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HOMOGENIZED PHONONIC PLATES AND WAVE DISPERSION

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Abstract: *We consider the problem of wave propagation in periodically heterogeneous composite plates with high contrasts in elastic coefficients. The unfolding method of homogenization is applied to obtain limit plate models. Due to the high contrast ansatz in scaling the elasticity coefficients of compliant inclusions, the dispersion properties are retained in the limit when the scale of the microstructure tends to zero. We study two plate models based on the Reissner-Mindlin theory and on the Kirchhoff-Love theory. We show that, when the size of the microstructures tends to zero, the limit homogeneous structure presents, for some wavelengths, a negative "mass density" tensor. This means that there exist intervals of frequencies in which there is no propagation of elastic waves, the so-called band-gaps.*

Keywords: ***phononic materials, plate models, homogenization, band gaps, wave dispersion***

1. Introduction

We consider problems of wave propagation in periodically heterogeneous plates with high contrasts in elastic coefficients. Following the approach of Ávila et al. (2008) and Rohan et al. (2009) we apply the unfolding method of homogenization Cioranescu et al. (2008) to obtain limit plate models. Two cases are studied: 1) according to the Reissner-Mindlin theory the plate deformation is described by the mid-plane deflections and by rotations of the plate cross-sections which account for the shear stress effects; 2) using the Kirchhoff-Love theory, the plate deflections are described by the bi-harmonic operator, thus neglecting the shear effects. In both cases we assume such heterogeneities which depend on the mid-plate coordinates only, but do not change with the transversal coordinate. As an example we can consider plates with soft cylindrical inclusions. Under such restrictions the homogenization is applied to the plate equations with the elastic coefficients defined as periodically fluctuating functions associated with the heterogeneities. Due to the high contrast ansatz in scaling the elasticity coefficients of inclusions, as employed in Ávila et al. (2008); Rohan and Miara (2011); Cimrman and Rohan (2009, 2010), dispersion properties are retained in the limit when the scale (the characteristic size) of the microstructure tends to zero.

We show that, when the size of the microstructures tends to zero, the limit homogeneous structure presents the phononic effect: for some wavelengths, a "mass density" tensor can be negative, see Rohan and Miara (2011). This means that there exist intervals of frequencies in which there is no propagation of elastic waves, the so-called band-gaps.

2. Heterogeneous plates

We consider heterogeneous structures associated with a given scale, say $\varepsilon_0 > 0$, which is the ratio between the characteristic lengths of the microscopic and the macroscopic description. There exist sequences of solutions of the plate problems characterized by scales $\varepsilon \to 0$. For any fixed $\varepsilon > 0$ we shall rely on the following essential material properties.

The fourth order, bi-dimensional elasticity tensor $\mathbb{C} = (C^{ijkl})$ is symmetric $C_{ijkl} = C_{klij} = C_{ikjl}$ and positive definite. In particular, for the sake of simplicity, we consider isotropic materials only, which

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are characterized by two Lamé parameters. The plate model according the Reissner-Mindlin theory involves also the shear modulus, here denoted by $\gamma > 0$, which is associated with one of the Lamé parameters. The mass density ρ is positive.

We treat periodic composite materials, so that the material coefficients $\mathbb{C}$, γ and ρ are periodically oscillating functions in $\mathbb{R}^2$; it will be described in detail in Section 3.1.

2.1. The Reissner–Mindlin plate model

The plate model can be derived by an asymptotic analysis of the elasticity problem imposed in $\Omega \times]-h, h[$, where $\Omega \subset \mathbb{R}^2$ is an open bounded domain with regular boundary $\partial\Omega$ and h is the plate thickness. In the time interval $[0, T]$ the plate undergoes the following two modes of displacements: the in-plane "membrane modes" described by $\boldsymbol{U} = (U_1, U_2) : [0, T] \times \bar{\Omega} \longrightarrow \mathbb{R}^2$, and the "off-plane" transversal deflections $W : [0, T] \times \bar{\Omega} \longrightarrow \mathbb{R}$; moreover the cross-sections undergo rotations $\boldsymbol{\Theta} = (\Theta_1, \Theta_2) : [0, T] \times \bar{\Omega} \longrightarrow \mathbb{R}^2$.

The displacement and rotation $(\boldsymbol{U}, W, \boldsymbol{\Theta})$ of the plate satisfy the equilibrium equations

$$\begin{cases} h\rho \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{U} - h\mathrm{div}\boldsymbol{\sigma}(\boldsymbol{U}) = \boldsymbol{T} \quad \text{in } \Omega \,, \\ h\rho \dfrac{\mathrm{d}^2}{\mathrm{d}t^2} W - h\mathrm{div}\boldsymbol{\tau}(W, \boldsymbol{\Theta}) = F \quad \text{in } \Omega, \\ \dfrac{h^3}{3}\rho \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{\Theta} - \dfrac{h^3}{3}\mathrm{div}\boldsymbol{\sigma}(\boldsymbol{\Theta}) + h\boldsymbol{\tau}(W, \boldsymbol{\Theta}) = \boldsymbol{M} \quad \text{in } \Omega, \\ \boldsymbol{U} = \boldsymbol{0} \,, \quad W = 0 \,, \quad \boldsymbol{\Theta} = \boldsymbol{0} \quad \text{on } \partial\Omega \,. \end{cases} \tag{1}$$

with $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ expressed by the following linear constitutive laws:

$$\begin{cases} \boldsymbol{\tau}(W, \boldsymbol{\Theta}) & := & \gamma(\nabla W - \boldsymbol{\Theta}), & \text{shear stress due to relative (to mid-plane) rotation of cross-s.} \\ \boldsymbol{\sigma}(\boldsymbol{\Theta}) & := & \mathbb{C}\boldsymbol{e}(\boldsymbol{\Theta}) \,, & \text{normal stress due to bending induced by rotations} \\ \boldsymbol{\sigma}(\boldsymbol{U}) & := & \mathbb{C}\boldsymbol{e}(\boldsymbol{U}) \,, & \text{normal stress due to the in-plane membrane modes.} \end{cases} \tag{2}$$

In general, we may consider a general decomposition of $\partial\Omega$ with respect tothe above displacements and rotations into the "Neumann" and "Dirichlet" parts of the boundary. However, for the sake of simplicity, we consider the fully supported and clamped plate.

The linearized deformation tensor $e(\boldsymbol{\Theta}) = (e_{ij}(\boldsymbol{\Theta}))$ is given by the symmetric gradient $\boldsymbol{e}(\boldsymbol{\Theta}) = 1/2\,(\partial_j \Theta_i + \partial_i \Theta_j)$, $i, j = 1, 2$.

We shall consider solutions in the form of harmonic stationary waves induced by harmonic loading

$$\boldsymbol{T}(x,t) = \boldsymbol{t}(x)\exp\{\mathrm{i}\omega t\} \,, \qquad F(x,t) = f(x)\exp\{\mathrm{i}\omega t\} \,, \qquad \boldsymbol{M}(x,t) = \boldsymbol{m}(x)\exp\{\mathrm{i}\omega t\} \,, \tag{3}$$

where ω is a given frequency, so that

$$\boldsymbol{U}(x,t) = \boldsymbol{u}(x)\exp\{\mathrm{i}\omega t\} \,, \qquad W(x,t) = w(x)\exp\{\mathrm{i}\omega t\} \,, \qquad \boldsymbol{\Theta}(x,t) = \boldsymbol{\theta}(x)\exp\{\mathrm{i}\omega t\} \,. \tag{4}$$

On substituting (3) into (1), we get the following equations governing the amplitudes $(\boldsymbol{u}, w, \boldsymbol{\theta})$:

$$\begin{cases} -\omega^2 h\rho\boldsymbol{u} - h\mathrm{div}\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{t} \quad \text{in } \Omega \,, \\ -\omega^2 h\rho w - h\mathrm{div}\boldsymbol{\tau}(w, \boldsymbol{\theta}) = f \quad \text{in } \Omega, \\ -\omega^2 \dfrac{h^3}{3}\rho\boldsymbol{\theta} - \dfrac{h^3}{3}\mathrm{div}\boldsymbol{\sigma}(\boldsymbol{\theta}) + h\boldsymbol{\tau}(w, \boldsymbol{\theta}) = \boldsymbol{m} \quad \text{in } \Omega, \\ \boldsymbol{u} = 0, \quad w = 0, \quad \boldsymbol{\theta} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5}$$

2.2. The Kirchhoff–Love plate model

The motion of the plate is given by the out-off plate deflections W and by the in-plane (membrane modes) displacements $\boldsymbol{U}$. Let us recall that this kind of plate does not admit any relative rotation of the plate cross-sections w.r.t. the plate mean surface, therefore, it is convenient rather for thin plates.

We shall consider solutions in the form of harmonic stationary waves induced by harmonic loading, see (3). Thus, in analogy with (4), for a given fixed frequency ω, the amplitudes $(\boldsymbol{u}, w)$ satisfy

$$\begin{cases} -\omega^2 h\rho\boldsymbol{u} - h\mathrm{div}\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{t} \quad \text{in } \Omega , \\ \dfrac{h^3}{3}\nabla\nabla : \boldsymbol{\Sigma}(w) - \omega^2\rho\left(hw - \dfrac{h^3}{3}\nabla\cdot\nabla w\right) = f - \nabla\cdot\boldsymbol{m} \quad \text{in } \Omega , \\ w = 0, \quad \boldsymbol{n}\cdot\nabla w = 0, \quad \boldsymbol{u} = 0 \quad \text{on } \partial\Omega , \end{cases} \tag{6}$$

where $\nabla\nabla v = (\partial^2 v/\partial x_i\partial x_j)$ is the 2nd order differential operator and stresses $\boldsymbol{\sigma}$ and $\boldsymbol{\Sigma}$ are given in terms of the elasticity tensor, as follows:

$$\begin{aligned} \boldsymbol{\sigma}(\boldsymbol{u}) &= \mathbb{C}\boldsymbol{e}(\boldsymbol{u}) , \\ \boldsymbol{\Sigma}(w) &= \mathbb{C}\nabla\nabla w = \boldsymbol{\sigma}(\nabla w) . \end{aligned} \tag{7}$$

In the rest of the paper we consider just the out-of-plane "deflection" and "rotation" modes of plate deformation, since, in the linear theory used here, there is no coupling between these modes and the "membrane" modes described by displacements $\boldsymbol{u}$. The "membrane" modes are driven by the same type of equations as in the 3D elasticity which was discussed in papers Ávila et al. (2008); Rohan et al. (2009); Cimrman and Rohan (2009, 2010).

3. Homogenization

We consider a plate made of a heterogeneous material, whereby its periodic structure is defined in the reduced $2D$ configuration directly. As usually, we use small parameter ε describing the characteristic size of the microstructure. The solutions of (5) and (6) depend upon ε, which will be indicated by the superscript $\square^\varepsilon$. Using asymptotic analysis, the limit model for $\varepsilon \to 0$ can be obtained which describes behaviour of the homogenized material. Details on the homogenization procedure are out of the scope in this short paper, interested readers are refered to associated publications Ávila et al. (2008); Rohan and Miara (2011), (Cioranescu et al., 2008).

3.1. Strongly heterogeneous periodic composite

We assume the plate Ω is constituted by the matrix Ω_m and by periodically distributed inclusions; their collection forms domain Ω_c. Thus, we consider

1. $\Omega_m, \Omega_c \subset \Omega \subset \mathbb{R}^2$ and $\Omega_m \cap \Omega_c = \emptyset$,
2. $\Omega = \Omega_m^\varepsilon \cup \Omega_c^\varepsilon \cup \Gamma^\varepsilon$, where $\Gamma^\varepsilon = \overline{\Omega_m^\varepsilon} \cap \overline{\Omega_c^\varepsilon}$.
3. matrix: Ω_m^ε is connected. As the result, inclusions Ω_c^ε are disconnected.

The microstructure is generated as a periodic lattice using the representative periodic cell (RPC) denoted by Y. For simplicity, we consider a rectangular RPC with the following definition: $Y = \Pi_{i=1}^2]0, \bar{y}_i[\subset \mathbb{R}^2$ ($\mathbb{R}$ being the set of real numbers) where $\bar{y}_i > 0$ can be chosen so that $|Y| = 1$. The RPC is decomposed in coherence with Ω, i.e. $Y_m \subset Y$ and $Y_c = Y \setminus \overline{Y_m}$ are strictly contained in Y.

The coordinates of any point in Ω can be split into a "coarse" part $\xi = (\xi_i)$ and a "fine" part $y = (y_i)$, also called fast and slow evolving parts: For a given finite $\varepsilon > 0$ we have the unique decomposition

$$\begin{aligned} x &\equiv \varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon\left\{\frac{x}{\varepsilon}\right\}_Y \\ &= \xi + \varepsilon y , \quad \text{where } y = \left\{\frac{x}{\varepsilon}\right\}_Y \in Y \quad \text{and } \xi = \varepsilon\left[\frac{x}{\varepsilon}\right]_Y \in \Omega , \end{aligned} \tag{8}$$

where $\xi_i = \varepsilon k_i \bar{y}_i$, $i = 1, 2$, $k_i \in \mathbb{Z}$ is the lattice coordinate. Such a decomposition is unique, once $Y \in \mathbb{R}^2$ is defined. Note that $[z_i]_Y$ is the integer part of $z_i/\bar{y}_i$ and $\{z_i\}_Y$ is the remainder.

Material parameters $\mathbb{C}^\varepsilon = (C^\varepsilon_{ijkl})$ and γ^ε are periodically oscillating. For the sake of simplicity we shall consider only piecewise constant material. To retain the phononic effect in the homogenized plate, following the analogous approach employed in the case of phononic 3D periodic structures, we introduce the scaling of the material coefficients in the inclusions:

$$\begin{aligned} \mathbb{C}^\varepsilon(x) &= \chi^\varepsilon_c(x)\varepsilon^2\mathbb{C}^c + \chi^\varepsilon_m(x)\mathbb{C}^m \ , \\ \gamma^\varepsilon(x) &= \chi^\varepsilon_c(x)\varepsilon^2\gamma^c + \chi^\varepsilon_m(x)\gamma^m \ , \end{aligned} \tag{9}$$

We recall the standard properties of constant tensors $\mathbb{C}^c, \mathbb{C}^m$ and coefficients γ^c, γ^m which usually are considered: there exist constants $0 < \underline{m} < \overline{m} < \infty$ independent of ε such that (recall $|\boldsymbol{e}|^2 = e_{ij}e_{ij}$),

$$\begin{aligned} & C^m_{ijkl}e_{ij}e_{kl} \geq \underline{m}|\boldsymbol{e}|^2 \quad \forall \boldsymbol{e} = (e_{ij}) \in \mathbb{R}^{2\times 2}, \quad e_{ij} = e_{ji} \ , \quad \sup_{x\in\Omega} |C^m_{ijkl}| \leq \overline{m} \ , \\ & \underline{m} \leq \gamma^m \leq \overline{m} \ , \end{aligned} \tag{10}$$

and in analogy for $\mathbb{C}^c$ and γ^c. The rotation-deflection coupling coefficient $\gamma^\varepsilon(x)$, i.e. the shear stiffness, is only relevant for the Reissner-Mindlin plate model. It is worth noting that $\mathbb{C}^\varepsilon(x)$ and $\gamma^\varepsilon(x)$ are positive definite for $\varepsilon > 0$ only.

The density of the two materials is assumed to be of the same order of magnitude, therefore we shall consider

$$\begin{aligned} & \rho^\varepsilon(x) = \chi^\varepsilon_c(x)\rho^c + \chi^\varepsilon_m(x)\rho^m \ , \\ & \underline{\rho} \leq \rho^s \leq \overline{\rho} \ , \end{aligned} \tag{11}$$

where $\underline{\rho}, \overline{\rho}$ are given positive real numbers.

Some preliminaries The homogenized model was obtained using the periodic unfolding method which is based on the unfolding operator $\mathcal{T}_\varepsilon : v \in L^1(\Omega;\mathbb{R}) \to L^1(\Omega \times Y;\mathbb{R})$ defined, as follows, see Cioranescu et al. (2008),

$$\mathcal{T}_\varepsilon(v)(x,y) = \ v(\varepsilon \left[\tfrac{x}{\varepsilon}\right] + \varepsilon y) \ , \qquad x \in \Omega, y \in Y \ .$$

We shall use $H^1_\#(Y)$ and $H^2_\#(Y)$, the spaces of periodic (scalar) functions,

$$\begin{aligned} H^1_\#(Y) &= \{v \in H^1(Y) \mid v \text{ is Y-periodic } \} \ , \\ H^2_\#(Y) &= \{w \in H^2(Y) | \ w, \nabla_y w \ \text{ are Y-periodic}\} \ , \end{aligned} \tag{12}$$

and the associated space of vector-valued functions $\mathbf{H}^1_\#(Y) = (H^1_\#(Y))^2$.

3.2. Reissner-Mindlin phononic plate

We apply the unfolding method of homogenization to obtain a limit model of the R-M plate for $\varepsilon \to 0$. The elastic standing waves are described by the solution of the following problem with the oscillating material coefficients: For a given frequency, find triplet $(w^\varepsilon, \boldsymbol{\theta}^\varepsilon) \in \left(H^1_0(\Omega)\right)^3$ such that

$$\begin{aligned} & - h\omega^2 \int_\Omega \rho^\varepsilon \left(w^\varepsilon z^\varepsilon + \frac{h^2}{3}\boldsymbol{\theta}^\varepsilon \cdot \boldsymbol{\psi}^\varepsilon\right) + h\int_\Omega [\gamma^\varepsilon(\nabla w^\varepsilon - \boldsymbol{\theta}^\varepsilon)]\cdot(\nabla z^\varepsilon - \boldsymbol{\psi}^\varepsilon) + \frac{h^3}{3}\int_\Omega [\mathbb{C}^\varepsilon e(\boldsymbol{\theta}^\varepsilon)] : e(\boldsymbol{\psi}^\varepsilon) \\ & = \int_\Omega (f z^\varepsilon + \boldsymbol{m}\cdot\boldsymbol{\psi}^\varepsilon) \ , \end{aligned} \tag{13}$$

for all $(z^\varepsilon, \boldsymbol{\psi}^\varepsilon) \in \left(H^1_0(\Omega)\right)^3$.

Here we present the homogenized "macroscopic" model which involves homogenized coefficients descibing the effective mass and elasticity coefficients.

The standing waves propagating in the homogenized plate are described in terms of amplitudes $(\boldsymbol{\theta}, w) \in \mathbf{H}_0^1(\Omega) \times \in H_0^1(\Omega)$ which satisfy the following equations:

$$
\begin{aligned}
& -\omega^2 \int_\Omega \left(\frac{h^3}{3} [\boldsymbol{\mathcal{M}}(\omega^2)\boldsymbol{\theta}] \cdot \boldsymbol{\psi} + h\mathcal{N}(\omega^2) w z \right) \\
& + \frac{h^3}{3} \int_\Omega [\boldsymbol{\mathcal{D}} \boldsymbol{e}_x(\boldsymbol{\theta})] : \boldsymbol{e}_x(\boldsymbol{\psi}) + h \int_\Omega [\boldsymbol{\mathcal{G}}(\nabla_x w - \boldsymbol{\theta})] \cdot (\nabla_x z - \boldsymbol{\psi}) \\
= & \int_\Omega \left([\boldsymbol{R}(\omega^2)\boldsymbol{m}] \cdot \boldsymbol{\psi} + S(\omega^2) f z \right) \quad \forall \boldsymbol{\psi} \in \mathbf{H}_0^1(\Omega), z \in H_0^1(\Omega) \; ,
\end{aligned}
\tag{14}
$$

where $\boldsymbol{\mathcal{D}}$ is the 4th order tensor of homogenized elasticity coefficients see (21), $\boldsymbol{\mathcal{G}}$ is the 2nd order tensor given in (22) describing the shear stiffness of the plate, $\boldsymbol{\mathcal{M}}(\omega^2)$ and $\mathcal{N}(\omega^2)$ are homogenized mass coefficients. Below we explain how these coefficients are computed.

Characteristic microscopic responses. The homogenized coefficients are expressed in terms of the characteristic responses, the so-called corrector basis functions. We proceed in analogy with Ávila et al. (2008) where the 3D elasticity dynamic problems were considered. For the sake of brevity we employ the elasticity bilinear form

$$
c_{Y_m}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}) = \int_{Y_m} (\mathbb{C}^m \boldsymbol{e}_y(\tilde{\boldsymbol{u}})) : \boldsymbol{e}_y(\tilde{\boldsymbol{v}}) \; . \tag{15}
$$

The following characteristic responses depend exclusively on properties of the stiffer material in Y_m; two problems for the so-called corrector basis functions are to be solved:

- Find $\tilde{\boldsymbol{\theta}}^{rs} \in \mathbf{H}_\#^1(Y)/\mathbb{R}$ such that

$$
c_{Y_m}\left(\tilde{\boldsymbol{\theta}}^{rs} + \boldsymbol{\Pi}^{rs}, \tilde{\boldsymbol{v}} \right) = 0 \quad \forall \tilde{\boldsymbol{v}} \in \mathbf{H}_\#^1(Y_m) \; . \tag{16}
$$

- Find $\tilde{w}^k \in H_\#^1(Y_m)/\mathbb{R}$ such that

$$
\int_{Y_m} \gamma^m \nabla_y (\tilde{w}^k + y_k) \cdot \nabla_y \tilde{z} = 0 \quad \forall \tilde{z} \in H_\#^1(Y_m) \; . \tag{17}
$$

To express the homogenized mass coefficients, we need the eigenfrequencies and eigenfunctions which describe vibration of the inclusions clamped into the matrix. Two eigenvalue problems with discrete spectra are solved:

- Find $(\boldsymbol{\Theta}^r, \lambda^r) \in \mathbf{H}_{\#0}^1(Y) \times \mathbb{R}$ for $r = 1, 2, \ldots$ such that (note $\boldsymbol{\Theta}^r = (\Theta_i^r)$)

$$
\fint_{Y_c} [\mathbb{C}^c \boldsymbol{e}_y(\boldsymbol{\Theta}^r)] : \boldsymbol{e}_y(\boldsymbol{\psi}) = \lambda^r \fint_{Y_c} \rho \boldsymbol{\Theta}^r \cdot \boldsymbol{\psi} \quad \forall \boldsymbol{\psi} \in \mathbf{H}_{\#0}^1(Y) \; , \tag{18}
$$

- Find $(W^r, \mu^r) \in H_{\#0}^1(Y_c) \times \mathbb{R}$ for $r = 1, 2, \ldots$ such that

$$
\fint_{Y_c} [\gamma^c \nabla_y W^r] \cdot \nabla_y \zeta = \mu^r \fint_{Y_c} \rho W^r \zeta \quad \forall \zeta \in H_{\#0}^1(Y_c) \; . \tag{19}
$$

The eigenfunctions are normalized, so that

$$
\fint_{Y_c} \rho \boldsymbol{\Theta}^r \cdot \boldsymbol{\Theta}^s = \delta_{rs} \; , \qquad \fint_{Y_c} \rho W^r W^s = \delta_{rs} \; . \tag{20}
$$

Homogenized coefficients. The homogenized plate elasticity is represented by the following two tensors:

- *Homogenized "in-plane" elasticity* $\boldsymbol{\mathcal{D}} = (\mathcal{D}_{ijkl})$:

$$
\begin{aligned}
\mathcal{D}_{ijkl} &= c_{Y_m}\left(\tilde{\boldsymbol{\theta}}^{kl} + \boldsymbol{\Pi}^{kl},\, \boldsymbol{\Pi}^{ij}\right) \\
&= c_{Y_m}\left(\tilde{\boldsymbol{\theta}}^{kl} + \boldsymbol{\Pi}^{kl},\, \tilde{\boldsymbol{\theta}}^{ij} + \boldsymbol{\Pi}^{ij}\right) .
\end{aligned}
\tag{21}
$$

  The symmetric expression is obtained due to (16).

- *Homogenized shear elasticity* $\boldsymbol{\mathcal{G}} = (\mathcal{G}_{kl})$ introduced as

$$
\begin{aligned}
\mathcal{G}_{kl} &= \int_{Y_m} \gamma^m \partial_l^y (\tilde{w}^k + y_k) \\
&= \int_{Y_m} \gamma^m \nabla_y (\tilde{w}^k + y_k) \cdot \nabla_y (\tilde{w}^l + y_l) .
\end{aligned}
\tag{22}
$$

  The symmetric expression is obtained due to (17).

Inertia of the homogenized plate is represented by the following two mass coefficients, see Ávila et al. (2008); Rohan et al. (2009) for derivation of the analogical mass coefficients in 3D elasticity problems:

$$
\begin{aligned}
\boldsymbol{\mathcal{M}}(\omega^2) &= \boldsymbol{I} \fint_Y \rho - \sum_r \frac{\omega^2}{\omega^2 - \lambda^r} \fint_{Y_c} \rho \boldsymbol{\Theta}^r \otimes \fint_{Y_c} \rho \boldsymbol{\Theta}^r , \\
\mathcal{N}(\omega^2) &= \fint_Y \rho - \sum_r \frac{\omega^2}{\omega^2 - \mu^r} \left| \fint_{Y_c} \rho W^r \right|^2 .
\end{aligned}
\tag{23}
$$

Influence of the load is weighted by the load coefficients which are computed by similar formulae

$$
\begin{aligned}
\boldsymbol{\mathcal{R}}(\omega^2) &= \boldsymbol{I} - \sum_r \frac{\omega^2}{\omega^2 - \lambda^r} \fint_{Y_c} \rho \boldsymbol{\Theta}^r \otimes \fint_{Y_c} \boldsymbol{\Theta}^r , \\
\mathcal{S}(\omega^2) &= 1 - \sum_r \frac{\omega^2}{\omega^2 - \mu^r} \fint_{Y_c} \rho W^r \fint_{Y_c} W^r .
\end{aligned}
\tag{24}
$$

As the result of our homogenization procedure, we obtain Problem (14) where $(w, \boldsymbol{\theta})$ are the local amplitudes of harmonic waves excited by harmonic "homogenized" loads with frequency ω. Let us note that, when for some ω the tensor $\boldsymbol{\mathcal{M}}(\omega)$ is positive definite and the scalar $\mathcal{N}(\omega)$ is positive, then also free structure vibrations (i.e. stationary waves in domain Ω) can be excited. However, $\boldsymbol{\mathcal{M}}^*(\omega)$ or $\mathcal{N}(\omega)$ may not by positive (definite) for some ω; for the "membrane mode", cf. Ávila et al. (2008) and Rohan et al. (2009), we proved existence of whole frequency intervals – *the band gaps* – where the positivity of $\boldsymbol{\mathcal{M}}(\omega)$ fails. An analogical result can be proved for the coupled rotational and deflection modes: in each interval of frequencies $\omega^2 \in (\lambda^r, \lambda^{r+1})$ given by (18) there exists a sub-interval of frequencies for which $\boldsymbol{\mathcal{M}}(\omega)$ is not positive. In such intervals, free vibration "rotation modes" are restricted, or completely suppressed. Also for the shear modes associated with the deflection w and the corresponding mass $\mathcal{N}(\omega) < 0$, in each interval of frequencies $\omega^2 \in (\mu^r, \mu^{r+1})$ there exist subintervals with restricted or suppressed wave propagation.

Thus, the band gaps for stationary waves can be predicted just upon analyzing positive definiteness of $\boldsymbol{\mathcal{M}}(\omega)$ and $\mathcal{N}^*(\omega)$. Although for the membrane mode $\boldsymbol{u}$ such band gaps prediction holds also for guided plane waves in infinite plates, see Rohan et al. (2009), for the coupled modes $\boldsymbol{q} := (\boldsymbol{\theta}, w)$ the dispersion analysis is more complex. Interesting applications can be found Vasseur et al. (2008).

3.3. Kirchhoff-Love phononic plate

In analogy with the Reissner-Mindlin plate model (13), we consider the elastic standing waves, cf. Ghergu et al. (2007) for the plate homogenization. We find solutions to the following problem with the oscillating material coefficients: For a given frequency, find deflection $w^\varepsilon \in H_0^2(\Omega)$ such that

$$
\begin{aligned}
-\omega^2 h \int_{\Omega_c^\varepsilon} \rho^c w^\varepsilon v - \omega^2 h \int_{\Omega_m^\varepsilon} \rho^m w^\varepsilon v - \omega^2 \frac{h^3}{3} \int_{\Omega_m^\varepsilon} \rho^m \nabla w^\varepsilon \cdot \nabla v - \omega^2 \frac{h^3}{3} \int_{\Omega_c^\varepsilon} \rho^c \nabla w^\varepsilon \cdot \nabla v \\
+ \frac{h^3}{3} \int_{\Omega_m^\varepsilon} \mathbb{C}^m \nabla\nabla w^\varepsilon : \nabla\nabla v + \varepsilon^2 \frac{h^3}{3} \int_{\Omega_c^\varepsilon} \mathbb{C}^c \nabla\nabla w^\varepsilon : \nabla\nabla v \\
= \int_\Omega (f v + \boldsymbol{m} \cdot \nabla v) \qquad \forall v \in H_0^2(\Omega) \;.
\end{aligned}
\tag{25}
$$

To present the homogenization result for $\varepsilon \to 0$, we proceed in analogy with the case of the Reissner-Mindlin plates. Using the unfolding method of homogenization Cioranescu et al. (2008) and obtain the following equation for the transversal deflections $w \in H_0^2(\Omega)$ such that

$$
\begin{aligned}
-\omega^2 h \int_\Omega \bar{\rho} w v - \omega^2 \frac{h^3}{3} \int_\Omega (\boldsymbol{\mathcal{M}}(\omega^2) \nabla w) \cdot \nabla v + \frac{h^3}{3} \int_\Omega (\boldsymbol{\mathcal{D}} \nabla\nabla w) : \nabla\nabla v \\
= \int_\Omega \big([\boldsymbol{R}(\omega^2) \boldsymbol{m}] \cdot \nabla v + f v \big) \quad \forall v \in H_0^2(\Omega) \;,
\end{aligned}
\tag{26}
$$

where $\bar{\rho}$ is the average density of both material components situated in Y. Above $\boldsymbol{\mathcal{D}}$ is the 4th order homogenized bending stiffness tensor defined below in (31) and $\boldsymbol{\mathcal{M}}(\omega^2)$ is the homogenized mass tensor computed using a similar expression to (23), see (32). The "effective material parameters" are defined in terms of the characteristic microscopic responses.

Characteristic microscopic responses. In contrast with the Reissner-Mindlin plates, the cross-section rotations in the Kirchhoff-Love theory are fully determined by the gradients of deflections; consequently only two instead of four microscopic problems must be solved. The corrector basis function $\tilde{w}^{kl} \in H_\#^2(Y_m)$ solves the following equation

$$
\int_{Y_m} \left[\mathbb{C}^m \nabla\nabla_{yy} (\tilde{w}^{kl} + \Pi^{kl}) \right] : \nabla\nabla_{yy} \tilde{v} = 0 \quad \forall \tilde{v} \in H_\#^2(Y) \;,
\tag{27}
$$

where $\Pi^{kl} = y_k y_l$. To compute the homogenized mass tensor, one needs to solve the local problem: find $(\lambda^r, \boldsymbol{\varphi}^r) \in \mathbb{R} \times \boldsymbol{W}(Y_c)$ satisfying

$$
\int_{Y_c} [\mathbb{C}^c \nabla_y \boldsymbol{\varphi}^r] : \nabla_y \boldsymbol{\vartheta} = \lambda^r \int_{Y_c} \rho \boldsymbol{\varphi}^r \cdot \boldsymbol{\vartheta} \qquad \forall \boldsymbol{\vartheta} \in \boldsymbol{W}(Y_c) \;,
\tag{28}
$$

where we employ the spaces of rotation-free vector fields:

$$
\boldsymbol{W}(Y_c) = \{ \boldsymbol{w} \in \mathbf{H}_\#^1(Y_c) |\ \nabla_y \times \boldsymbol{w} = 0 \} \;.
\tag{29}
$$

Obviously, due to the ellipticity of the operator in (28), functions $\{\boldsymbol{\varphi}^r\}_r$ are orthogonal; we use the standard normalization

$$
\int_{Y_c} \rho \boldsymbol{\varphi}^r \cdot \boldsymbol{\varphi}^s = \delta_{rs} \;.
\tag{30}
$$

Homogenized coefficients The homogenized Kirchhoff-Love plate model involves the following material coefficients defined in terms of the characteristic responses just introduced:

- *Homogenized elastic coefficients* $\boldsymbol{\mathcal{D}} = (\mathcal{D}_{ijkl})$

$$
\begin{aligned}
\mathcal{D}_{ijkl} &= \fint_{Y_m} [\mathbb{C}^m \nabla\nabla_{yy} (\tilde{w}^{kl} + \Pi^{kl})] : \nabla\nabla_{yy} \Pi^{ij} \\
&= \fint_{Y_m} [\mathbb{C}^m \nabla\nabla_{yy} (\tilde{w}^{kl} + \Pi^{kl})] : \nabla\nabla_{yy} (\pi^{ij} + \Pi^{ij})
\end{aligned}
\tag{31}
$$

  where the symmetric expression follows due to (27);

- *Homogenized mass tensor* $\boldsymbol{\mathcal{M}} = (\mathcal{M}_{ij})$

$$\boldsymbol{\mathcal{M}}(\omega^2) = \boldsymbol{I} \fint_Y \rho - \sum_{r\geq 1} \frac{\omega^2}{\omega^2 - \lambda^r} \fint_{Y_c} \rho \boldsymbol{\varphi}^r \otimes \fint_{Y_c} \rho \boldsymbol{\varphi}^r \; . \tag{32}$$

- *Homogenized load coefficient* $\boldsymbol{\mathcal{R}} = (\mathcal{R}_{ij})$

$$\boldsymbol{\mathcal{R}}(\omega^2) = \boldsymbol{I} - \sum_{r\geq 1} \frac{\omega^2}{\omega^2 - \lambda^r} \fint_{Y_c} \rho \boldsymbol{\varphi}^r \otimes \fint_{Y_c} \boldsymbol{\varphi}^r \; . \tag{33}$$

By virtue of the right-hand side expression in (32), $\boldsymbol{\mathcal{M}}(\omega^2)$ can be negative, or negative semi-definite for some frequencies ω. In such a case, wave propagation can be restricted or even suppressed for modes characterized by the deflection gradient $\boldsymbol{\psi} := \nabla w$ being the eigenvector associated with the non-positive eigenvalue of $\boldsymbol{\mathcal{M}}(\omega^2)$. However, the theory explained in Ávila et al. (2008) for the standard 3D elasticity must be adapted because the first left hand side term in (26) does not change its sign and contributes to the positive inertia even for negative $\boldsymbol{\mathcal{M}}(\omega^2)$.

4. Conclusions

We presented homogenized models of wave propagation in strongly heterogeneous plates, considering the Reissner-Mindlin (R-M) and the Kirchhoff-Love (K-L) theories; while the first one takes into account shear effects related to rotations of the plate cross-section with respect tothe mid-plane, the second theory neglects this phenomenon, thus, being convenient for thin plates only. The homogenization results reveal dispersion properties for the homogenized R-M plates: we claim that there exist bands of frequencies for which the wave equations admit evanescent solutions only, at least for certain polarizations. There is remarkable difference between the R-M and K-L models: while for R-M the wave polarization is determined by components of $(\boldsymbol{\theta}, w)$, i.e. the rotation and deflection, for K-L there is just a scalar wave associated with the deflection w. Existence of the band gap effect for the K-L plates is to be examined in a more detail.

The phononic effect, in general, is associated with vibration modes excited at the "microscopic" level. By virtue of definitions (23) and (32), these modes determine "positivity", or "negativity" of the homogenized masses; in Ávila et al. (2008) we described how this observation can be employed to predict band gaps. The classical method of the band gap identification is based on analysis of guided waves, thus, upon construction of dispersion curves; it is necessary to compute frequencies for selected wave numbers ranging the Brillouin zone, cf. Rohan et al. (2009).

In a forthcoming publication we will study dispersion properties and band gaps distributions for some basic microstructures. An important restriction of both presented models is related to the transversal isotropy: here only cylindrical inclusions are admissible, although their shapes can be arbitrary. To treat more general composite plates with e.g. spheroidal inclusions, the homogenization procedure must be applied to a 3D composite with thickness proportional to ε, i.e. to the microstructure scale.

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