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THE LAMINAR FLOW SOLUTION IN THE PLANE BY EIGENMODE EXPANSION

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Abstract: *The paper deals with non-stationary laminar flow solution of an incompressible fluid. The Navier-Stokes equation is used for the description of this motion and it is solved by means of an expansion into a series of eigenmodes of vibration. A mathematical model, which assumes planar flow with specific boundary conditions, can be generalized to the spatial problem with different types of boundary conditions. The influence of the individual eigenmodes of vibration on the form of unsteady flow is evident.*

Keywords: ***Navier-Stokes equation, modal analysis, eigenmodes***

1. Introduction

The paper proposes a solution to a non-stationary laminar flow of incompressible fluid using an expansion into a series of eigenmodes of vibration. Pressure drop is chosen as the boundary conditions for this problem. After finding the equation for the velocity function, individual eigenmodes of vibration, partial sums and their time development were then drawn.

2. Problem definition

The Navier-Stokes equation (without the convective term) was used as the mathematical model of the aforementioned flow:

$$\rho \frac{\partial c_i}{\partial t} + \frac{\partial p}{\partial x_i} - \mu \frac{\partial^2 c_i}{\partial x_j^2} = 0. \qquad (1)$$

While solving this equation, only the case of the planar flow in a pipe was considered (*fig. 1*):

$$\rho \frac{\partial c_1}{\partial t} + \frac{\partial p}{\partial x_1} - \mu \frac{\partial^2 c_1}{\partial x_2^2} = 0. \qquad (2)$$

The flow with the pressure drop was considered and hence the pressure at the ends of pipe was chosen as the boundary conditions:

$$\begin{aligned} x_1 = 0 &: \ p(0,t) = p_1(t), \\ x_1 = L &: \ p(L,t) = p_2(t). \end{aligned} \qquad (3)$$

In order to solve this problem, initial conditions for velocity and pressure are required. Initial velocity was chosen as zero and initial pressure was prescribed by using a general function φ, which depends only on the position in axis x_1.

$$\begin{aligned} t = 0 : p(x_1, 0) &= \varphi(x_1, 0), \\ c_1(x_2, 0) &= 0. \end{aligned} \qquad (4)$$

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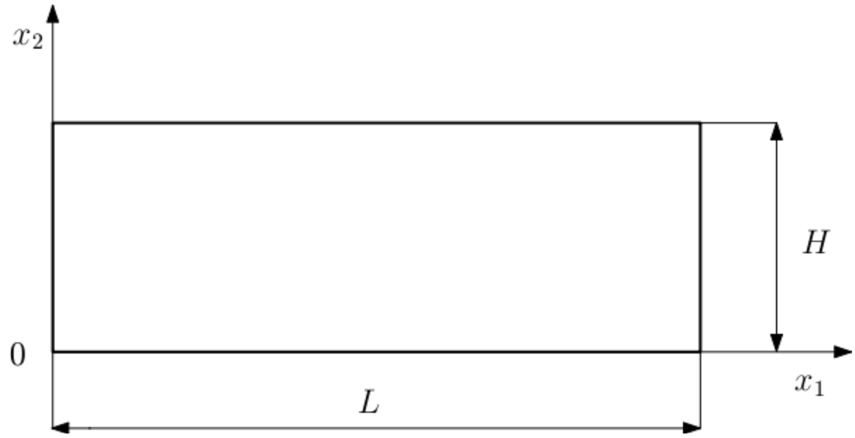


Fig. 1: Planar pipe

On comparing the dependencies of velocity and pressure with the variables in equation (2), it was found out that pressure derivative is a time dependent function. Using boundary conditions (3), the equation for pressure was obtained:

$$p(x_1, t) = p_1(t) - \frac{p_1(t) - p_2(t)}{L}\, x_1. \tag{5}$$

Equation (2) was then transformed into the following form:

$$\frac{\partial c_1}{\partial t} - \nu\, \frac{\partial^2 c_1}{\partial x_2^2} = \frac{p_1(t) - p_2(t)}{\rho L}. \tag{6}$$

The solution of the homogeneous part of this equation was considered to be in the form:

$$c_1(x_2, t) = e^{st}\, w(x_2), \tag{7}$$

where s is the eigenvalue and w is the eigenvector of velocity.

Using equation (7), the partial differential equation (6) was transformed to the ordinary differential equation:

$$sw - \nu\, \frac{\partial^2 w}{\partial x_2^2} = 0, \tag{8}$$

with zero boundary conditions for the eigenmode shape of velocity:

$$\begin{aligned} x_2 &= 0 : w = 0, \\ x_2 &= H : w = 0. \end{aligned} \tag{9}$$

The solution of equation (6) is discussed in further detail in section 2.3. The following sections give important properties of eigenvalue and eigenmode shapes of velocity, which are later used to find the solution.

2.1. Eigenvalue and eigenmode shapes of velocity

In order to solve equation (8), it is required to find out more information about the form of eigenvalue s. An estimate was performed which showed that the eigenvalue is negative real number:

$$s = -\,\nu\, \frac{\int_0^H \frac{\partial w}{\partial x_2} \frac{\partial w^*}{\partial x_2}\, dx_2}{\int_0^H w\, w^*\, dx_2} = -\,\alpha, \qquad \alpha \in \mathbb{R}^+. \tag{10}$$

Another fact that needs to be considered is that the eigenvectors are orthogonal. This was observed after comparing equation (8) for the k-th and l-th term of eigenmode shapes of velocity w_k and w_l^*:

$$(s_k - z_l) \int_0^H w_k \, w_l^* \, dx_2 = 0, \tag{11}$$

where z_l is the eigenvalue for the l-th term of eigenmode shapes of velocity.

From this equation two conclusions can be made:

- Eigenvalues are equal for the same indices:

$$s_k = z_l, \quad \text{for } \; k = l. \tag{12}$$

- Eigenvectors are orthogonal real functions:

$$\int_0^H w_k \, w_l^* \, dx_2 = 0. \tag{13}$$

2.2. Determination of the velocity function

The general solution of homogeneous partial differential equation of second order (8):

$$sw - \nu \frac{\partial^2 w}{\partial x_2^2} = 0, \tag{14}$$

has the form:

$$w(x_2) = A \, sinh\left(\sqrt{\frac{s}{\nu}} \, x_2\right) + B \, cosh\left(\sqrt{\frac{s}{\nu}} \, x_2\right). \; A, B \in \mathbb{C}. \tag{15}$$

Using the first boundary condition (9), that is:

$$x_2 = 0 : w = 0, \tag{16}$$

it was found out that constant B is zero. Hence the eigenmodes will be only a function of hyperbolic sine:

$$w(x_2) = A \, sinh\left(\sqrt{\frac{s}{\nu}} \, x_2\right). \tag{17}$$

From the second boundary condition the following expression was obtained

$$sinh\left(\sqrt{\frac{s}{\nu}} \, H\right) = 0. \tag{18}$$

Using the property (10) that the eigenvalue is a real number, an expression for constant α was derived:

$$\alpha = \nu \left(\frac{k\pi}{H}\right)^2. \tag{19}$$

Using the boundary conditions, an equation for the eigenvalue and also the form of the eigenmodes was found. However, the constant A in the function for eigenmode shapes of velocity has still to be determined. This constant was found by using the orthogonality condition:

$$\int_0^H w_k \, w_k^* \, dx_2 = 1. \tag{20}$$

After that, the constant for the k-th term of eigenmode shapes of velocity was found to be of the form:

$$A_k = -\,i\sqrt{\frac{2}{H}}. \tag{21}$$

Final form of the eigenmode shapes of velocity is solely the sine function:

$$w_k = \sqrt{\frac{2}{H}}\, sin\left(\frac{k\pi}{H}\, x_2\right). \tag{22}$$

2.3. Eigenmode expansion

In the previous section, only the homogeneous part of equation (8) was discussed. Here the complete solution of equation (6) will be considered:

$$\frac{\partial c_1}{\partial t} - \nu\,\frac{\partial^2 c_1}{\partial x_2^2} = \frac{p_1(t) - p_2(t)}{\rho L}, \tag{23}$$

accompanied by the boundary and initial conditions:

$$\begin{aligned} x_2 = 0 :&\quad c_1(0,t) = 0,\\ x_2 = H :&\quad c_1(H,t) = 0,\\ t = 0 :&\quad c_1(x_2,0) = 0. \end{aligned} \tag{24}$$

Solution of this equation was approached by using an expansion into a series of eigenmodes of vibration in the form:

$$c_1(x_2,t) = \sum_{k=1}^{\infty} a_k(t)\, w_k(x_2). \tag{25}$$

Substituting this form of velocity function into equation (23) the following equation was obtained:

$$\sum_{k=1}^{\infty}\left(\frac{\partial a_k(t)}{\partial t} - s_k\, a_k(t)\right) w_k = \frac{p_1(t) - p_2(t)}{\rho L}. \tag{26}$$

This equation can be simplified according to the orthogonality (13) and orthonormality (20) condition to the form:

$$\frac{\partial a_k(t)}{\partial t} - s_k\, a_k(t) = \sqrt{\frac{2}{H}}\,\frac{H}{k\pi}\,\frac{p_1(t) - p_2(t)}{\rho L}\,[1 - (-1)^k]. \tag{27}$$

This last equation accompanied by the initial condition:

$$t = 0 : a(0) = 0, \tag{28}$$

was solved using Laplace transformation. During an inverse transformation, convolution theorem was used and for the next computation constant change of pressure Δp was considered.

Expression for time dependent function $a_k(t)$ is:

$$a_k(t) = \sqrt{\frac{2}{H}}\,\frac{H^3}{(k\pi)^3\rho L\nu}\,[1 - (-1)^k]\,(1 - e^{s_k t})\,\Delta p. \tag{29}$$

Substituting the equation for time dependent function (29) back into equation (25) the following expression was obtained:

$$c_1(x_2,t) = \frac{2H^2\Delta p}{\pi^3 \rho L \nu} \sum_{k=1}^{\infty} \frac{1-(-1)^k}{k^3} \left(1-e^{s_k t}\right) \sin\left(\frac{k\pi}{H} x_2\right), \tag{30}$$

where eigenvalue is:

$$s_k = -\nu \left(\frac{k\pi}{H}\right)^2. \tag{31}$$

2.4. Result comparison

The result obtained was confirmed by comparing the maximal velocity achieved through the calculation (30) with the exact value. But nevertheless it was a restricted case. The considered time interval t was large enough and the equation was approximated only by the first eigenmode:

$$c_1\left(\frac{H}{2}\right) = \frac{4}{\pi^3} \frac{H^2}{\rho L \nu} \Delta p \doteq \frac{H^2}{7.75\, \rho L \nu} \Delta p. \tag{32}$$

While exact value (Šob, 2002) for velocity in the middle of pipe is:

$$c_1\left(\frac{H}{2}\right) = \frac{H^2}{8\, \rho L \nu} \Delta p. \tag{33}$$

Mentioned results differ only by 4%, hence it can be said that approximation by the first eigenmode is quite accurate. By increasing the number of eigenmodes, a more precise value for the maximal velocity will be obtained.

2.5. Graphical assessment

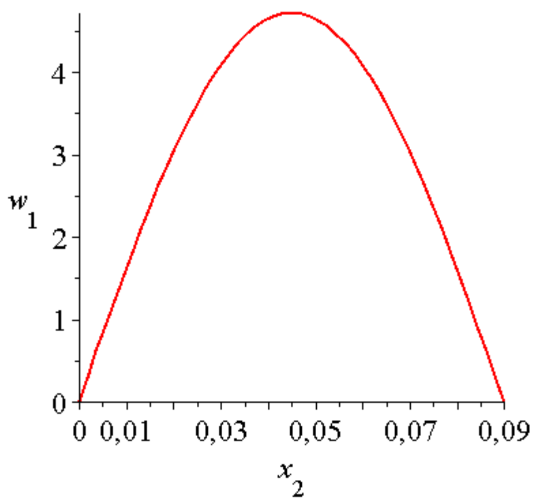
For the following graphical assessment these values for the variables were used:

$H = 0.09$ m, $\rho = 1000\ \mathrm{kg \cdot m^{-3}}$,

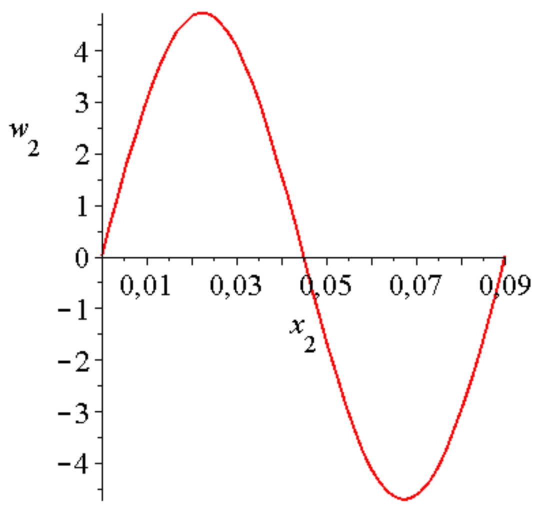
$L = 12$ m, $\nu = 10^{-6}\ \mathrm{m^2 \cdot s^{-1}}$.

$\Delta p = 300$ Pa,

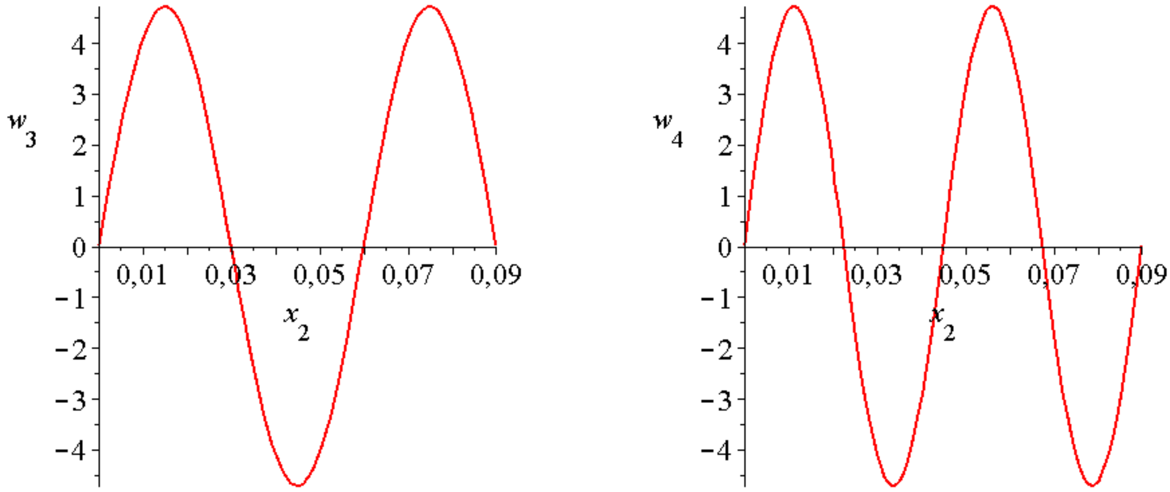
From the equation (30) it can be easily seen that the velocity profile is composed only from odd eigenmode shapes of velocity, because the time dependent function $a_k(t)$ for even eigenmodes is zero. The first four eigenmodes are drawn in the following figures:



(a) First



(b) Second



(c) Third *(d) Fourth*

Fig. 2: Eigenmodes of vibration

Final form of velocity function is given by equation (30). The partial sums of this series are plotted against time $t = 200$ s. This stretch of time was chosen randomly, however it was not chosen to be long enough for the stabilization of the velocity profile.

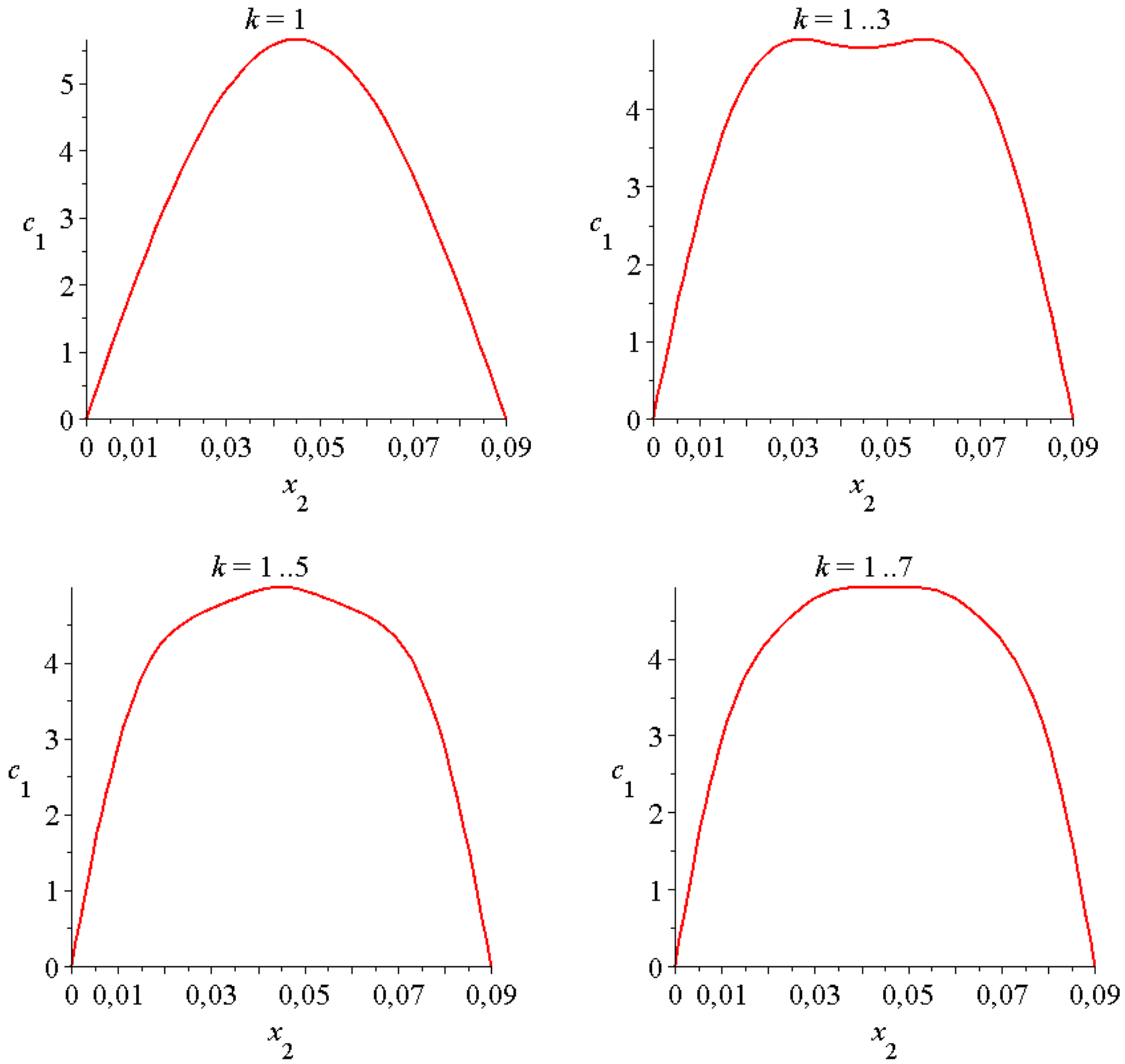


Fig. 3: The partial sums of velocity function

From these graphs, it can be shown that after summing seven eigenmodes, the shape of the velocity profile is getting closer to profile of laminar flow despite the fact that the profile is not completely stabilized. Odd eigenmodes cause symmetric velocity profile which is expected by our case of start-up flow.

Next the time dependence of the velocity when twenty eigenmode of vibrations are summed was then observed.

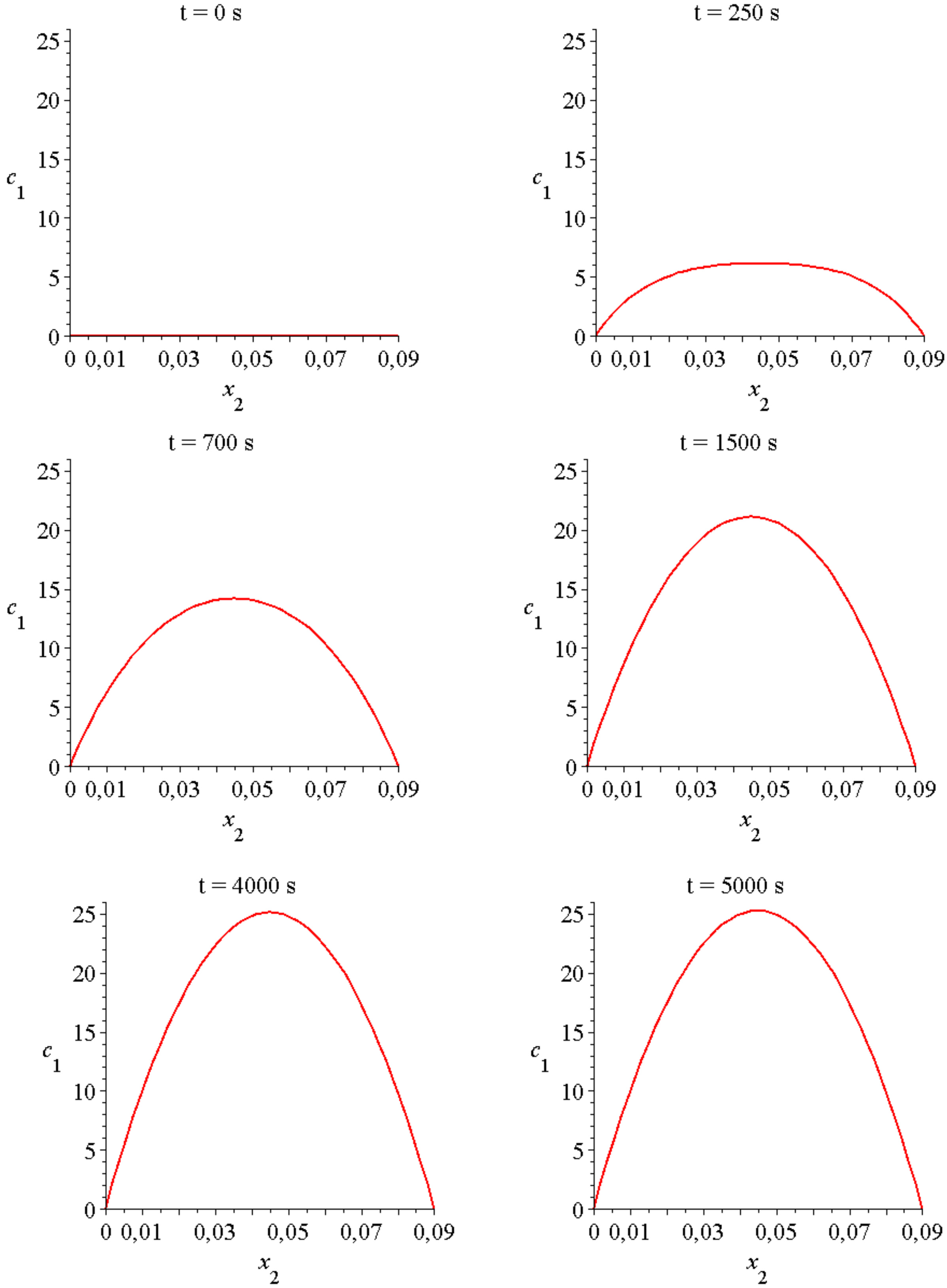


Fig. 4: Time dependence for ten odd eigenmode shapes of velocity

2.6. Comparison

From literature, it is common knowledge that for a time step long enough, the flow is considered as stationary, hence equation (2) has the form:

$$\frac{\partial p}{\partial x_1} - \mu \frac{\partial^2 c_1}{\partial x_2^2} = 0. \tag{34}$$

Analytical solution of this equation with zero boundary condition for velocity on walls of the pipe is a quadratic function of coordinate x_2:

$$c_1(x_2) = \frac{\Delta p}{2\mu L}(Hx_2 - x_2^2). \tag{35}$$

This expression can also be obtained from equation (30) for a long enough time t:

$$c_1(x_2) = \frac{4H^2 \Delta p}{\pi^3 \rho L \nu} \sum_{\substack{l=1 \\ k=2l-1}}^{\infty} \frac{1}{k^3} \sin\left(\frac{k\pi}{H} x_2\right). \tag{36}$$

For summation of the series equation (Kadlec & Kufner, 1969) was used:

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^3} = \frac{2\pi^2 x - 3\pi x^2 + x^3}{12}. \tag{37}$$

This sum of series holds for all summing indices, so it is necessary to transform this expression to only odd summing indices. To obtain only odd summing indices, the property that sum over all summing indices is equal to sum over odd and even indices was used. Hence using equation (37), expression (35) was obtained, which is the same to the analytical solution of equation (34).

3. Conclusions

This paper described the solution of the Navier-Stokes equation for non-stationary flow of an incompressible fluid. This solution is based on an expansion into a series of eigenmodes of vibration. The proposed approach considers the solution of velocity function as a combination of eigenmode shapes of velocity and time dependent function. Specific solution to the presented problem depends only on the odd eigenmode shapes of velocity since even eigenmodes were zero. The graphical part of this paper illustrated the achieved results.

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References

Brdička, M., Samek, L. & Sopko, B. (2000) *Mechanika kontinua*. Academia, Prague.

Kadlec, J. & Kufner, A. (1969) *Fourierovy řady*. Academia, Prague.

Krausová, H. (2009) *Navier-Stokesova rovnice – řešení laminárního proudění*. Bachelor's thesis. BUT, FME, Brno.

Pochylý, F. & Fialová, S. (2009) *Řešení laminárního proudění rozvojem podle vlastních tvarů kmitu*. Research work, BUT, FME, Brno.

Šob, F. (2002) *Hydromechanika*. Academic Publisher CERM, Brno.