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COMPARISON OF TWO POSSIBLE APPROACHES TO INVERSE LAPLACE TRANSFORM APPLIED TO WAVE PROBLEMS

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Abstract: *This paper concerns the investigation of non-stationary wave phenomena in a thin elastic disc under radial impact by means of analytical methods. When the method of integral transforms is used for solving the system of PDEs describing a wave problem solved, one has to overcome the problem of inverse transform. This work focuses on two possible approaches to the inverse Laplace transform. Using the existing analytical solution of the problem, the classic method making use of the residue theorem and the method based on the numerical inverse Laplace transform are compared. Advantages and disadvantages of both approaches, mainly from computational point of view, are discussed and demonstrated.*

Keywords: ***thin disc, radial impact, non-stationary wave problem, analytical solution, inverse Laplace transform.***

1. Introduction

The utilization of analytical methods by the investigation of stationary and non-stationary wave problems in solids brings one significant benefit - the possibility of detailed insight into physical phenomena occurred. This main advantage of this approach can be then used for further analyses related for instance to dispersion and attenuation behaviour, to conditions under which specific types of waves can propagate etc. Consequently, the results of such analyses can be utilized for solving of forward or inverse problems of real components, equipments and structures. On the other hand, it is clear that the application of analytical methods is considerably limited by geometry and material properties of solids studied and by initial and boundary conditions assumed.

The application of Fourier method (separation of variables) in combination with appropriate integral transform represents a classic method for solving the system of PDEs describing a wave problem solved (Graff (1975)). In such cases, the inverse integral transform is one of the primary tasks of the process of analytical solution evaluation. Laplace transform represents one of the most used transform in time domain. There exist two possible approaches to its inversion, analytical and numerical. The first mentioned methods are based on the exact evaluation of Bromwich integral defining the inverse Laplace transform (ILT). This is usually done by the help of Cauchy's residue theorem (see Achenbach (1975)). The use of residue theorem is quite limited, usually to problems of elastic solids, e.g. the existence of branch points in problems of viscoelastic solids makes the inverse process much more complicated. On the other hand, the second mentioned methods making use of the numerical evaluation of Bromwich integral are more general and can be applied to more complicated problems. There exist more than one hundred algorithms for numerical inverse Laplace transform (NILT), from simple ones (see Duffy (2004)) to more sophisticated procedures which usually include sequence accelerators to improve the convergence of numerical process (see Cohen (2007)). The main disadvantage of NILT methods lies in the fact that they distort exact analytical solution. The utilization of modern Computer Algebra Systems (CASs), like Maple, Mathematica etc., which enable to perform difficult symbolic manipulations and multi-precision computations, is one of the possibilities how to overcome this problem.

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The main aim of this work is to show the possibility of application of selected NILT algorithm to a specific wave problem without the lost of analytical results accuracy and to demonstrate its robustness and efficiency. In particular, the problem of an elastic thin disc under radial impact is chosen for this purpose. The exact analytical solution, which can be found in Brepta and Červ (1978), is derived in detail in Červ (1974). The exact analytical formulae for displacement components and other mechanical quantities are derived by the help of residue theorem in the last-mentioned work.

2. Analytical solution of chosen wave problem

In this section we will formulate the wave problem used for the testing of analytical and numerical approach to ILT at first. Then a brief description of technique used in Červ (1974) for the derivation of analytical solution will be given and resulting formulae for displacement components transforms will be presented.

2.1. Problem formulation, governing equations

Let us assume a thin elastic disc of constant thickness, of finite radius r_1 and of homogeneous isotropic material properties described by Young modulus E and Poisson's ratio ν. This disc is loaded in radial direction by a uniformly distributed pressure of amplitude σ_0 acting on a part of its rim defined by the angle $2\alpha_0$ (see Fig. 1). The time history of applied load is described by Heaviside function in time so it invokes non-stationary wave phenomena in the disc studied. Introducing the polar coordinates r and φ as depicted in Fig. 1 and taking into account previous description, the external load can be expressed as

$$\sigma_r(r,\varphi,t)|_{r=r_1} = \begin{cases} \sigma_0 H(t) & \text{for } \varphi \in \langle -\alpha_0, \alpha_0 \rangle, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The hyperbolic system of PDEs representing equations of motion of the disc can be derived from momentum conservation formulated for a disc element and using appropriate constitutive and strain-displacement relations. It consists of two coupled PDEs for unknown functions of radial $u_r(r,\varphi,t)$ and circumferential $u_\varphi(r,\varphi,t)$ displacement components (see Červ (1974)). Introducing the well known relation between shear modulus G and parameters E and ν which holds under the assumption of material isotropy, it is useful to rewrite the original system of equations to a system for new unknown functions of dilatation $\Delta_d(r,\varphi,t)$ and rotation $\omega_z(r,\varphi,t)$ of the form

$$\frac{\partial^2 u_r}{\partial t^2} = {c_3}^2 \frac{\partial \Delta_d}{\partial r} - \frac{2{c_2}^2}{r}\frac{\partial \omega_z}{\partial \varphi}, \qquad \frac{\partial^2 u_\varphi}{\partial t^2} = \frac{{c_3}^2}{r}\frac{\partial \Delta_d}{\partial \varphi} + 2{c_2}^2 \frac{\partial \omega_z}{\partial r}, \tag{2}$$

where $\Delta_d(r,\varphi,t)$ and $\omega_z(r,\varphi,t)$ are defined by formulae

$$\Delta_d = \frac{\partial u_r}{\partial r} + \frac{1}{r}\left(u_r + \frac{\partial u_\varphi}{\partial \varphi}\right), \quad \omega_z = \frac{1}{2}\left[\frac{\partial u_\varphi}{\partial r} + \frac{1}{r}\left(u_\varphi - \frac{\partial u_r}{\partial \varphi}\right)\right]. \tag{3}$$

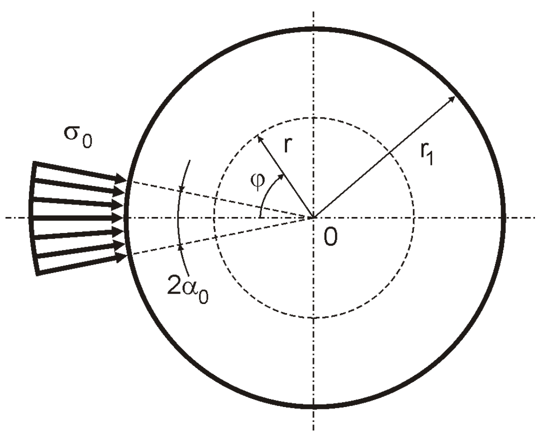


Fig. 1: The geometry of solved problem

Constants c_2 and c_3 correspond to the velocity of equivoluminal (shear) waves and to the velocity of dilatational waves in two-dimensional continuum, respectively (see Červ (1974)).

To complete the problem formulation, initial and boundary conditions have to be defined. For simplicity, the zero initial conditions for both displacement components and their time derivatives are considered. With respect to previous problem description, boundary conditions for stress components can be expressed as

$$\tau_{r\varphi}|_{r=r_1} = 0 \quad \text{and} \quad \sigma_r|_{r=r_1} = -\frac{2\alpha_0\sigma_0}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\sin(n\alpha_0)\cos(n\varphi)}{n\alpha_0}\right)H(t)\,, \tag{4}$$

when the expansion of (1) to the Fourier cosine series is used.

2.2. Derivation of final formulae for integral transforms of displacement components

The system (2) will be now solved by the application of Laplace transform in time domain following by Fourier method in spatial domain. When the Laplace transform is applied to (2), taking into account zero initial conditions, we obtain a system of PDEs for Laplace transforms of original time dependent functions, which can be converted to the system of uncoupled Bessel's type equations for the Laplace transforms of dilatation $\bar{\Delta}_d(r,\varphi,p)$ and rotation $\bar{\omega}_z(r,\varphi,p)$

$$\frac{\partial^2\bar{\Delta}_d}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{\Delta}_d}{\partial r} - \frac{p^2}{c_3^2}\bar{\Delta}_d + \frac{1}{r^2}\frac{\partial^2\bar{\Delta}_d}{\partial\varphi^2} = 0\,, \qquad \frac{\partial^2\bar{\omega}_z}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{\omega}_z}{\partial r} - \frac{p^2}{c_2^2}\bar{\omega}_z + \frac{1}{r^2}\frac{\partial^2\bar{\omega}_z}{\partial\varphi^2} = 0\,, \tag{5}$$

where $p \in \mathcal{C}$. The solution of system (5) can be then found using the separation of spatial variables r and φ and resulting relations for $\bar{\Delta}_d(r,\varphi,p)$ and $\bar{\omega}_z(r,\varphi,p)$ can be derived (for more details see Červ (1974)). Introducing these formulae into (3) we obtain required relations for integral transforms of radial $\bar{u}_r(r,\varphi,p)$ and circumferential $\bar{u}_\varphi(r,\varphi,p)$ displacement components in the form

$$\begin{aligned}
\bar{u}_r(r,\varphi,p) = &-\frac{c_3^2}{p^2}J_1\!\left(\frac{ip}{c_3}r\right)\left(\frac{ip}{c_3}\right)P_0(p) - \frac{1}{p^2}\sum_{n=1}^{\infty}\left\{{c_3}^2\left[\frac{n}{r}J_n\!\left(\frac{ip}{c_3}r\right) - \right.\right.\\
&\left.\left. - \left(\frac{ip}{c_3}\right)J_{n-1}\!\left(\frac{ip}{c_3}r\right)\right]P_n(p) + 2\,c_2^2\,\frac{n}{r}J_n\!\left(\frac{ip}{c_2}r\right)Q_n(p)\right\}\cos(n\varphi)\,,\\
\bar{u}_\varphi(r,\varphi,p) = &-\frac{1}{p^2}\sum_{n=1}^{\infty}\left\{2\,c_2^2\left[\frac{n}{r}J_n\!\left(\frac{ip}{c_2}r\right) - \left(\frac{ip}{c_2}\right)J_{n-1}\!\left(\frac{ip}{c_2}r\right)\right]Q_n(p) + \right.\\
&\left. + c_3^2\,\frac{n}{r}J_n\!\left(\frac{ip}{c_3}r\right)P_n(p)\right\}\sin(n\varphi)\,,
\end{aligned} \tag{6}$$

in which the symbol J_n denotes Bessel function of the first kind and n-th order.

The complex functions $P_n(p)$ $(n = 0,1,2,\ldots)$ and $Q_n(p)$ $(n = 1,2,\ldots)$ are unknown for now and they can be derived by the help of boundary conditions (4). Rewriting these conditions using constitutive and strain-displacement equations in term of displacement components and introducing (6) into their Laplace transforms, we obtain a system of algebraic equations for $P_n(p)$ and $Q_n(p)$. The final formulae of these functions, which are derived in detail in Červ (1974), are quite complicated and can be found in compact form in Brepta and Červ (1978). Subsequently, Laplace transforms of other mechanical quantities (e.g. velocity components, stress components etc.) can be derived on the basis of (6) with relative ease (see Brepta and Červ (1978)).

3. Inverse Laplace transform

This section deals with the basic description of two confronted approaches to ILT applied to the wave problem formulated above. The results of radial velocity transform inversion back to time domain are presented, analysed and discussed. This quantity is suitable for the purpose of this work because it is able to register steep fronts of waves propagated in the disc such that appropriate accuracy analysis of methods being compared can be made.

3.1. Analytical approach to ILT

The analytical ILT procedure is based on the Cauchy's residue theorem (see Achenbach (1975)) which can be written for $p \in \mathcal{C}$ in the form

$$\oint f(p)dp = 2\pi i \sum_{a_i \in A} \operatorname*{Res}_{p=a_i} f(p), \tag{7}$$

i.e. the value of a contour integral of an arbitrary analytic (holomorphic) function $f(p)$ for any enclosed contour is equal to the sum of residues in poles $a_i \in A$, where A is the set of poles contained inside the contour. It can be proved that both integral transforms (6) and other mechanical quantities transforms are holomorphic in complex plane except of their isolated singular points (poles) and hence the application of (7) is possible. To do so, we have to apply the Cauchy's integral theorem at first (Achenbach (1975)) to make the Bromwich-Wagner integration path of the integral defining ILT enclosed.

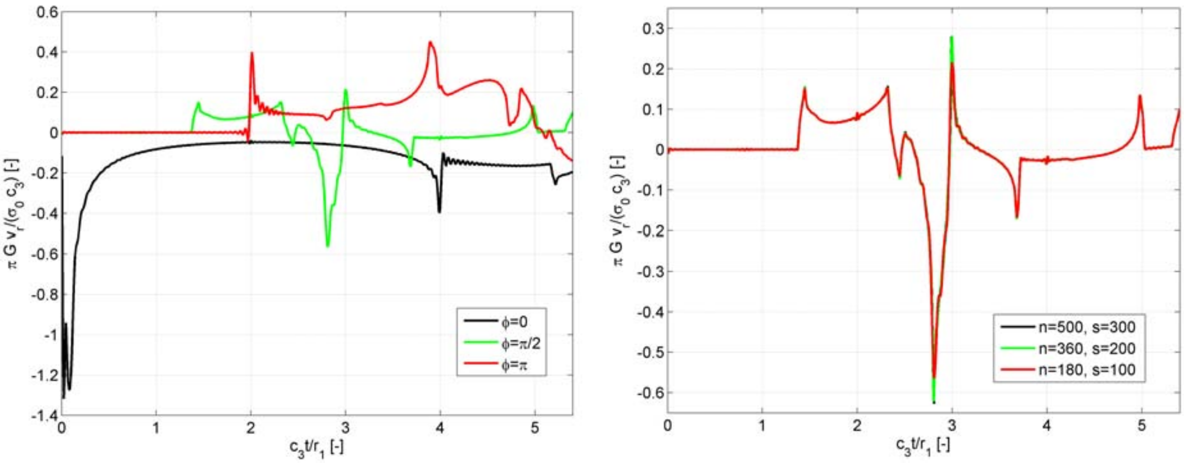
Using mentioned technique one can derive final exact analytical formula for required radial velocity $v_r(r, \varphi, t)$. Due to the complexity of this relation (see Červ (1974)) it is not possible to present it in this work and we confine only to some remarks regarding this solution. The first note is related to singular points of transforms (6). As shown in the last-mentioned work, these points are simple poles, i.e. complex functions (6) have no essential singularities. These poles can be found as the roots of appropriate frequency equation and for different wavenumber values they represent points of dispersion curves. This means that the use of analytical approach to ILT requires the determination of dispersion curves at first which causes added demands on CPU time. Additionally, it should be mentioned here that the accuracy of dispersion curves determination significantly influences the accuracy of analytical results obtained. The number of terms which are summed during the evaluation process is another important factor which affects the correctness of analytical results. The analytical formula for $v_r(r, \varphi, t)$, as well as the formulae of others quantities, contains two infinite sums: the first one follows from the application of Fourier method (see relations (6)) and the second one represents the summation over an infinite number of singular points (dispersion curves) following from (7). For subsequent usage in the following text, let us denote corresponding summation indices by n and s in sequence.

In spite of the presence of mentioned "numerical factors", the knowledge of actual formula in time domain, which follows from this ILT approach, enable us to determine the physical meaning of each term of derived analytical solution which can be then used for other analyses. In this case, the function $v_r(r, \varphi, t)$ consists of two parts: the first one represents the long-term (stationary) effect of inertial forces following from constant acceleration of all disc particles and the second one expresses the transient wave component of solution (see Červ (1974)).

The evaluation of analytical formula for $v_r(r, \varphi, t)$ has been done using the above mentioned procedure for following material and geometric parameters: $r_1 = 0.05\text{m}$, $\rho = 7800\text{kg}\,\text{m}^{-3}$, $\nu = 0.3$ and $E = 2.07 \cdot 10^{11}\text{Pa}$. The disc response to radial load specified by $\sigma_0 = 1\text{Pa}$ and $\alpha_0 = \pi/60$ have been studied in time interval $t \in \langle 0, 50 \rangle\,\mu\text{s}$ with constant step $\Delta t = 0.05\mu\text{s}$ at the disc rim and for $\varphi \in \{0, \pi/2, \pi\}$. Dimensionless plots of v_r time histories in selected points are depicted in Fig. 2. These results were obtained by computations performed in the system Maple using 17 significant digits. Fig. 2(a) presents the results in all three selected points for $n = 180$ and $s = 100$. It can be deduced from these curves, mainly from the oscillating character of v_r in the vicinity of steep fronts, that the values of n and s are too low for the solution to be able to represent the disc response correctly. This follows not only from the comparison of curves from Fig. 2(a) and Fig. 4(b) corresponding to $\varphi = \pi$ but also from Fig. 2(b) in which the results for $\varphi = \frac{\pi}{2}$ and for different values of n and s are presented. It is clear from this figure that results for $n = 360$, $s = 200$ and $n = 500$, $s = 300$ are nearly identical. But this conclusion can not be made in case of $\varphi = \pi$ as was verified by additional computations. Finally, one should mention that increasing number of significant digits used does not lead to higher accuracy of analytical results, but causes significant increase of total CPU time, as proved by computations carried out with 34 significant digits.

3.2. Numerical approach to ILT

Numerical approach to ILT consists in the numerical evaluation of Bromwich integral. As stated in Abate and Valkó (2004), algorithms for numerical inverse Laplace transform (NILT) can be usually divided



(a) (b)

Fig. 2: Results obtained by analytical approach to ILT, dimensionless time plots of v_r: (a) results for $n = 180$, $s = 100$ in three different points at the disc rim, (b) the influence of n and s for $\varphi = \frac{\pi}{2}$

into following four categories considering the methods which are based on: algorithms making use of Fourier series or deformation of Bromwich-Wagner contour and algorithms based on Gaver functionals or Laguerre functions. Many of them include the problem of infinite series, usually with low rate of convergence. In such cases, the methods are combined with a suitable sequence accelerator. A great overview of different NILT algorithms and accelerators is given in Cohen (2007).

Based on our previous experiences acquired by the analytical solution of analogous wave problem of a viscoelastic disc (see Adámek and Valeš (2011a), Adámek and Valeš (2011b)), the combination of FFT-based algorithm and non-linear Wynn's epsilon accelerator (ε-algorithm) was used. This method was adopted mainly from Brančík (1999) where it is used for NILT by the study of wave phenomena in electric circuits. The basic idea of this method consists in a discrete formula of Bromwich integral (see Brančík (1999)) such that the functional value of f in a discrete time kT ($k = 0, 1, \ldots, N-1$), where T is a sampling period in time domain, can be approximated by relation

$$f(kT) \approx f^k = C^k \left\{ 2\mathrm{Re}\left[\sum_{n=0}^{\infty} F_n\, E_n^k \right] - F_0 \right\}, \tag{8}$$

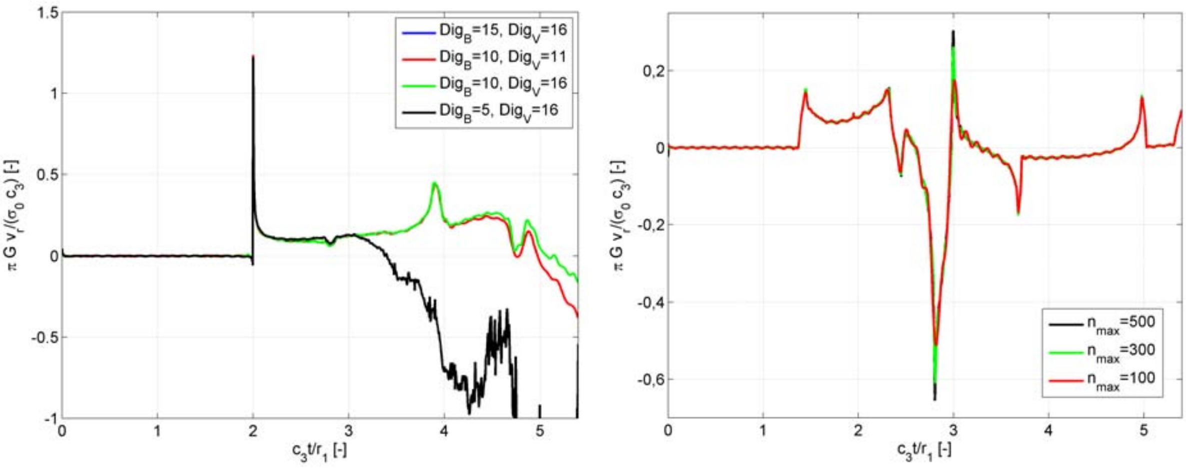
where

$$C^k = \frac{\Omega}{2\pi}\, e^{ckT}, \quad F_n = F(c - in\Omega), \quad E_n^k = e^{-ikTn\Omega}, \quad \Omega = \frac{2\pi}{NT} \quad \text{and} \quad c \approx \alpha - \frac{\Omega}{2\pi} \log E_r, \tag{9}$$

in which E_r denotes the desired relative error and α is an exponential order of the real function $f(t)$ (for more details see Brančík (1999) or Cohen (2007)).

Considering relations (8) and (9), the NILT procedure can be divided into two basic steps: in the first one, the functional values of the transform F in specific complex points are calculated; the second step involves the calculation of f^k including the infinite summation using chosen ε-algorithm. First numerical computations performed have shown that the first phase of NILT procedure is sensitive to cumulative numerical errors, therefore it must be carried out precisely to avoid the lost of "analytical" results accuracy. The system Maple, which enables to perform symbolic operations and multi-precision computations, has been used for this purpose. The second step of NILT process, which has not so high precision demands, has been implemented into the Matlab environment, in which the multi-dimensional array operations are much more faster compared to Maple. Using the combination of two mentioned systems, we obtain quite effective and stable tool for NILT of required formulae.

All numerical computations have been done for parameters stated in the section 3.1.. Performed computations have shown that the accuracy of evaluation process depends mainly on the number of digits used during the first phase of evaluation procedure mentioned above, namely on the number of



(a) (b)

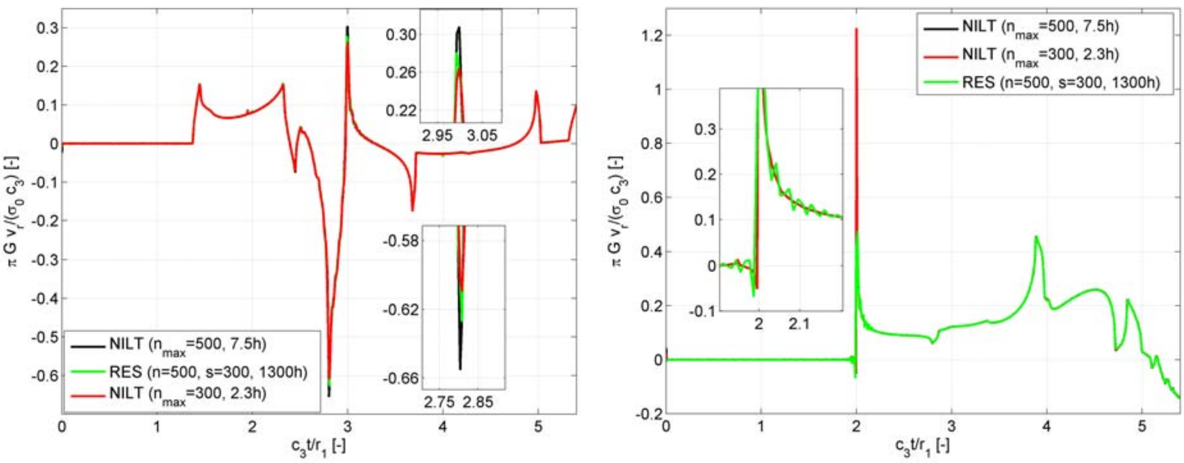
Fig. 3: Results obtained by NILT, dimensionless time plots of v_r: (a) the influence of number of digits used during computations for $\varphi = \pi$, (b) the influence of n_{max} for $\varphi = \frac{\pi}{2}$

digits Dig_B used for the computation of Bessel function values and number of digits Dig_V used for following operations needed for the determination of functional values of radial velocity transform V_r. The second significant factor, which is obvious from (6), is represented by the number of terms n_{max} which are summed in the infinite sum. The influence of mentioned factors is clear from Fig. 3 in which dimensionless time plots of v_r for different values of φ are presented. The detailed analyses have shown that if $Dig_B < 10$ and $Dig_V < 16$, the results for longer times are of pure accuracy (see Fig. 3(a)). In particular, when $Dig_V < 16$ and $Dig_B \geq 10$ the functional values of V_r for $n \geq 75$ are due to a large number of operations calculated incorrectly (their imaginary or real parts tend to infinity). On the other hand when $Dig_B < 10$ the increasing value of Dig_V can not compensate the errors of Bessel function values. If we use $Dig_B = 10$ and $Dig_V = 16$ or $Dig_B = 15$ and $Dig_V = 16$, we obtain nearly identical results (the green curve coincides with the blue one, see Fig. 3(a)). Further increasing of Dig_B and Dig_V does not bring significantly better results and leads to slow increasing of CPU time. Based on these results, further computations have been done using $Dig_B = 15$ and $Dig_V = 16$.

The dependence of ”analytical” results on the second main factor n_{max} is obvious for $\varphi = \frac{\pi}{2}$ from Fig. 3(b). It can be said that the low number of summed terms leads to the solution oscillation and to the reduction of dominant peaks in the disc response. This is caused by the fact that low value of n_{max} act as a ”frequency filter”. But contrary to previous factor, it does not cause the distortion for long times, so the time shape of v_r for different values of n_{max} is preserved in the whole time interval studied. Finally, one should mention that analogously to the method presented in the section 3.1., the right number of terms which could be summed to obtain results of required accuracy depends on the position of points in which the responses are studied, both in radial and tangential direction.

4. Results comparison and discussion

In this section, we present the comparison of ”analytical” results achieved by means of both methods described. Fig. 4 confronts the most accurate results obtained using the analytical ILT approach (curves *RES*), i.e. for $n = 500$ and $s = 300$, with those resulted from the application of chosen NILT algorithm (curves *NILT*) and of a comparable accuracy, i.e. for $n_{max} = 500$ and $n_{max} = 300$. It is evident that the dimensionless time plots of v_r are in good agreement both for $\varphi = \frac{\pi}{2}$ and for $\varphi = \pi$ in the whole time interval of interest. The main discrepancies occur in times corresponding to dominant peaks of v_r, as obvious from detailed views in Fig. 4(a) and from Fig. 4(b). The curves representing results of NILT coincide in the last mentioned figure. Additionally, the results of NILT are probably of higher accuracy, which is indicated by slightly oscillating character of curve *RES* in the vicinity of dominant peak, see the detail in Fig. 4(b).



(a) (b)

Fig. 4: Comparison of results obtained by analytical ILT and by NILT: (a) $\varphi = \frac{\pi}{2}$, (b) $\varphi = \pi$

Based on these results, we can make the first conclusion, such that the "analytical" results of comparable quality can be achieved by both approaches. But analogous conclusion can not be made in the case of total CPU time required by each method. These times in hours are stated in the legends of both figures in Fig. 4. Their values correspond to the computations performed on one 2.66 GHz processor. NILT approach requires only several hours depending on the value of n_{max} whereas the method based on the residue theorem needs non-comparably longer time to obtain results of the same accuracy. This extreme difference is mainly caused by the necessity of dispersion curves computation and by doing the double summation without the usage of a suitable sequence accelerator. Moreover, the total CPU time of NILT method could be significantly reduced if the process of Bessel functions evaluation, which consumes more than 95 % of computational time, is speed up by the help of well-known recurrent formulae for Bessel functions. But taking into account the increase of cumulative numerical errors, this improvement can be used only when the number of terms n_{max}, which have to be summed to achieved results of required accuracy, is approximately up to 35. This is applicable only in special cases when the response is investigated near the disc center, i.e. for small values of r.

If we should summarize advantages and disadvantages of both approaches tested, we can say that the analytical method making use of residue theorem, contrary to the NILT based method, gives us the insight into the physical meaning of each term of the analytical solution in time domain. But its numerical implementation is much more time demanding compared to the obtaining results of the same accuracy by the help of NILT procedure. Another significant advantage of the second mentioned method lies in the possibility of its application to the larger set of problems. Since this approach requires the knowledge of the solution in transform domain only, which can be derived easier than that in time domain, it can be used for more complicated wave problems, from geometrical, material and boundary/initial conditions point of view. But this method should be used cautiously because, as stated e.g. in Abate and Valkó (2004), there does not exist any universal NILT algorithm suitable for arbitrary problem so the verification of correctness and accuracy of obtained results by another method is important.

5. Conclusion

This work concerned the comparison of two different approaches to the matter of inverse Laplace transform in the solving process of a chosen non-stationary wave problem. It is demonstrated on the problem of a radial impact on an elastic disc that the results of required accuracy can be obtained not only by the help of an analytical method making use of residue theorem but also by strictly numerical procedure based on the combination of FFT algorithm and Wynn's epsilon algorithm. Presented results clearly show that the numerical approach to ILT does not cause the lost of analytical solution accuracy and that it is much more efficient than the analytical one. Furthermore, it can be applied to other more complicated wave problems, as proved by several works of authors dealing with the non-stationary wave problems of elastic and viscoelastic beams and discs.

Acknowledgments

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