeneous system in unknowns q_1, q_2. This system will have a nontrivial solution if the determinant is zero. It leads after some algebra to the desired secular equation that is quartic one in $\rho \cdot c^2$. All the coefficients of the secular equation are real and depend on material constants only.

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